Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 1

ASO: Special Relativity

May 11, 2020

Question 1. Galilean group.

A Galilean transformation between the coordinate systems of two inertial frames R and R' is given by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & H_{11} & H_{12} & H_{13} \\ v_2 & H_{21} & H_{22} & H_{23} \\ v_3 & H_{31} & H_{32} & H_{33} \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where $H \in \mathrm{SO}(3)$.

(a) Show that (v_1, v_2, v_3) are the components in frame R of the vector which gives the velocity of the origin of R'.

(b) Show that the composition of two such transformations in again a Galilean transformation. Find the inverse of a Galilean transformation and show that it is again a Galiean transformation.

(c) Let (t, x_1, y_1, z_1) and (t_2, x_2, y_2, z_3) be the coordinates relative to R of two events. Show that under Galilean transformations

(A) The temporal separation $t_2 - t_1$ is invariant.

(B) The spatial distance $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (z_2 - z_1)^2}$ is invariant provided that the events are simultaneous, i.e. $t_2 = t_1$.

Explain why statement (B) is not true for events that are not simultaneous. Show that, conversely, any coordinate transformation of the form

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

with properties (A) and (B), where $M \in M_{4\times 4}(\mathbb{R})$ has positive determinant, is a Galilean transformation.

Proof. I shall use the convention that $\boldsymbol{r} = (x, y, z)$ represents a 3-vector and $\boldsymbol{X} = (t, \boldsymbol{r}) = (t, x, y, z)$ represents a 4-vector.

(a) At time t, the 4-vecctor of the origin of R' in R' is $\boldsymbol{X}' = (t + c_0, 0, 0, 0)$. By applying a Galilean transformation $\boldsymbol{G}$, in the frame R it is

$$\boldsymbol{X} = \boldsymbol{G}(\boldsymbol{X}') = (t, v_1 t' + c_1, v_2 t' + c_2, v_3 t' + c_3) = (t, v_1 t + v_1 c_0 + c_1, v_2 t + v_2 c_0 + c_2, v_3 t + v_3 c_0 + c_3)$$

The velocity of the origin of R' in R:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (v_1, v_2, v_3)$$

(b) Suppose that $\boldsymbol{G}_1, \boldsymbol{G}_2$ are two Galilean transformations such that:

$$\begin{pmatrix} t \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_1 & H_1 \end{pmatrix} \begin{pmatrix} t' \\ \boldsymbol{r}' \end{pmatrix} + \begin{pmatrix} c_0 \\ \boldsymbol{c} \end{pmatrix} \qquad \begin{pmatrix} t' \\ \boldsymbol{r}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_2 & H_2 \end{pmatrix} \begin{pmatrix} t'' \\ \boldsymbol{r}'' \end{pmatrix} + \begin{pmatrix} d_0 \\ \boldsymbol{d} \end{pmatrix}$$

Then their composition $\boldsymbol{G}_1 \circ \boldsymbol{G}_2$ is given by

$$\begin{pmatrix} t \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_1 & H_1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_2 & H_2 \end{pmatrix} \begin{pmatrix} t'' \\ \boldsymbol{r}'' \end{pmatrix} + \begin{pmatrix} d_0 \\ \boldsymbol{d} \end{pmatrix} \right) + \begin{pmatrix} c_0 \\ \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_1 + H\boldsymbol{v}_2 & H_1 H_2 \end{pmatrix} \begin{pmatrix} t'' \\ \boldsymbol{r}'' \end{pmatrix} + \begin{pmatrix} c_0 + d_0 \\ \boldsymbol{c} + \boldsymbol{v}_2 d_0 + H_2 \boldsymbol{d} \end{pmatrix}$$

Since $H_1, H_2 \in \mathrm{SO}(3)$, $H_1 H_2 \in \mathrm{SO}(3)$. We deduce that $\boldsymbol{G}_1 \circ \boldsymbol{G}_2$ is also a Galilean transformation.

