Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3
B1.1: Logic

July 31, 2020

Question 1 α

Show that L_0 and the sequent calculus SQ are equivalent. I.e., using the language $\mathcal{L}_0 = \mathcal{L}[\{\neg, \rightarrow\}]$ of propositional calculus, for all $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ and for all $\phi \in \text{Form}(\mathcal{L}_0)$:

$$\Gamma \vdash_{L_0} \phi \text{ if and only if } \Gamma \vdash_{SQ} \phi.$$

Proof. We first show that L_0 implies SQ. The deduction rules Ass and MP are trivial. DT is implied by the Deduction Theorem in L_0. We only need to prove PC:

Suppose that $\Delta \cup \{\neg\psi\} \vdash_{L_0} \chi$ and $\Delta' \cup \{\neg\psi\} \vdash_{L_0} \neg\chi$. Hence $\Delta \cup \Delta'\{\neg\psi\} \vdash_{L_0} \{\chi, \neg\chi\}$. By Deduction Theorem, we have

1:	$\Delta \cup \Delta' \vdash_{L_0} (\neg\psi \rightarrow \chi)$	[DT]
2:	$\Delta \cup \Delta' \vdash_{L_0} (\neg\psi \rightarrow \neg\chi)$	[DT]
3:	$\vdash_{L_0} (\chi \rightarrow (\neg\chi \rightarrow \neg(\psi \rightarrow \psi)))$	[Sheet 2 Question 5.(i)]
4:	$\Delta \cup \Delta' \vdash_{L_0} (\neg\psi \rightarrow (\neg\chi \rightarrow \neg(\psi \rightarrow \psi)))$	[HS 1,3]
5:	$\vdash_{L_0} ((\neg\psi \rightarrow (\neg\chi \rightarrow \neg(\psi \rightarrow \psi))) \rightarrow ((\neg\psi \rightarrow \neg\chi) \rightarrow (\neg\psi \rightarrow \neg(\psi \rightarrow \psi))))$	[A2]
6:	$\Delta \cup \Delta' \vdash_{L_0} ((\neg\psi \rightarrow \neg\chi) \rightarrow (\neg\psi \rightarrow \neg(\psi \rightarrow \psi)))$	[MP 4,5]
7:	$\Delta \cup \Delta' \vdash_{L_0} (\neg\psi \rightarrow \neg(\psi \rightarrow \psi))$	[MP 2,6]
8:	$\vdash_{L_0} ((\neg\psi \rightarrow \neg(\psi \rightarrow \psi)) \rightarrow ((\psi \rightarrow \psi) \rightarrow \psi))$	[A3]
9:	$\Delta \cup \Delta' \vdash_{L_0} ((\psi \rightarrow \psi) \rightarrow \psi)$	[MP 7,8]
10:	$\vdash_{L_0} (\psi \rightarrow \psi)$	[Theorem]
11:	$\Delta \cup \Delta' \vdash_{L_0} \psi$	[MP 9,10]

In L0 we do not vary the set of premises in one proof, so we usually state the premises at the beginning and omit it from the exact lines of the proof

✓

Next we show that SQ implies L_0. Proof of MP is trivial. Proof of A1:

1:	$\{\alpha, \beta\} \vdash_{SQ} \alpha$	[Ass]
2:	$\alpha \vdash_{SQ} (\beta \rightarrow \alpha)$	[DT]
3:	$\vdash_{SQ} (\alpha \rightarrow (\beta \rightarrow \alpha))$	[DT]

✓

Proof of A2:

