Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 4

C2.1: Lie Algebras

20 December, 2021

Assume throughout the problems that we work over a field $\Bbbk$ *which is algebraically closed of characteristic zero, and all Lie algebras and representations are finite dimensional over* $\Bbbk$*, unless the contrary is explicitly stated.*

Question 1

(i) Show directly that if $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective homomorphism of semisimple Lie algebras and $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, then $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x) \in \mathfrak{g}_2$.

(ii) Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1, \phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{g}_2$.

(*For this part you may assume the fact, stated in lectures, that if $x = s + n$ is the Jordan decomposition of x and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then $\rho(s)$ is semsimple and $\rho(n)$ is nilpotent.*)

Proof. (i) The homomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ induces a representation $\psi : \mathfrak{g}_1 \to \mathfrak{gl}(\mathfrak{g}_2)$, defined by $\psi(x) = \operatorname{ad}\phi(x)$. For $y \in \mathfrak{g}_2$, since ϕ is surjective, there is $z \in \mathfrak{g}_1$ such that $y = \phi(z)$. Then

$$\psi(x)(y) = [\phi(x), y] = [\phi(x), \phi(z)] = \phi([x, z]) = \phi(\operatorname{ad}(x)(z))$$

- $\phi(s)$ commutes with $\phi(n)$: $[\phi(s), \psi(n)] = \phi([s, n]) = 0$.
- Since n is nilpotent, there exists $k \in \mathbb{N}$ such that $(\operatorname{ad} n)^k = 0$. Then for $y \in \mathfrak{g}_2$

$$\psi(n)^k(y) = \psi(n)^{k-1} \circ \phi(\operatorname{ad}(n)(z)) = \phi(\operatorname{ad}(n)^k(z)) = 0$$

  Hence $\psi(n)$ is nilpotent. $\phi(n)$ is ad-nilpotent.
- Since s is semisimple, the minimal polynomial p of $\operatorname{ad} s \in \mathfrak{gl}(\mathfrak{g}_1)$ has distinct roots over $\Bbbk$. Note that

$$p(\psi(s))(y) = \psi(p(s))(y) = \phi(\operatorname{ad}(p(s))(z)) = \phi(p(\operatorname{ad} s)(z)) = 0$$

  Hence $p(\psi(s)) = 0$. The minimal polynomial of $\psi(s)$ divides p. Therefore the minimal polynomial of $\psi(s)$ must have distinct roots. We deduce that $\psi(s)$ is semisimple. $\phi(s)$ is ad-semisimple.

This is more subtle than it seems.

Now by the uniqueness of abstract Jordan decomposition, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$.

(ii) Since $\psi : \mathfrak{g}_1 \to \mathfrak{gl}(\mathfrak{g}_2)$ is a representation, $\psi(s)$ is semisimple and $\psi(n)$ is nilpotent. Then $\phi(s)$ is ad-semisimple and $\phi(n)$ is ad-nilpotent. Clearly $[\phi(s), \phi(n)] = 0$. By the uniqueness of abstract Jordan decomposition, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$. □

$\phi(s)$ is semisimple on $\psi(\mathfrak{g}_1)$, not entirely on $\mathfrak{g}_2$.

Question 2

(i) Show that if V is a (finite dimensional) representation of $\mathfrak{sl}_2$ and $V = \bigoplus_{k\in\mathbb{Z}} V_k$ is the decomposition of V into (generalised) eigenspaces of h, then the number of irreducible constituents of V is equal to $\dim(V_0) + \dim(V_1)$

(ii) Show that if $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is the Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$, and α, β and $\alpha + \beta$ are all in Φ, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(*Hint: Use the representation theory of $\mathfrak{sl}_2$. You may assume all root spaces are 1-dimensional.*)

