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Problem Sheet 3

C2.1: Lie Algebras

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Assume throughout the problems that we work over a field $\Bbbk$ which is algebraically closed of characteristic zero, and all Lie algebras and representations are finite dimensional over $\Bbbk$, unless the contrary is explicitly stated.

Question 1

Let V be an n-dimensional $\Bbbk$-vector space and let $\mathscr{F} = (0 = F_0 < F_1 < \ldots < F_n = V)$ be a complete flag in V. Let $\mathfrak{b} = \mathfrak{b}_{\mathscr{F}} = \{x \in \mathfrak{gl}(V) : x(F_i) \subseteq F_i, \forall i, 1 \leqslant i \leqslant n\}$.

(a) Find a Cartan subalgebra of $\mathfrak{b}$.

(b) Describe the associated Cartan decomposition of $\mathfrak{b}$.

Proof. (a) Know from Sheet 2 $\mathfrak{g} = \mathfrak{gl}(V)$ has $\mathfrak{h} = \bigoplus_i \mathrm{End}(L_i)$ where $F_i = \bigoplus_{k=1}^{i} L_i$. Let $L_i = \Bbbk e_i$. Then $\mathfrak{h}$ is a set of diagonal matrices with respect to $\{e_1, \ldots, e_n\}$. $N_{\mathfrak{gl}(V)}(\mathfrak{h}) = \mathfrak{h}$.

Note: If $\mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{g}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{gl}(V)$, then it is also a Cartan subalgebra of $\mathfrak{a}$.

Since $\mathfrak{h} \subseteq \mathfrak{b}_{\mathscr{F}}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{b}_{\mathscr{F}}$.

(b)

$$\mathfrak{gl}(V) = \mathrm{Hom}\left(\bigoplus_i L_i, \bigoplus_j L_j\right) = \bigoplus_{i,j} \mathrm{Hom}(L_i, L_j) = \bigoplus_{i,j} L_i^\vee \otimes L_j$$

$\mathfrak{h}^\vee$ has a basis $\{\varepsilon_i : 1 \leqslant i \leqslant n\}$, where $\varepsilon_i : \mathfrak{h} \to \Bbbk$, $\bigoplus_k f_k \mapsto f_i$.

For $\mathfrak{b}_{\mathscr{F}}$ we have $\mathfrak{b}_{\mathscr{F}} = \bigoplus_{1 \leqslant i \leqslant j \leqslant n} L_i^\vee \otimes L_j = \mathfrak{h} \oplus \bigoplus_{1 \leqslant i < j \leqslant n} L_i^\vee \otimes L_j$. □

Question 2

Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$, so that $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, and let $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices respectively.

(a) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and that the restriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is nondegenerate.

[*Hint: It is probably useful to calculate the values of the Killing form on matrix coefficients.*]

(b) Calculate $\mathfrak{n}_+^\perp$.

(c) Describe the radical of the restriction of κ to $\mathfrak{h}$ and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is nondegenerate.

Proof. (a) $\mathrm{ad}(E_{ij})\,\mathrm{ad}(E_{k\ell})(E_{\alpha\beta}) = \mathrm{ad}(E_{ij})(\delta_{\ell\alpha}E_{k\beta} - \delta_{\ell\beta}E_{k\alpha}) = \delta_{kj}\delta_{\ell\alpha}E_{i\beta} - \delta_{i\beta}\delta_{\ell\alpha}E_{kj} - \delta_{\beta k}\delta_{\alpha j}E_{i\ell} + \delta_{\beta k}\delta_{\ell i}E_{\alpha j}$

$$\begin{aligned}\mathrm{tr}\big(\mathrm{ad}(E_{ij})\,\mathrm{ad}(E_{k\ell})\big) &= \sum_{\alpha,\beta} \big(\delta_{kj}\delta_{\ell\alpha}\delta_{i\alpha} - \delta_{i\beta}\delta_{\ell\alpha}\delta_{k\alpha}\delta_{j\beta} - \delta_{\beta k}\delta_{\alpha j}\delta_{i\alpha}\delta_{\ell\beta} + \delta_{\beta k}\delta_{\ell i}\delta_{\beta j}\big) \\ &= \begin{cases} \begin{cases} 2n-2 & i = k = \ell \\ -2 & i \neq k = \ell \\ 0 & k \neq \ell \end{cases} & i = j \\ 2n\delta_{i\ell}\delta_{jk} & i \neq j \end{cases}\end{aligned}$$