1:	$\alpha \vdash_{SQ} \alpha$	[Ass]
2:	$(\alpha \rightarrow \beta) \vdash_{SQ} (\alpha \rightarrow \beta)$	[Ass]
3:	$(\alpha \rightarrow (\beta \rightarrow \gamma)) \vdash_{SQ} (\alpha \rightarrow (\beta \rightarrow \gamma))$	[Ass]
4:	$\{\alpha, (\alpha \rightarrow \beta)\} \vdash_{SQ} \beta$	[MP 1,2]
5:	$\{\alpha, (\alpha \rightarrow (\beta \rightarrow \gamma))\} \vdash_{SQ} (\beta \rightarrow \gamma)$	[MP 1,3]
6:	$\{\alpha, (\alpha \rightarrow \beta), (\alpha \rightarrow (\beta \rightarrow \gamma))\} \vdash_{SQ} \gamma$	[MP 4,5]
7:	$\{(\alpha \rightarrow \beta), (\alpha \rightarrow (\beta \rightarrow \gamma))\} \vdash_{SQ} (\alpha \rightarrow \gamma)$	[DT]
8:	$(\alpha \rightarrow (\beta \rightarrow \gamma)) \vdash_{SQ} ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$	[DT]
9:	$\vdash_{SQ} ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$	[DT]

✓

Proof of A3:

1:	$\{(\neg\beta \rightarrow \neg\alpha), \neg\beta\} \vdash_{SQ} \neg\beta$	[Ass]
2:	$\{(\neg\beta \rightarrow \neg\alpha), \neg\beta\} \vdash_{SQ} (\neg\beta \rightarrow \neg\alpha)$	[Ass]
3:	$\{(\neg\beta \rightarrow \neg\alpha), \neg\beta\} \vdash_{SQ} \neg\alpha$	[MP 1,2]
4:	$\{\alpha, \neg\beta\} \vdash_{SQ} \alpha$	[Ass]
5:	$\{\alpha, (\neg\beta \rightarrow \neg\alpha)\} \vdash_{SQ} \beta$	[PC 3,4]
6:	$(\neg\beta \rightarrow \neg\alpha) \vdash_{SQ} (\alpha \rightarrow \beta)$	[DT]
7:	$\vdash_{SQ} ((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta))$	[DT]

✓

We have shown that the axioms and rules of L_0 and SQ are equivalent. Hence $\Gamma \vdash_{L_0} \phi$ if and only if $\Gamma \vdash_{SQ} \phi$. □

Question 2 α

The Four Colour Theorem asserts that if a region in the plane is divided into finitely many countries, then each country may be coloured either red, green, blue, or yellow in such a way that no two countries with a common border (of positive length) get the same colour. Use the Compactness Theorem to show that this remains true even if there are countably infinitely many countries.

Proof. We assume that every country is a connected set on $\mathbb{R}^2$ with positive Lebesgue measure. Hence there are at most countably many countries. We enumerate them as $c_1, c_2, c_3, \ldots$. For each c_n, we associate it with two propositional variables p_{2n-1} and p_{2n}. For each assignment v, we extend it on $\mathscr{C} := \{c_n : n \in \mathbb{Z}_+\}$ by defining:

$$\tilde{v}(c_n) = \begin{cases} A, & v(p_{2n-1}) = T \text{ and } v(p_{2n}) = T \\ B, & v(p_{2n-1}) = T \text{ and } v(p_{2n}) = F \\ C, & v(p_{2n-1}) = F \text{ and } v(p_{2n}) = T \\ D, & v(p_{2n-1}) = F \text{ and } v(p_{2n}) = F \end{cases}$$

where A, B, C, D are colours.

Let

$$\mathscr{A} = \{(m, n) : c_n \text{ and } c_m \text{ are adjacent}\}.$$

Consider

$$\Gamma := \{\phi_{m,n} = \neg((p_{2n-1} \leftrightarrow p_{2m-1}) \wedge (p_{2n} \leftrightarrow p_{2m})) : (m, n) \in \mathscr{A}\},$$

where $\leftrightarrow$ is a binary connective defined by

$$(\phi \leftrightarrow \psi) \equiv ((\phi \to \psi) \wedge (\psi \to \phi)).$$

Note that, by truth table, $\tilde{v}(\phi_{n,m}) = F$ if and only if $\tilde{v}(c_n) = \tilde{v}(c_m)$. A colouring of $\mathscr{C}$ uniquely determines an assignment v. Hence there exists a correct colouring if and only if Γ is satisfiable. By Compactness Theorem, Γ is satisfiable if and only it is finitely satisfiable. But any finite subset of Γ only concerns finitely many countries, on which the Four Colour Theorem hold. Hence the Four Colour Theorem holds for countably many countries. □

Question 3

State and prove the unique readability theorem for predicate calculus

(a) for terms,

(b) for atomic formulae,

(c) for all formulae.