For the inverse of the Galilean transformation $\boldsymbol{G}$, notice that

$$\begin{pmatrix} 1 & 0 \\ \boldsymbol{v} & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -H^{-1}\boldsymbol{v} & H^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{0} & I_3 \end{pmatrix}$$

We have

$$\begin{pmatrix} t \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_1 & H_1 \end{pmatrix} \begin{pmatrix} t' \\ \boldsymbol{r}' \end{pmatrix} + \begin{pmatrix} c_0 \\ \boldsymbol{c} \end{pmatrix} \implies \begin{pmatrix} t - c_0 \\ \boldsymbol{r} - \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{v}_1 & H_1 \end{pmatrix} \begin{pmatrix} t' \\ \boldsymbol{r}' \end{pmatrix}$$
$$\implies \begin{pmatrix} t' \\ \boldsymbol{r}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -H^{-1}\boldsymbol{v} & H^{-1} \end{pmatrix} \begin{pmatrix} t - c_0 \\ \boldsymbol{r} - \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -H^{-1}\boldsymbol{v} & H^{-1} \end{pmatrix} \begin{pmatrix} t \\ \boldsymbol{r} \end{pmatrix} + \begin{pmatrix} -c_0 \\ H^{-1}(c_0\boldsymbol{v} - \boldsymbol{c}) \end{pmatrix}$$

Since $H \in \mathrm{SO}(3)$, $H^{-1}\,\mathrm{SO}(3)$. We deduce that $\boldsymbol{G}^{-1}$ is also a Galilean transformation.

We conclude that the set of all Galilean transformations forms a group, namely the Galilean group.

(c) (A). Under a Galilean transformation from R to R', we have $t_1' = t_1 + c_0$ and $t_2' = t_2 + c_0$ for some $c_0 \in \mathbb{R}$. Hence $t_2' - t_1' = t_2 - t_1$. The temporal separation is invariant.

(B). Under a Galilean transformation from R to R', we have $\boldsymbol{r}_1' = \boldsymbol{v}t_1 + H\boldsymbol{r}_1 + \boldsymbol{c}$ and $\boldsymbol{r}_2' = \boldsymbol{v}t_2 + H\boldsymbol{r}_2 + \boldsymbol{c}$ for some $\boldsymbol{v}, \boldsymbol{c} \in \mathbb{R}^3$ and $H \in \mathrm{SO}(3)$. If $t_1 = t_2$, then

$$||\boldsymbol{r}_2' - \boldsymbol{r}_1'|| = ||\boldsymbol{v}(t_2 - t_1) + H(\boldsymbol{r}_2 - \boldsymbol{r}_1)|| = ||H(\boldsymbol{r}_2 - \boldsymbol{r}_1)|| = ||\boldsymbol{r}_2 - \boldsymbol{r}_1||$$

since orthogonal matrix H preserves Euclidean norm. Hence the spatial separation is invariant. If $t_1 \neq t_2$, then $||\boldsymbol{r}_2' - \boldsymbol{r}_1'|| = ||\boldsymbol{v}(t_2 - t_1) + \boldsymbol{r}_2 - \boldsymbol{r}_1|| \neq ||\boldsymbol{r}_2 - \boldsymbol{r}_1||$ in general.

Conversely, consider a coordinate transformation with properties (A) and (B). We write M in the following block matrix form:

$$M = \begin{pmatrix} a & B \\ \boldsymbol{v} & D \end{pmatrix}$$

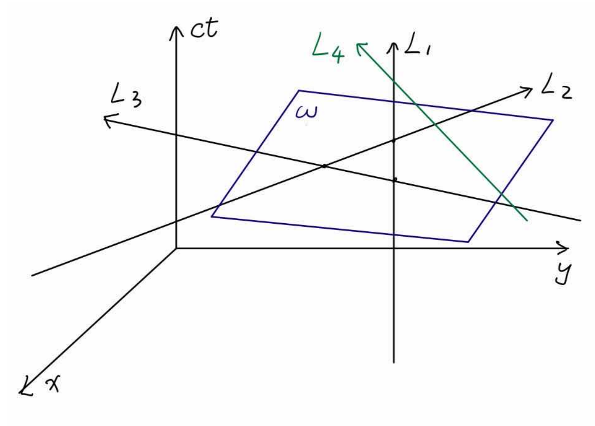
where $a \in \mathbb{R}, B \in M_{1\times 3}(\mathbb{R}), \boldsymbol{v} \in \mathbb{R}^3, D \in M_{3\times 3}(\mathbb{R})$.