Let $\mathfrak{h} = \bigoplus_i \mathbb{C}E_i i$. Then $\kappa(E_{ii}, E_{jj}) = 2(n-1)\delta_{ij} - 2$. Hence κ is non-degenerate on $\mathfrak{n}_+ \oplus \mathfrak{n}_-$.

Since $\mathfrak{gl}_n = (\mathfrak{n}_+ \oplus \mathfrak{n}_-) \oplus^\perp \mathfrak{h}$, $\mathrm{rad}(\kappa^{\mathfrak{gl}_n}) = \mathrm{rad}(\kappa^{\mathfrak{gl}(V)}|_{\mathfrak{h}})$.

$\kappa(E_{ii}, E_{jj}) = 2(n-1)\delta_{ij} - 2$ shows that if $z = \sum_i \lambda_i \varepsilon_i \in \mathfrak{h} \cap \mathfrak{h}^\perp = \mathrm{rad}(\kappa^{\mathfrak{gl}_n})$, then

$$0 = \kappa(\sum_i \lambda_i \varepsilon_i, \varepsilon_j) = 2(n-1)\lambda_j - 2\sum_{i \neq j} \lambda_i \implies (n-1)\lambda_j = \sum_{i \neq j} \lambda_i \implies \lambda_1 = \cdots = \lambda_n$$

Hence $z \in \mathbb{C}\,\mathrm{id}_n$. □

Question 3

Show that the Killing form for $\mathfrak{sl}_n$ is given by

$$\kappa(x, y) = 2n \cdot \mathrm{tr}(x \cdot y)$$

(where $\mathrm{tr}(x.y)$ denotes the ordinary trace, i.e. the trace form for the vector representation).

Proof. For $x, y \in \mathfrak{gl}_n$, $\mathfrak{t}_n(x, y) = \mathrm{tr}(xy)$. Use the calculation in Question 2 or the fact that $\mathfrak{sl}_n$ is simple (if and only if ad is an irrep of $\mathfrak{sl}_n$).

$\mathfrak{t}_n$ is an invariant bilinear form.

$$\mathrm{Bil}(\mathfrak{sl}_n)^{\mathfrak{sl}_n} \cong \mathrm{Hom}(\mathfrak{sl}_n, \mathfrak{sl}_n^\vee)^{\mathfrak{sl}_n} \cong \mathrm{Hom}_{\mathfrak{sl}_n}(\mathfrak{sl}_n, \mathfrak{sl}_n^\vee)$$

As $\mathfrak{sl}_n$ is simple, Schur's Lemma shows that $\mathrm{Hom}_{\mathfrak{sl}_n}(\mathfrak{sl}_n, \mathfrak{sl}_n^\vee)$ is 1-dimensional. Hence $\mathrm{Bil}(\mathfrak{sl}_n)^{\mathfrak{sl}_n} = \mathbb{C}\mathfrak{t}_n$.