Proof. (a) Unique Readability Theorem for Terms:

For each term t of $\mathcal{L}^{\text{FOPC}}$, exactly one of the following holds:

(i) $t = x_i$ for a unique $i \in \mathbb{N}$;

(ii) $t = f_i^{(k)}(t_1, .., t_k)$ for unique terms $t_1, .., t_k$ and a unique function symbol $f_i^{(k)}$.

We first prove that **any proper initial substring of a term in $\mathcal{L}^{\text{FOPC}}$ is not a term**. We use induction on the length of the term. Suppose that the result holds for all terms of length less than n.

Suppose that t is a term in $\mathcal{L}^{\text{FOPC}}$. $n = 1$ is trivial. Assume that $n > 1$. Then $t = f_i^{(k)}(t_1, \ldots, t_k)$ for some terms $t_1, .., t_k$ and some function symbol $f_i^{(k)}$. The proper initial substring of t is one of the following:

(1) $f_i^{(k)}$;

(2) $f_i^{(k)}(t_1, \ldots, t_\ell$ where $\ell \in \{0, \ldots, k\}$;

(3) $f_i^{(k)}(t_1, \ldots, t_\ell,$ where $\ell \in \{1, \ldots, k\}$;

(4) $f_i^{(k)}(t_1, \ldots, t_\ell, s$ where $\ell \in \{0, \ldots, k-1\}$ and s is a proper initial substring of $t_{\ell+1}$.

(1) is not a term, because it has length 1 and is not a variable symbol. (3) is not a term because a term longer than 1 must end with).

Suppose that (2) is a term. There exists function symbol $f_j^{(m)}$ and terms $u_1, \ldots, u_m$ such that

$$f_i^{(k)}(t_1, \ldots, t_\ell = f_j^{(m)}(u_1, \ldots, u_m).$$

Then $f_i^{(k)} = f_j^{(m)}$. In particular $i = j$ and $k = m$. If $t_1 \neq u_1$, then either t_1 is a proper initial substring of u_1, or u_1 is a proper initial substring of t_1. But t_1 and u_1 are terms of length shorter than n. By induction hypothesis, this is impossible. Hence $t_1 = u_1$. Inductively, we have $t_p = u_p$ for all $p \in \{1, ..., \ell\}$. This is impossible, as the length of $f_i^{(k)}(u_1, ..., u_k)$ is longer than $f_i^{(k)}(u_1, ..., u_\ell$.

Suppose that (4) is a term. There exists function symbol f_j^m and terms $u_1, ..., u_m$ such that

$$f_i^{(k)}(t_1, ..., s = f_j^{(m)}(u_1, ..., u_m).$$

As above, we can prove that $i = j$, $k = m$, and $t_p = u_p$ for all $p \in \{1, ..., \ell\}$. Hence

$$s = u_{\ell+1}, ..., u_k)$$

In particular, $u_{\ell+1}$ is a poper initial substring of s. This is impossible because the length of s is shorter than n.

This completes the induction. Now we return to the proof of unique readability.

Suppose that t is a term. t is a variable symbol if and only if it has length 1. Suppose that $t = f_i^{(k)}(t_1, .., t_k) = f_j^{(l)}(s_1, ..., s_l)$ for some terms $t_1, ..., t_k, s_1, ..., s_l$. Then $f_i^{(k)} = f_j^{(l)}$. Hence $i = j$ and $k = l$. If $t_1 \neq s_1$, than either t_1 is a proper initial substring of s_1, or s_1 is a proper initial substring of t_1, both of which are impossible. Thus $t_1 = s_1$. Inductively, we have $t_n = s_n$ for each $n \in \{1, ..., k\}$. Hence the term t is uniquely readable. ✓

(b) Unique Readability Theorem for Atomic Formulae:

For each atomic formula α of $\mathcal{L}^{\mathrm{FOPC}}$, exactly one of the following holds:

(i) $\alpha = P_i^{(k)}(t_1, ..., t_k)$ for unique terms $t_1, .., t_k$ and a unique predicate symbol $P_i^{(k)}$;

(ii) $\alpha = t_1 \doteq t_2$ for unique terms t_1 and t_2.