Proof. (i) Since $\mathfrak{sl}_2(\Bbbk)$ is semisimple, by Weyl's Theorem or Question 7 of Sheet 3, V is completely reducible. Suppose that $V = \bigoplus_{i=1}^{\ell} W_i$, where each W_i is a simple $\mathfrak{sl}_2(\Bbbk)$-submodule. By Question 6 of Sheet 3, each $W_i \cong V(\lambda)$ for some $\lambda \in \mathbb{Z}$. If λ is odd, there exists a unique $w_i \in W_i \setminus \{0\}$ (up to rescaling) such that $h(w_i) = 0$. Then $w_i \in V_0$. If λ is even, there exists a unique $w_i \in W_i \setminus \{0\}$ (up to rescaling) such that $h(w_i) = w_i$. Then $w_i \in V_1$.

In summary, we must have $\ell = \dim V_0 + \dim V_1$.

(ii) By Lemma 4.11, we know that (for arbitrary Lie algebra $\mathfrak{g}$) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

If $\beta = \lambda\alpha$ for some $\lambda \in \mathbb{C}$, then by Lemma 6.40, $\lambda = \pm 1$. Since $\alpha + \beta = (\lambda + 1)\alpha \in \Phi$, we also have $\lambda + 1 = \pm 1$. This is impossible. Then β and α are linearly independent in $H^\vee$. Consider the α-string through β:

$$\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$$

The root space $\mathfrak{g}_\alpha$ is 1-dimensional. Let $\mathfrak{sl}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ be the 3-dimensional subalgebra

$$\mathfrak{sl}_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

Since each root space is 1-dimensional, $L := \bigoplus_{i=-p}^{q} \mathfrak{g}_{\beta+i\alpha}$ is a $\mathfrak{sl}_\alpha$-submodule of $\mathfrak{g}$.

$h_\alpha \in \mathfrak{sl}_\alpha$ acts on $\mathfrak{g}_{\beta+i\alpha}$ with eigenvalue $(\beta + i\alpha)(h_\alpha)$. We know that $\alpha(h_\alpha) = 2$ and $\beta(h_\alpha) = p - q$. Then the eigenvalues of h_α on L form the set $\{-p-q, -p-q+2, \ldots, p+q-2, p+q\}$. In the set there is exactly one number equals to 0 or 1, which implies that the 0-weight and 1-weight subspace of L is 1-dimensional. By (i), L has one irreducible component, so itself is a simple $\mathfrak{sl}_\alpha$-module.

For $x \in \mathfrak{g}_\beta \setminus \{0\}$, $(\operatorname{ad} e_\alpha)(x) \in \mathfrak{g}_{\alpha+\beta}$. Since $x \in L$ is not of highest weight of h_α, $(\operatorname{ad} e_\alpha)(x) \neq 0$. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$. We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. □

Question 3

Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra $\mathfrak{g}$ is inner.

(*Hint: A derivation lets you construct a semi-direct product.*)

Proof. Let $\delta\colon \mathfrak{g} \to \mathfrak{g}$ be a derivation of $\mathfrak{g}$. Then we have a Lie algebra homomorphism $\varphi\colon \mathfrak{gl}_1 \to \mathfrak{g}$ given by $\varphi(a) = a\delta$, from which we can construct a semi-direct product $\mathfrak{h} := \mathfrak{gl}_1 \ltimes \mathfrak{g}$ with the Lie bracket

$$[(a,x),(b,y)] := \big(0, [x,y] + a\delta(y) - b\delta(x)\big)$$

$\mathfrak{h}$ affords a representation of $\mathfrak{g}$ given by

$$x \cdot (a,y) := (0, (\operatorname{ad} x)(y))$$

Then $\mathfrak{g}$ is a $\mathfrak{g}$-submodule of $\mathfrak{h}$. Since $\mathfrak{g}$ is semsimple, by Weyl's Theorem, $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{l}$, where $\mathfrak{l}$ is a 1-dimensional $\mathfrak{g}$-module. Then (*by the analogy of 5-lemma in Lie algebra?*) we have $\mathfrak{l} \cong \mathfrak{gl}_1$. Fix $(1,x) \in \mathfrak{l}$. For $(0,y) \in \mathfrak{g} \leqslant \mathfrak{h}$, we have

$$0 = [(1,x),(0,y)] = (0, [x,y] - \delta(y))$$

$\mathfrak{gl}_1$ is just k the base field, $\mathfrak{gl}_1 = \mathfrak{gl}(L)$ for any 1-dim v-space V.