Then we have $t_2' - t_1' = a(t_2 - t_1) + B(\boldsymbol{r}_2 - \boldsymbol{r}_1)$. The invariance of temporal separation implies that $a = 1$, $B = 0$. If $t_1 = t_2$, then $\boldsymbol{r}_2' - bmr_1' = D(\boldsymbol{r}_2 - \boldsymbol{r}_1)$. The invariance of spatial distance implies that $D \in \mathrm{O}(3)$. But we know that $\det M = a \det D = \det D > 0$. Hence $D \in \mathrm{SO}(3)$. We conclude that this coordinate transformtion is Galilean. □

Question 2. Space-time diagrams.

Four ghosts travel in straight lines at different constant velocities across a flat (two-dimensional) field. Five of the six possible pairs of ghosts pass through each other at different times. Show that the sixth pair must also pass through each other at some point. It will be helpful to think in terms of a space-time diagram.

Proof. Consider their space-time coordinates $(t, x, y) \in \mathbb{R}^3$. The motion of the four particles are represented by four straight lines $L_1, L_2, L_3, L_4 \subseteq \mathbb{R}^3$. The intersections of two lines are the event that two particles pass through each other. Without loss of generality we assume that L_1, L_2, L_3 intersect mutually. Then they lie in the same plane $\omega \subseteq \mathbb{R}^3$. Since L_4 intersects with two of L_1, L_2, and L_3, it has at least two points in ω. Hence $L_4 \subseteq \omega$. If the statement in the question is false, then L_4 doesnot intersect with one of L_1, L_2, and L_3. For example if L_4 does not intersect L_1, then $L_1 /\!/ L_4$ because they are coplanar. But it means that the particles 1 and 4 have the same velocity, which is a contradiction. Hence L_4 must intersect all other lines in the space-time diagram. □



Question 3. Relativistic Doppler effect.

Two observers O and O' travel along the same straight line in space at constant speeds. The first sends out two light signals separated by an interval τ measured on O 's clock. What is the interval between the times (according to O') when these signals are received at O' if both light signals are emitted

(a) When O' is approaching O ?

(b) When O' is receding from O?

If the light making up the signal has frequency ω as measured by O, what is the frequency as measured by O' in cases (a) and (b) respectively?

Solution. Suppose that O' has velocity v in the frame of O. We set up the coordinates (t, x) of O and (t', x') of O' such that their origins coincide.

Suppose that at $t = t_1$, O sends the first signal. By Lorentz transformation, this event in the frame of O' has coordinates:

$$t'_1 = \gamma t_1 \qquad x'_1 = -\gamma v t_1.$$

At $t_2 = t_1 + \tau$, O sends the second signal. By Lorentz transformation, this event in the frame of O' has coordinates:

$$t'_2 = \gamma(t_1 + \tau) \qquad x_2 = -\gamma v(t_1 + \tau).$$

The time when O' receives the two signals:

$$s'_1 = t'_1 + \frac{0 - x'_1}{c} = \gamma t_1 + \frac{\gamma v t_1}{c} \qquad s'_2 = t'_2 + \frac{0 - x'_2}{c} n = \gamma(t_1 + \tau) + \frac{\gamma v(t_1 + \tau)}{c}$$

Hence the temporal separation in the frame O' is $\tau' = s'_2 - s'_1 = \gamma\left(1 + \frac{v}{c}\right)\tau = \sqrt{\frac{c+v}{c-v}}\,\tau.$

If O' is approaching O, then $v < 0$ and $\tau' < \tau$. If O' is receding from O, then $v > 0$ and $\tau' > \tau$.