It is clear that the cases are mutually exclusive. If α is an atomic formula where $\alpha = P_i^{(k)}(t_1, ..., t_k)$, then α is uniquely readable similar to (a).(ii). If $\alpha = t_1 \doteq t_2 = s_1 \doteq s_2$ for some terms t_1, t_2, s_1, s_2. If $t_1 \neq s_1$, then either t_1 is a proper initial substring of s_1, or s_1 is a proper initial substring of t_1, both of which is impossible. Then $t_1 = s_1$ and hence $t_2 = s_2$. Therefore the atomic formula α is uniquely readable.

More simply, the atomic formula t1 = t2 has one unique equality symbol in it, so the position of that symbol uniquely determines t1 and t2

(c) Unique Readability Theorem for Formulae:

For each formula ϕ of $\mathcal{L}^{\mathrm{FOPC}}$, exactly one of the following holds:

(i) ϕ is an atomic formula;

(ii) $\phi = \neg\psi$ for a unique formula ψ;

(iii) $\phi = (\psi \to \chi)$ for unique formulae ψ and χ;

(iv) $\phi = \forall x_i \alpha$ for a unique formula α and a unique variable symbol x_i.

We first prove that **any proper initial substring of a formula in $\mathcal{L}^{\mathrm{FOPC}}$ is not a formula**. We use induction on the length of the formula. Suppose that the result holds for all formulae of length less than n.

Suppose that ϕ is a formula in $\mathcal{L}^{\mathrm{FOPC}}$ of length n.

(i) If ϕ is an atomic formula, then it is one of (b).(i) or (b).(ii). For the case (b).(i), we have proven the result. For the case (b).(ii), the proper initial substring of ϕ is one of the following:

(1) s (proper initial substring of t_1);

(2) t_1;

(3) $t_1 \doteq$;

(4) $t_1 \doteq s$ where s is a proper initial substring of t_2.

(1) and (2) are clearly not atomic formulae, as they do not contain predicate symbols or $\doteq$. (3) is not an atomic formulae, as atomic formulae cannot end with $\doteq$. Suppose that (4) is an atomic formula. By unique readability s is a term, which contradicts the result proven in (a).

It helps if you write down the most fundamental observation that atomic formulae cannot be read as non-atomic formulae (and vice versa) because atomic formulae have no connectives/quantifiers

(ii) If $\phi = \neg\psi$, then the proper initial substring of ϕ is one of the following:

(1) $\neg$;

(2) $\neg\alpha$ where α is a proper initial substring of ψ.

(1) is not a formula. By induction hypothesis, since ψ has length less than n, then α is not a formula. Hence $\neg\alpha$ is not a formula either.

(iii) If $\phi = (\psi \to \chi)$, then the proper initial substring of ϕ is one of the following:

(1) $($;

(2) $(\alpha$ where α is a proper initial substring of ψ;

(3) $(\psi$;

(4) $(\psi \to$;

(5) $(\psi \to \alpha$ where α is a proper initial substring of χ;

(6) $(\psi \to \chi$.

You mean (4)?

(1) and (3) are clearly not formulae. Suppose that (2) is a formula. Since it begins with $($, there exists formulae β and γ such that $(\alpha = (\beta \to \gamma)$. Then β is a proper initial substring of α. But α is of length less than n. By induction hypothesis this is impossible. Similarly we can prove that none of (3),(5),(6) are formulae.

(iv) If $\phi = \forall x_i \alpha$, then the proper initial substring of ϕ is one of the following:

(1) $\forall$;

(2) $\forall x_i$;

(3) $\forall x_i \beta$ where β is a proper initial substring of α.