If $h = \mathrm{diag}(h_1, ..., h_n)$, then $\mathrm{tr}\big(\mathrm{ad}(h)^2\big) = \sum_{i,j} (h_i - h_j)^2 = 2n \sum_i h_i^2$ since $\sum_i h_i = 0$. □

The next few questions of this exercise sheet classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall that if we let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then e, f and h give a basis of $\mathfrak{sl}_2$ with relations

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h$$

Hence, a representation of $\mathfrak{sl}_2(\mathbb{C})$ consists of a vector space V over $\mathbb{C}$ together with three endomorphisms E, F and H satisfying

$$HE - EH = 2E, \quad HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

(We recover the representation $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ by setting $\phi(e) = E, \phi(f) = F$ and $\phi(h) = H$.) We will also need a partial ordering on k : since k has characteristic zero it contains a copy of $\mathbb{Q}$, and we will say that $a < b$ if $b - a \in \mathbb{Q}_{>0}$. If $I \subseteq \mathrm{k}$ is a finite subset of k we say $\lambda \in I$ is maximal if $\lambda < \mu$ implies $\mu \notin I$. In the rest of this problem set we always assume that V is *finite dimensional.*

Question 4

(a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^k E = E(H - \lambda)^k$$

(Here $\lambda \in \mathbb{C}$ and we write λ instead of $\lambda \cdot \mathrm{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ-eigenspace of H, then Ev belongs to the generalised $(\lambda + 2)$-eigenspace.

(b) Deduce a similar statement for the action of F on the generalised eigenspaces of H.

(c) Let λ be an eigenvalue for H which is a maximal element of the set of eigenvalues of H in the sense described above. Use a) to show that $EV_\lambda = 0$.

(d) Use b) to deduce that for large enough n we have $F^n(v) = 0$.

Proof. The result is already familiar to us from the theory of angular momentum in quantum mechanics.

(a) We use induction on k. For $k = 0$ this is trivial. Suppose it holds for $k-1$. Then

$$(H-(\lambda+2))^k E = (H-(\lambda+2))E(H-\lambda)^{k-1} = (EH+2E)(H-\lambda)^{k-1} - (\lambda+2)E(H-\lambda)^{k-1} = E(H-\lambda)^k$$

which completes the induction.

If $v \in V_\lambda$, then $(H-\lambda)^k v = 0$. So $(H-(\lambda+2))^k Ev = 0$. We have $Ev \in V_{\lambda+2}$.

(b) For F we have

$$(H-(\lambda-2))^k F = E(H-\lambda)^k$$

If $v \in V_\lambda$, then $Fv \in V_{\lambda-2}$.

(c) Suppose that λ is maximal among the eigenvalues of H. If $v \in V_\lambda$, then $Ev \in V_{\lambda+2}$. By assumption $V_{\lambda+2} = \{0\}$. Hence $EV_\lambda = \{0\}$.

(d) Let $\lambda_1, ..., \lambda_n$ be the set of eigenvalues of V. We have the primary decomposition of V

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}$$

Let $v = \sum_{i=1}^n v_i \in V$, where each $v_i \in V_{\lambda_i}$. If $F^n v_i \neq 0$ for infinitely many $n \in \mathbb{N}$, then $\lambda_i - 2n$ is an eigenvalue of H for infinitely many n, which is impossible as V is finite-dimensional. Hence $F^n v_i = 0$ for sufficiently large n. It follows that $F^n v = 0$ for sufficiently large n. □

Question 5

(a) Show the relation (for $n \geqslant 1$)

$$HF^n = F^n H - 2nF^n$$

(b) Show ($n \geqslant 1$ as before)

$$EF^n = F^n E + nF^{n-1}H - n(n-1)F^{n-1}$$

(c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^n F^n v = nE^{n-1}F^{n-1}(H-(n-1))v = n!\prod_{i=1}^{n}(H-(i-1))v$$

(d) Let λ be a maximal eigenvalue of H (in the above sense) and let V_λ denote the generalised λ-eigenspace. Use 4(d) and (c) to deduce that H acts diagonalisably on V_λ and that λ is a non-negative integer.