If τ is the period of the electromagnetic wave, then $\omega\tau = 2\pi$. We have $\omega' = \sqrt{\frac{c-v}{c+v}}\,\omega$. The sign of v indicates if O' is approaching or receding form O. □

Question 4. Barn door paradox.

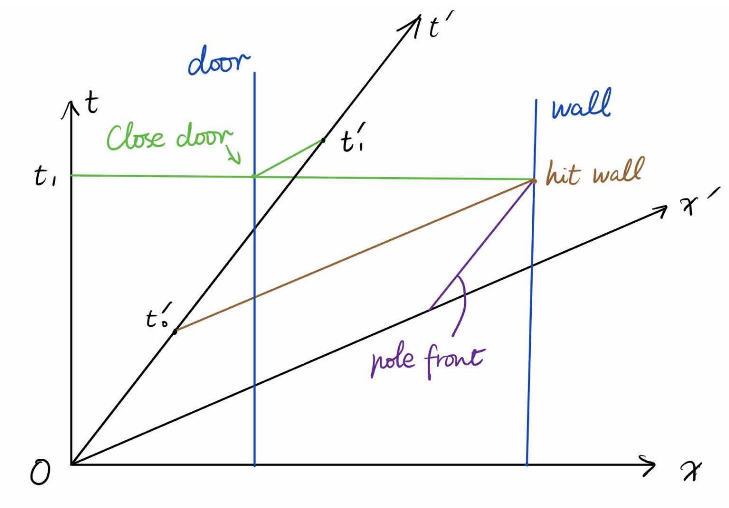
An pole-vaulter carrying a 15-foot-long pole runs with speed $\sqrt{3}c/2$ towards a barn that is 10 feet long. Exactly when the front of the pole reaches the far wall of the barn, a child standing by the barn door closes it. Explain, with the aid of a space-time diagram, how this is possible when in the athlete's coordinate system, the pole has a length 15 feet but the room is only 5 feet deep.

If the athlete holds the pole by its back end, are they (i) outside the barn, (ii) at the barn door, or (iii) inside the barn door when they first feel the shock of the tip of the pole striking the far wall?

Solution. Let (t, x) be the frame observed by the child and (t', x') be the frame observed by the athlete (or the back end of the pole). At $t' = t'_0$, the athlete observes that the front end of the pole hits the wall, when he is outside the room. The same event is observed by the child at $t = t_1$. He immediately closes the door, which is observed by the athlete at $t' = t'_1 > t'_0$, when he is inside the room.

The process is only possible if we assume that the pole is contractible (special relativity does not allow the existence of rigid bodies). The shock travels along the rod at a speed of $\sqrt{\frac{E}{\rho}} \ll c$. The athelete is inside the room when he feels the shock.

The space-time diagram is shown as follows: □



Question 5. Symmetries of the wave equation.

Consider the wave equation in one spatial dimension describing waves propagating at speed c,

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

(a) Consider the homogeneous Galilean transformation in one spatial dimension,

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

Show that the wave equation is not invariant under this transformation, i.e., there are solutions $\phi(x, t)$ equation that are not solutions of the corresponding equation in x' and t'.

(b) Consider a transformation of the form

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

where $p, q, r,$ and s are constant. Find necessary and sufficient conditions on $p, q, r,$ and s to ensure that the wave equation in one spatial dimension is invariant, that is, that

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} - \frac{\partial^2\phi}{\partial x'^2}$$

Assuming that p and s are both positive, show that we can rewrite this as the standard Lorentz transformation in one spatial dimension:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v)\begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}$$

Proof. For convenience we consider the general transformation with the form given in part (b):

$$\begin{cases} t = pt' + qx' \\ x = rt' + sx' \end{cases}$$

By chain rule we have

$$\partial_{t'} = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial x}{\partial t'}\partial_x = p\partial_t + r\partial_x \qquad \partial_{x'} = \frac{\partial x}{\partial x'}\partial_x + \frac{\partial t}{\partial x'}\partial_x = q\partial_t + s\partial_x$$