(1) and (2) are clearly not formulae. Suppose that (3) is a formula. Then there exists variable symbol x_j and formula γ such that $\forall x_i \beta = \forall x_j \gamma$. Immediately we have $i = j$ and $\beta = \gamma$. But α is of length less than n. By induction hypothesis β is not formula. We obtain a contradiction.

After listing all the cases, we finish the induction. Now we return to the proof of unique readability.

It is clear that all cases are mutually exclusive. The unique readability for Cases (i), (ii) and (iv) is trivial. For Case (iii), suppose that $\phi = (\psi_1 \to \chi_1) = (\psi_2 \to \chi_2)$. If $\psi_1 \neq \psi_2$, then either ψ_1 is a proper initial substring of ψ_2, or ψ_2 is a proper initial substring of ψ_1, both of which are impossible. Hence $\psi_1 = \psi_2$ and $\chi_1 = \chi_2$. ϕ is uniquely readable. □

Question 4 β

Let $\mathcal{L} = \{f\}$ be a first-order language containing a unary function symbol f, and no other non-logical symbols. Write down sentences ϕ and ψ of $\mathcal{L}$ such that for any $\mathcal{L}$-structure $\mathcal{A} = \langle A, f_{\mathcal{A}} \rangle$

(i) $\mathcal{A} \vDash \phi$ if and only if $f_{\mathcal{A}}$ is injective;

(ii) $\mathcal{A} \vDash \psi$ if and only if $f_{\mathcal{A}}$ is surjective.

Write down a sentence χ of $\mathcal{L}$ which is satisfiable in some structure with an infinite domain but is false in every structure with a finite domain. What can you say about the size of the domains of the models of the sentence $\neg\chi$?

Write down a sentence ρ such that whenever $\mathcal{A} \vDash \rho$ and A is finite, then A contains an even number of elements, and, further, every finite set with an even number of elements is the domain of some model of ρ. What can you say about the size of the domains of the models of the sentence $\neg\rho$?

Proof. (i) $\phi = \forall x_0 \forall x_1 (f(x_0) \doteq f(x_1) \to x_0 \doteq x_1)$;

(ii) $\psi = \forall x_0 \exists x_1 f(x_1) \doteq x_0$.

Since every set-endomorphism over a finite set is injective if and only if it is surjective, the sentence

$$\chi = (\phi \wedge \neg\psi) = (\forall x_0 \forall x_1 (f(x_0) \doteq f(x_1) \to x_0 \doteq x_1) \wedge \exists x_1 \forall x_0 \neg f(x_1) \doteq x_0)$$

can never be satisfied in a structure with finite domain. However, it can be satisfied in $\langle \mathbb{N}, \cdot^2 \rangle$.

$\neg\chi$ can be satisfied in a structure with infinite or finite domain. In natural language, $\neg\chi$ is satisfied if there exists a set-endomorphism which is either surjective or is not injective.

Here cannot be arrow. Should be conjunction!

$$\rho = \forall x_0 \exists x_1 (\neg x_0 \doteq x_1 \to (f(x_0) \doteq x_1 \wedge f(x_1) \doteq x_0))$$

If $\mathcal{A} \vDash \rho$, then for any element x_0 in A there exists a unique distinct element x_1 in A that pairs with x_0 via $f_{\mathcal{A}}$. In other words, $f_{\mathcal{A}}$ induces an equivalence relation on A such that every equivalence class is a doubleton. In particular, if A is finite, then $\operatorname{card} A$ is an even number.