Hence $\delta(y) = -[x,y]$. We conclude that $\delta = -\operatorname{ad} x$ is inner. □

Question 4

Suppose that $\mathfrak{g}$ is a Lie algebra and (V,ρ) is a faithful finite-dimensional representation (so that we may think of $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(V)$). Show that if V is irreducible and $\operatorname{tr}(\rho(x)) = 0$ for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is semisimple.

Proof. The radical $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ is a solvable ideal. By Lie's Theorem there exists $v \in V \setminus \{0\}$ and linear $\lambda\colon \operatorname{rad}\mathfrak{g} \to \mathsf{k}$ such that $\rho(z)(v) = \lambda(z)v$ for all $z \in \operatorname{rad}\mathfrak{g}$. Let $U := \{v \in V : \forall z \in \operatorname{rad}\mathfrak{g}\ (\rho(z)(v) = \lambda(z)v)\}$. Then $U \neq \{0\}$ by Lie's Theorem.

For $x \in \mathfrak{g}$, $z \in \operatorname{rad}\mathfrak{g}$, and $v \in U$, we have

$$\rho(z) \circ \rho(x)(v) = \rho(x) \circ \rho(z)(v) - \rho([x,z])(v) = \rho(x)(\lambda(z)v) - 0 = \lambda(z)\rho(x)(v)$$

Hence $\rho(x)(v) \in U$. Then U affords a subrepresentation of ρ. Since ρ is irreducible, we must have $U = V$.

For $z \in \operatorname{rad}\mathfrak{g}$, we have

$$0 = \operatorname{tr}\big(\rho(z)\big) = \lambda(z)\dim V \implies \lambda(z) = 0$$

Since ρ is faithful, we have $z = 0$. Hence $\operatorname{rad}\mathfrak{g} = \{0\}$. $\mathfrak{g}$ is semisimple. □

Question 5

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ be the symplectic Lie algebra. Show that $\mathfrak{h}$, the space of matrices in $\mathfrak{g}$ which are diagonal, is a Cartan subalgebra, and thus find the roots of $\mathfrak{sp}_{2n}$.

(Optional: Do the same for the Lie algebras $\mathfrak{so}_{2n}$.)

Proof. Recall that $\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} : x^\top S + Sx = 0\}$, where

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \qquad J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

Let $A^\perp$ be the matrix obained by flipping A anti-diagonally. If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}_{2n}$, then $B = B^\perp$, $C = C^\perp$, and $D = -A^\perp$. Hence

$$\mathfrak{h} = \left\{\begin{pmatrix} D & 0 \\ 0 & -D^\perp \end{pmatrix} : D = \operatorname{diag}\{a_1, \dots, a_n\},\ a_1, \dots, a_n \in \mathrm{k}\right\}$$

It is clear that $\mathfrak{h}$ is Abelian, and hence nilpotent. Let $X \in N_{\mathfrak{sp}_{2n}}(\mathfrak{h})$. We write $X = A + B$, where A is the diagonal part and B is the off-diagonal part. Then $[X, D] = [B, D] \in \mathfrak{h}$ for all $D \in \mathfrak{h}$. Note that, for $i \neq j$, $[E_{ij}, E_{kk}]$ has zero diagonal entries. Hence $[B, D] = 0$ for all $D \in \mathfrak{h}$. Somehow we can prove that $B = 0$. Hence $X \in \mathfrak{h}$. $N_{\mathfrak{sp}_{2n}}(\mathfrak{h}) = \mathfrak{h}$. We deduce that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sp}_{2n}$.