Proof. (a) We use induction on n. For $n = 1$ this is the given commutator $[H, F] = -2F$. Suppose that it holds for $n-1$. Then for n

$$HF^n = (F^{n-1}H - 2(n-1)F^{n-1})F = F^{n-1}HF - 2(n-1)F^n = F^n H - 2F^n - 2(n-1)F^n = F^n H - 2nF^n$$

(b) Again we use induction on n. For $n = 1$ this is the given commutator $[E, F] = H$. Suppose that it holds for

$n-1$. Then for n

$$\begin{aligned}EF^n &= \left(F^{n-1}E+(n-1)F^{n-2}H-(n-1)(n-2)F^{n-2}\right)F\\ &= F^{n-1}EF+(n-1)F^{n-2}HF-(n-1)(n-2)F^{n-1}\\ &= F^{n-1}(FE+[E,F])+(n-1)F^{n-2}(FH+[H,F])-(n-1)(n-2)F^{n-1}\\ &= F^nE+F^{n-1}H+(n-1)F^{n-1}H-2(n-1)F^{n-1}-(n-1)(n-2)F^{n-1}\\ &= F^nE+nF^{n-1}H-n(n-1)F^{n-1}\end{aligned}$$

(c) By (b) We have

$$\begin{aligned}E^nF^nv &= E^{n-1}\left(F^nE+nF^{n-1}H-n(n-1)F^{n-1}\right)v\\ &= nE^{n-1}F^{n-1}Hv-n(n-1)E^{n-1}F^{n-1}v\\ &= nE^{n-1}F^{n-1}(H-(n-1))v\end{aligned}$$

For $n=1$, we have $EFv=(FE+H)v=Hv$. Inductively we have

$$E^nF^nv=n!\prod_{i=1}^{n}(H-(i-1))v$$

(d) Let m_H be the minimal polynomial of H on V_λ. By 4.(d), there exists $n\in\mathbb{N}$ such that $F^nv=0$ for all $v\in V_\lambda$. Then

$$V_\lambda\ni 0=E^nF^nv=n!\prod_{i=1}^{n}(H-i+1)v$$

Hence $f(x):=\prod_{i=1}^{n}(x-i+1)$ is an annihilating polynomial of H on V_λ. $f(x)$ splits implies that $m_H(x)$ splits. Hence H is diagonalisble over V_λ. We have $m_H(x)=x-\lambda$ which is a disivor of $f(x)$. Hence $\lambda\in\{0,\dots,n\}$ is a non-negative integer. □

Question 6

(a) Let λ be a maximal eigenvalue of H as in the previous question, and choose a non-zero vector $v\in V_\lambda$. We know by Questions 2 and 3 that $Ev=0$ and that λ is an non-negative integer. Show the relations:

$$HF^kv=(\lambda-2k)F^kv$$
$$EF^kv=k(\lambda-(k-1))F^{k-1}v$$

Deduce that $F^{\lambda+1}v=0$ and that the F^iv for $0\leqslant i\leqslant\lambda$ are linearly independent and span a simple submodule of V.

(b) Check that the above relations define an $\mathfrak{sl}_2(\mathbb{C})$-module for any non-negative integer λ. Deduce that there is (up to isomorphism) a unique simple module $V(\lambda)$ of dimension $\lambda+1$ for all non-negative integers λ.

Proof. (a) We use induction on k to prove that $HF^kv=(\lambda-2k)F^kv$. For $k=0$ this is trivial. Suppose that it holds for $k-1$. Then

$$HF^kv=(FH-2F)F^{k-1}v=FHF^{k-1}v-2F^kv=F(\lambda-2(k-1))F^{k-1}v-2F^kv=(\lambda-2k)F^kv$$

We use induction on k to prove that $EF^kv=k(\lambda-(k-1))F^{k-1}v$. For $k=1$:

$$EFv=Hv=\lambda v$$

Suppose that it holds for $k-1$. Then

$$\begin{aligned} EF^k v &= (FE+H)F^{k-1}v \\ &= FEF^{k-1}v + HF^{k-1}v \\ &= F(k-1)(\lambda-(k-2))F^{k-2}v + (\lambda-2(k-1))F^{k-1}v \\ &= k(\lambda-(k-1))F^{k-1}v \end{aligned}$$

When $k=\lambda+1$, the first equation gives $HF^{\lambda+1}v = -(\lambda+2)F^{\lambda+1}v$. If $F^{\lambda+1}v \neq 0$, then $-F^{\lambda+1}v$ is en eigenvector of H associated with the eigenvalue $\lambda+2$. This contradicts the maximality of λ. Hence $F^{\lambda+1}v = 0$.