The negation of $\neg\rho$ is

So this is also wrong

$$\neg\rho = \exists x_0 \forall x_1 (\neg x_0 \doteq x_1 \wedge \neg(f(x_0) \doteq x_1 \wedge f(x_1) \doteq x_0))$$

It can be satisfied by structures with arbitrary size of domain. □

Proof? It helps to say that any domain with function f that does not pair up the elements (e.g. map everything to the same element) satisfies ¬ρ
By the way, if you have checked you would have realised that your ¬ρ is written wrongly (and hence ρ is wrong) because it cannot be satisfied

Question 5 α-

Let $\mathcal{L} = \{P\}$ be a first-order language with a binary relation symbol P as only non-logical symbol. By exhibiting three suitable $\mathcal{L}$-structures prove (informally) that no two of the following sentences logically implies the other one:

(i) $\forall x \forall y \forall z (P(x, y) \to (P(y, z) \to P(x, z)))$

(ii) $\forall x \forall y (P(x, y) \to (P(y, x) \to x \doteq y))$

(iii) $(\forall x \exists y P(x, y) \to \exists y \forall x P(x, y))$

Proof. 1. (i) and (ii) does not imply (iii):

(i) and (ii) are satisfiable in the model $\langle \mathbb{R}; \leqslant \rangle$. (iii) is not satisfiable in $\langle \mathbb{R}, \leqslant \rangle$, since $\tilde{v}(\forall x \exists y P_{\leqslant}(x, y)) = T$ for all assignments v ("for all real numbers, there exists one no small than it") and $\tilde{v}(\exists y \forall x P_{\leqslant}(x, y)) = F$ for all assignments v ("there exists a real number which is not smaller than all real numbers"). ✓

2. (i) and (iii) does not imply (ii):

Consider the model $\mathcal{A} = \langle \mathbb{C}; P_{\mathcal{A}} \rangle$, where $P_{\mathcal{A}}(x, y)$ if and only if $|x| > |y|$. It is clear that $P_{\mathcal{A}}$ is transitive and is not antisymmetric. So (i) is satisfiable and (ii) is not satisfiable in the model. Since $|0| > |y|$ is false for all $y \in \mathbb{C}$, $\forall x \exists y P_{\mathcal{A}}(x, y)$ is not satisfiable in $\mathcal{A}$. Hence (iii) is satisfiable in $\mathcal{A}$.

(ii) is also satisfied — because for no x, y we have |x| > |y| and |y| > |x| together, so (ii) is vaguely true. P(x, y) := |x| ≥ |y| should work instead — but it satisfies (iii) in a different way

3. (ii) and (iii) does not imply (i):

Consider the model $\mathcal{A} = \langle \mathbb{N}; P_{\mathcal{A}} \rangle$, where $P_{\mathcal{A}}(x, y)$ if and only if $x = y + 1$. It is clear that $P_{\mathcal{A}}$ is antisymmetric and is not transitive. So (i) is not satisfiable and (ii) is satisfiable in the model. Since $0 = y + 1$ is false for all $y \in \mathbb{N}$, $\forall x \exists y P_{\mathcal{A}}(x, y)$ is not satisfiable in $\mathcal{A}$. Hence (iii) is satisfiable in $\mathcal{A}$. □

✓

Question 6 α-

Let $\mathcal{L} = \{f, c\}$ be a first-order language containing as non-logical symbols the unary function f and the constant symbol c. Let $\mathcal{L}_1 = \mathcal{L} \cup \{P\}$, where P is a unary predicate symbol. Consider the following strings of $\mathcal{L}_1$:

$$\phi: \quad ((P(x_0) \wedge \forall x_0 (P(x_0) \to P(f(x_2)))) \to \forall x_2 P(x_0))$$
$$\psi: \quad ((P(c) \wedge \forall x_0 (P(x_0) \to P(f(x_0)))) \to \forall x_0 P(x_0))$$

(a) Prove that both ϕ and ψ are formulae of $\mathcal{L}_1$ and indicate the free and bound occurences of variables in them. Which of these formulae are sentences?

(b) Describe the collection of closed terms of $\mathcal{L}$ and of $\mathcal{L}_1$. (A term is called *closed* if it contains no occurences of variables.)

(c) Characterise those $\mathcal{L}$-structures $\mathcal{A} = \langle A; f_{\mathcal{A}}; c_{\mathcal{A}} \rangle$ where the domain A is an infinite set, and where for every unary relation $P_{\mathcal{A}}$ on A, $\langle A; f_{\mathcal{A}}; c_{\mathcal{A}}; P_{\mathcal{A}} \rangle \vDash \psi$

Proof. (a) It is easy to show that ϕ and ψ are formulae by following the definition step by step. ψ is a sentence because it has no free variables. ϕ is not a sentence because x_0 and x_2 have free occurences in ϕ.