$\mathfrak{h}$ has a basis $\{E_{k,k} - E_{2n+1-k,2n+1-k} : k \in \{1, \dots, n\}\}$. Let $\{\varphi_1, \dots, \varphi_n\}$ be the dual basis of it.

$\mathfrak{sp}_{2n}$ has the decomposition

$$\mathfrak{sp}_{2n} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_\delta \oplus \mathfrak{g}_\varepsilon$$

where

$$\mathfrak{g}_\alpha = \left\{\begin{pmatrix} D & 0 \\ 0 & -D^\perp \end{pmatrix} : \forall\, i \in \{1, \dots, n\}\ D_{ii} = 0\right\} = \operatorname{span}\{E_{i,j} - E_{2n+1-i,2n+1-j} : 1 \leqslant i \neq j \leqslant n\}$$

$$\mathfrak{g}_\beta = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B = B^\perp,\ \forall\, i \in \{1, \dots, n\}\ B_{i,2n+1-i} = 0\right\} = \operatorname{span}\{E_{i,n+j} + E_{n+1-j,2n+1-i} : 1 \leqslant i + j \leqslant n\}$$

$$\mathfrak{g}_\gamma = \left\{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C = C^\perp,\ \forall\, i \in \{1, \dots, n\}\ C_{i,2n+1-i} = 0\right\} = \operatorname{span}\{E_{n+i,j} + E_{2n+1-j,n+1-i} : 1 \leqslant i + j \leqslant n\}$$

$$\mathfrak{g}_\delta = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B = B^\perp,\ \forall\, i \neq 2n+1-j\ B_{ij} = 0\right\} = \operatorname{span}\{E_{i,2n+1-i} : 1 \leqslant i \leqslant n\}$$

$$\mathfrak{g}_\varepsilon = \left\{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C = C^\perp,\ \forall\, i \neq 2n+1-j\ C_{ij} = 0\right\} = \operatorname{span}\{E_{n+i,n+1-i} : 1 \leqslant i \leqslant n\}$$

If $X = \sum_{i=1}^n \lambda_i (E_{i,i} - E_{2n+1-i,2n+1-i})$, then

$$[X, E_{i,j}] = (\lambda_i - \lambda_j) E_{i,j}, \qquad [X, E_{i,n+j}] = (\lambda_i + \lambda_j) E_{i,n+j}, \qquad [X, E_{n+i,j}] = -(\lambda_i + \lambda_j) E_{n+i,j}$$

This gives you the Cartan decomp. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

The weights are given by

$$\alpha_{i,j} = \varphi_i - \varphi_j, \qquad \beta_{i,j} = \varphi_i + \varphi_j, \qquad \gamma_{i,j} = -\varphi_i - \varphi_j, \qquad \delta_i = 2\varphi_i, \qquad \varepsilon_i = -2\varphi_{n-i}$$

You can use this to show $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

We simply observe that Φ has a base $\{\varphi_1 - \varphi_2, \ldots, \varphi_{n-1} - \varphi_n, 2\varphi_n\}$. For convenience we write $\sigma_i := \varphi_i - \varphi_{i+1}$ and $\tau := 2\varphi_n$. Next we shall calculate the Cartan matrix.

Let $e_{\sigma_i} = E_{i,i+1} - E_{2n+1-i,2n-i}$. Then

$$h_{\sigma_i} = [E_{i,i+1} - E_{2n+1-i,2n-i}, E_{i+1,i} - E_{2n-i,2n+1-i}] = E_{i,i} - E_{i+1,i+1} + E_{2n+1-i,2n+1-i} - E_{2n-i,2n-i}$$