Hence

$$\frac{1}{c^2}\partial_{t'}^2 - \partial_{x'}^2 = \frac{1}{c^2}(p\partial_t + r\partial_x)^2 - (q\partial_t + s\partial_x)^2 = \left(\frac{p^2}{c^2} - q^2\right)\partial_t^2 + \left(\frac{r^2}{c^2} - s^2\right)\partial_x^2 + 2\left(\frac{pr}{c^2} - qs\right)\partial_x\partial_t$$

With $p = 1$, $q = 0$, $r = v$, $s = 1$, we have

$$\frac{1}{c^2}\partial_{t'}^2 - \partial_{x'}^2 = \frac{1}{c^2}\partial_t^2 + \left(\frac{v^2}{c^2} - 1\right)\partial_x^2 + \frac{2v}{c^2}\partial_x\partial_t$$

We see that the wave equation is invariant if and only if $v = 0$ (or $v \ll c$ in physics, which gives the low speed approximation).

In general, the wave equation is invariant if and only if:

$$p^2 = c^2q^2 + 1 \qquad r^2 = c^2(s^2 - 1) \qquad pr = c^2qs$$

For positive p, s, this is equivalent to

$$p = s \qquad r = c^2q \qquad s^2 - q^2c^2 = 1.$$

We have

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} s & cq \\ cq & s \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Change of variable: $s = \cosh\eta$, $cq = \sinh\eta$. Let $v/c = \tanh\eta$. Then $\cosh\eta = \dfrac{c}{\sqrt{c^2 - v^2}} = \gamma$. We have

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \cosh\eta\begin{pmatrix} 1 & \tanh\eta \\ \tanh\eta & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma\begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix}$$

□

Question 6. Characterizing Lorentz transformation.

Consider a linear transformation

$$\begin{pmatrix} ct \\ x \end{pmatrix} = L\begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Show that L takes the form of the standard one-dimensional Lorentz transformation,

$$L = \gamma(v) \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix}$$

if and only if

(i) the top left entry in L is positive,

(ii) $\det L > 0$, and

(iii) L obeys the pseudo-orthogonality condition

$$L^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. "$\Longrightarrow$": Suppose that L is the standard one-dimensional Lorentz transformation. Then trivially the top left entry of L is positive. The determinant $\det L = \gamma^2(1 - v^2/c^2) = 1 > 0$. For the pseudo-orthogonality condition:

$$L^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L = \gamma^2 \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} = \gamma^2 \begin{pmatrix} 1 - v^2/c^2 & 0 \\ 0 & v^2/c^2 - 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

"$\Longleftarrow$": Suppose that

$$L = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

satisfies the three conditions.

For the pseudo-orthogonality condition:

$$L^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p^2 - r^2 & pq - rs \\ pq - rs & q^2 - s^2 \end{pmatrix}$$

Hence a, b, c, d satisfy the condition:

$$pq - rs = 0 \qquad p^2 - r^2 = 1 \qquad q^2 - s^2 = -1$$

which implies that

$$p^2 = s^2 \qquad q^2 = r^2 \qquad s^2 - r^2 = 1$$

The condition (i) and (ii) requires that $p > 0$ and $ps - qr > 0$.

If $s < 0$, then $p = -s > 0$. $pq = rs$ implies that $qr < 0$. Then $\det L = ps - qr = -s^2 + r^2 = -1 < 0$, which is a contradiction.

We must have $s > 0$. Then $p = s > 0$. $pq = rs$ implies that $q = r$. The rest of the procedure is identical to that in Question 5: Let $s = \cosh\eta$, $r = \sinh\eta$, $v/c = \tanh\eta$. We shoould obtain that

$$L = \gamma \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix}$$

That is, L is a standard one-dimensional Lorentz transformation. □