Note that $v, Fv, ..., F^\lambda v$ correspond to different eigenvalues of H. To show that they are linearly independent, it suffices to show that none of them is zero. Let $i \in \{0,...,\lambda\}$ be a minimal integer such that $F^i v = 0$. Then

$$0 = EF^i v = i(\lambda - i + 1)F^{i-1}v$$

As $F^{i-1}v \neq 0$, we have $i = 0$ or $\lambda - i + 1 = 0$, both of which are impossible. Hence $\{v, Fv, ..., F^\lambda v\}$ is linearly independent.

Let $e_i := F^i v$. We claim that $W := \operatorname{span}\{e_0, ..., e_\lambda\}$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module. Suppose that $I \triangleleft W$ is a non-zero ideal. Let $v = \sum_{i=k}^{\lambda} a_i e_i \in I$ where $a_k \neq 0$. Then $F^{\lambda-k}v = F^{\lambda-k}a_k e_k = a_k e_\lambda$. Hence $e_\lambda \in I$. Note that $Ee_i \in \operatorname{span}\{e_{i-1}\} \setminus \{0\}$ for $i \in \{1,...,\lambda\}$. Hence $e_0, ..., e_\lambda \in I$. We have $I = W$. W is a simple submodule of V.

(b) For $\lambda \in \mathbb{N}$, let $V(\lambda)$ be a complex vector space such that $E, F, H \in \operatorname{End}(V(\lambda))$ satisfies the relations given above.

Let $e_i = F^i v$. Then we have

$$Fe_i = \begin{cases} e_{i+1} & 0 \leq i < \lambda \\ 0 & i = \lambda \end{cases}, \qquad Ee_i = \begin{cases} i(\lambda-i+1)e_{i-1} & 0 < i \leq \lambda \\ 0 & i = 0 \end{cases}, \qquad He_i = (\lambda - 2i)e_i$$

It is straightforward to check that $[E,F]e_i = He_i$, $[H,E]e_i = 2Ee_i$, and $[H,F]e_i = -2Fe_i$ for all $i \in \{0,...,\lambda\}$. Hence $V(\lambda) = \operatorname{span}\{e_0, ..., e_\lambda\}$ affords a representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $(\lambda+1)$.

To show that it is unique, suppose that W is a simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $(\lambda+1)$. Let μ be the largest eigenvalue of H. The discussion above shows that W has a simple submodule of dimension $(\mu+1)$ isomorphic to $V(\mu)$. If W is simple, then $\mu = \lambda$ and hence $W \cong V(\lambda)$ as $\mathfrak{sl}_2(\mathbb{C})$-modules. □

Question 7

Let V be an arbitrary finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.

(a) Let $\lambda \in \mathbb{Z}$ be maximal amongst the eigenvalues of H, and let $V_\lambda \subseteq V$ denote the λ-eigenspace. Suppose that V has the property that $Ev = 0$ implies that $v \in V_\lambda$. Show that V is completely reducible.

(b) Consider the endomorphism $c = EF + FE + \frac{1}{2}H^2$. Show that c commutes with E, F and H.(c is called the Casimir element.) Deduce from Schur's lemma that c acts as a scalar on any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Compute the scalar with which c acts on $V(m)$.

(c) Show that V is completely reducible.

[*This result also follows from the more general complete reducibility result due to Weyl, but it is nice to see a more explicit proof for an algebra as small as* $\mathfrak{sl}_2$.]