You should indicate which of the occurrences are free

(b) In $\mathcal{L}$ and $\mathcal{L}_1$, a closed term is one of the following:

(i) c,

(ii) $f(\chi)$, where χ is a closed term. ✓

(c) We claim that all $\mathcal{L}$-structures satisfying the property are isomorphic. In other words, for any two such models $\mathcal{A} = \langle A; f_{\mathcal{A}}; c_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B; f_{\mathcal{B}}; c_{\mathcal{B}} \rangle$, there exists a bijection $\sigma : A \to B$ such that $\sigma(c_{\mathcal{A}}) = c_{\mathcal{B}}$ and $\sigma(f_{\mathcal{A}}(x)) = f_{\mathcal{B}}(\sigma(x))$ for all $x \in A$. In particular, every such model is isomorphic to the Peano system of natural numbers $\langle \mathbb{N}; ++; 0 \rangle$.

Let $\mathcal{A} = \langle A; f_{\mathcal{A}}; c_{\mathcal{A}} \rangle$ be a model with the property. Then for any $x \in A$, there exists a closed term $t_{\mathcal{A}}$ such that $x = t_{\mathcal{A}}$. If not, suppose that there exists $y \in A$ such that $y \neq t_{\mathcal{A}}$ for all closed terms $t_{\mathcal{A}}$. Consider a unary relation $P_{\mathcal{A}}$ on A such that $P_{\mathcal{A}}(x)$ for all $x \in A \backslash \{y\}$. Then we have $\mathcal{A} \models P(c)$. Assume that $\mathcal{A} \models \forall x_0 (P(x_0) \to P(f(x_0)))$. By substition, $\mathcal{A} \models P(f(c))$. Inductively, we have $\mathcal{A} \models P(t)$ for all closed terms t. However, since $P(y)$ is false, $\mathcal{A} \not\models \forall x_0 P(x_0)$. Hence $\mathcal{A} \not\models \psi$, which is a contradiction. This is not correct! Suppose that y = f(z), where z also do not correspond to a closed term, then P_A = A\{y} does not satisfy the condition!

Correct proof is to directly consider P_A = {x in A | x is expressible by a closed term}, and show that P_A satisfies ψ

Next, we claim that every $x \in A$ is equal to a unique closed term $t_{\mathcal{A}}$. Note that the closed terms of $\mathcal{L}_1$ has a natural bijection to the natural numbers. We define $t^{(n)}$ to be the closed term with n function symbols. Suppose that there exists $y \in A$ such that $y = t^{(n_1)}_{\mathcal{A}} = t^{(n_2)}_{\mathcal{A}}$. Without loss of generality we assume that $n_1 > n_2$. For $n \geqslant n_2$, $t^{(n)}_{\mathcal{A}} = t^{(n'+n_2)}_{\mathcal{A}}$ where $n_2 \leqslant n' \leqslant n_1 - 1$ and $n \equiv n' \mod (n_1 - n_2)$. Then A has at most n_1 distinct elements: $t^{(0)}_{\mathcal{A}}, \ldots, t^{(n_1)}_{\mathcal{A}}$. This contradicts that A is an infinite set.

For $x \in A$, we define $\sigma(x)$ to be the number of function symbols of the closed form t such that $x = t_{\mathcal{A}}$. Following the discussion above, $\sigma : A \to \mathbb{N}$ is naturally a bijection. For $x \in A$, if $x = t^{(n)}_{\mathcal{A}}$, then $\sigma(f_{\mathcal{A}}(x)) = \sigma(f_{\mathcal{A}}(t^{(n)}_{\mathcal{A}})) = \sigma(t^{(n+1)}_{\mathcal{A}}) = n + 1 = \sigma(t^{(n)}_{\mathcal{A}}) + 1 = \sigma(x) + +$. This completes the proof. □