Hence

$$\langle \sigma_i, \sigma_j \rangle = (\varphi_i - \varphi_{i+1})(h_{\sigma_j}) = -\delta_{i,j-1} + 2\delta_{i,j} - \delta_{i,j+1}$$

and

$$\langle \tau, \sigma_i \rangle = 2\varphi_i(h_{\sigma_i}) = -2\delta_{i-1,n}$$

Let $e_\tau = E_{n,n+1}$. Then $h_\tau = [E_{n,n+1}, E_{n+1,n}] = E_{n,n} - E_{n+1,n+1}$. Hence

$$\langle \sigma_i, \tau \rangle = \sigma_i(h_\tau) = (\varphi_i - \varphi_{i+1})(E_{n,n} - E_{n+1,n+1}) = -\delta_{i-1,n}$$

The Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}$$

The Dyakin diagram is given by

$$\bullet \;\text{——}\; \bullet \;\cdots\cdots\; \bullet \;\text{——}\; \bullet \Longleftarrow \bullet$$

good.

We conclude that $\mathfrak{sp}_{2n}$ has type C_n. $\square$

Question 6

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subseteq \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of $\mathfrak{g}$ in terms of Φ. (In particular, the dimension of $\mathfrak{g}$ is determined by the root system).

Proof. The Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Since $\mathfrak{g}$ is semisimple, we know that each $\mathfrak{g}_\alpha$ is 1-dimensional, and Φ spans $\mathfrak{h}^\vee$. Hence

$$\dim \mathfrak{g} = \dim \mathfrak{h} + |\Phi| = \dim \mathfrak{h}^\vee + |\Phi| = \dim \operatorname{span} \Phi + |\Phi|$$ $\square$

Question 7

Let V be a $\mathbb{Q}$-vector space. A lattice in V is a discrete subgroup [1] $Q \subseteq V$ which spans V over $\mathbb{Q}$. Equivalently, a lattice is a subgroup Q of V of the form

$$\left\{\sum_i \lambda_i \beta_i : \lambda_i \in \mathbb{Z}\right\}$$

where $\{\beta_i\}_{i=1}^n$ is a basis of V. (You do not have to prove this). Assume that V is equipped with an positive definite inner product $(-,-)$. A lattice $Q \subseteq V$ is called *integral* if $(\alpha,\beta) \in \mathbb{Z}$ for all $\alpha,\beta \in Q$. A lattice Q is called *even* if $(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$

(i) Show that an even lattice is integral.

(ii) Let $Q \subseteq V$ be an even lattice. Assume that the set $R_Q = \{\alpha \in Q : (\alpha,\alpha) = 2\}$ spans V. Show that R_Q is a root system in V.

(iii) Let $V = \bigoplus_{i=1}^r \mathbb{Q}e_i$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Let

$$\Gamma_r = \left\{\sum a_i e_i : \sum a_i \in 2\mathbb{Z} \text{ and either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}\cdot\right\}$$

Show that Γ_r is an even lattice if r is divisible by 8 .

(iv) Consider $\Gamma = \Gamma_8 \subseteq \mathbb{Q}^8$. Show that V is spanned by the vectors $v \in \Gamma$ such that $(v,v) = 2$, and describe the roots in the resulting root system R_Γ

(v) Consider the functional $t \in V^*$ given by

$$t = \sum_{i=1}^{7} (i-1)e_i^* + 23e_8^*$$

where $\{e_i^*\}_{i=1}^8$ is the dual basis to $\{e_i\}_{i=1}^8$. Show that $0 \neq t(\alpha) \in \mathbb{Z}$ for all $\alpha \in R_\Gamma$. Calculate the set of roots $\alpha \in R_\Gamma$ with $t(\alpha) = 1$ and check it is a basis of V. Compute the matrix of the inner product with respect to this basis. (This step is similar to the proof that a root system has a base.)

(vi) Let H_7 denote the hyperplane V orthogonal to $e_7 + e_8$. Show that $R_\Gamma \cap H_7$ is a root system of and find a basis for H_7 contained in it (*hint: start with the basis in the previous part.*)

(vii) Let H_6 be the subspace orthogonal to $e_6 + e_7 + 2e_8$ and $e_7 + e_8$. Show that $R_\Gamma \cap H_6$ is a root system and calculate a basis for H_6 contained in it.

Proof. (i)

□