Proof. (a) Let W be a maximal submodule (ordered by dimension) of V which is completely reducible. Suppose that $W \neq V$. Then $V/W \neq 0$. Let μ be the largest eigenvalue of $\widetilde{H} \in \operatorname{End}(V/W)$. Then there exists $v \in V \setminus W$ such that $Hv = \mu v$ (because μ is also an eigenvalue of H) and $\widetilde{H}(v+W) = \mu v + W$. Then $Ev \in V_{\mu+2}$ and

$\widetilde{E}(v+W)=W$, which imply that $Ev\in V_{\mu+2}\cap W$. Since $v\notin W$, we must have $Ev=0$. By assumption $v\in V_\lambda$. Hence $\mu=\lambda$. Let $U=\operatorname{span}\{v,\dots,F^\lambda v\}$ be a simple submodule of V. Then $U\cap W=\{0\}$ by simplicity. Hence $U\oplus W$ is a larger completely reducible submodule of V than W, which is a contradiction. In conclusion, $V=W$ is completely reducible.

(b) We compute the commutators:

$$[c,H]=[E,H]F+E[F,H]+F[E,H]+[F,H]E=-2EF+2EF-2FE+2FE=0$$
$$[c,E]=E[F,E]+[F,E]E+\frac{1}{2}H[H,E]+\frac{1}{2}[H,E]H=-EH-HE+HE+EH=0$$
$$[c,F]=[E,F]F+F[E,F]+\frac{1}{2}H[H,F]+\frac{1}{2}[H,F]H=HF+FH-HF-FH=0$$

Hence c commutes with E,F,H, and hence commutes with any element of $\mathfrak{sl}_2(\mathbb{C})$. Let V be a simple $\mathfrak{sl}_2(\mathbb{C})$-module. We have $c\cdot(\alpha\cdot v)=\alpha\cdot(c\cdot v)$ for any $\alpha\in\mathfrak{sl}_2(\mathbb{C})$ and $v\in V$. In particular $c\in\operatorname{End}(V)$. Since V is irreducible and $\mathbb{C}$ is algebraically closed, by Schur's Lemma, c acts on V as a scalar.

Let $V=V(m)$. Consider $v\in V(m)$ with highest weight. That is $Ev=0$ and $Hv=mv$. Then Q6(a) shows that

$$cv=(EF+FE+\frac{1}{2}H^2)v=(2EF-H+\frac{1}{2}H^2)v=(\frac{1}{2}m^2+m)v$$

Therefore $c=\left(\frac{1}{2}m^2+m\right)\operatorname{id}$ on $V(m)$.

(c) Consider the primary decomposition of V into generalised eigenspaces (which are $\mathfrak{sl}_2(\mathbb{C})$-submodules of V) of c:

$$V=\bigoplus_{i=1}^{\ell}\ker(c-\lambda_i\operatorname{id})^{\dim V}$$

To show that V is completely reducible, it suffices to show that each generalised eigenspace is completely reducible. We may assume that c has a unique eigenvalue μ on V.

Let λ be the largest eigenvalue of H in V. Then V has a simple submodule isomorphic to $V(\lambda)$. By (b) we have $c=\left(\frac{1}{2}\lambda^2+\lambda\right)\operatorname{id}$ restricting on $V(\lambda)$. Hence $\mu=\frac{1}{2}\lambda^2+\lambda$.

Suppose that $v\in\ker E$. Then

$$\left(\frac{1}{2}\lambda^2+\lambda\right)v=\mu v=c\cdot v=\left(\frac{1}{2}H^2+H\right)v$$

Hence

$$\frac{1}{2}(H-\lambda)(H+\lambda+2)v=0$$

It follows that $(H+\lambda+2)v\in V_\lambda$. Since the generalised eigenspaces of H are $(H+\lambda+2)$-invariant, we must have $v\in V_\lambda$. Now by (a), V is completely reducible. □