Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 1

B4.2: Functional Analysis II

Fantastic work! These solutions
are almost perfect and were a pleasure to mark

27 January, 2020

Question 1

Let $(X, \|\cdot\|)$ be a real norm vector space satisfying the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all } x, y \in X$$

Define

$$f(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \text{ for } x, y \in X$$

Show that

(a) $f(x,y) = f(y,x)$

(b) $f(x+z,y) = f(x,y) + f(z,y)$.

(c) $f(\alpha x, y) = \alpha f(x,y)$ for all $\alpha \in \mathbb{R}$.

Conclude that $f(x,y)$ defines an inner product on X.

Proof. (a) For $x, y \in X$:

$$f(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \frac{1}{4}\left(\|y+x\|^2 - (-1)^2\|y-x\|^2\right) = f(y,x)$$

(b) For $x, y, z \in X$:

$$\begin{aligned}
4\left(f(x+z,y) - f(x,y) - f(z,y)\right) &= \|x+z+y\|^2 - \|x+z-y\|^2 - \|x+y\|^2 + \|x-y\|^2 - \|z+y\|^2 + \|z-y\|^2 \\
&= \|x+z+y\|^2 - \|x+z-y\|^2 + \frac{1}{2}\left(\|x+z-2y\|^2 + \|x-z\|^2\right) - \frac{1}{2}\left(\|x+z+2y\|^2 + \|x-z\|^2\right) \\
&= \|x+z+y\|^2 - \|x+z-y\|^2 + \frac{1}{2}\left(\|x+z-2y\|^2 - \|x+z+2y\|^2\right) \\
&= \left(\|x+z+y\|^2 + \|y\|^2\right) - \left(\|x+z-y\|^2 + \|y\|^2\right) + \frac{1}{2}\left(\|x+z-2y\|^2 - \|x+z+2y\|^2\right) \\
&= \frac{1}{2}\left(\|x+z+2y\|^2 + \|x+z\|^2\right) - \frac{1}{2}\left(\|x+z\|^2 + \|x+z-2y\|^2\right) + \frac{1}{2}\left(\|x+z-2y\|^2 - \|x+z+2y\|^2\right) \\
&= 0
\end{aligned}$$

Hence $f(x+z,y) = f(x,y) + f(z,y)$.

(c) We proceed the proof in the order $\mathbb{Z}_+ \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$.

- For $\alpha \in \mathbb{Z}_+$, $f(\alpha x, y) = \alpha f(x,y)$ follows from (b) and induction on α.
- Note that

  $$f(-x,y) = \frac{1}{4}\left(\|-x+y\|^2 - \|-x-y\|^2\right) = \frac{1}{4}\left(\|x-y\|^2 - \|x+y\|^2\right) = -f(x,y)$$

  Hence for $\alpha \in \mathbb{Z}$, if $\alpha < 0$, then

  $$f(\alpha x, y) = -f(-\alpha x, y) = -(-\alpha) f(x,y) = \alpha f(x,y)$$

  If $\alpha = 0$, then

  $$f(\alpha x, y) = f(0,y) = f(0+0,y) = f(0,y) + f(0,y) \implies f(0,y) = 0 = 0 f(x,y)$$

- For $\alpha \in \mathbb{Q}$, there exists $m, n \in \mathbb{Z}$ such that $\alpha = m/n$. Note that

  $$f(x,y) = f\left(n \cdot \frac{1}{n}x, y\right) = nf\left(\frac{1}{n}x, y\right) \implies f\left(\frac{1}{n}x, y\right) = \frac{1}{n} f(x,y)$$

  Therefore

  $$f(\alpha x, y) = f\left(\frac{m}{n}x, y\right) = mf\left(\frac{1}{n}x, y\right) = \frac{m}{n} f(x,y) = \alpha f(x,y)$$

- For $\alpha \in \mathbb{R}$, there exists a sequence $\{q_n\} \subseteq \mathbb{Q}$ such that $q_n \to \alpha$ as $n \to \infty$. We claim that $\lim_{n\to\infty} \|q_n x + y\| = \|\alpha x + y\|$.

By triangular inequality,

$$\left| \|q_n x + y\| - \|\alpha x + y\| \right| \leq \|q_n x - \alpha x\| \leq |q_n - \alpha| \, \|x\| \to 0$$

as $n \to \infty$, which proves the claim. Similarly we also have $\lim_{n\to\infty} \|q_n x - y\| = \|\alpha x - y\|$. Finally,

$$f(\alpha x, y) = \frac{1}{4}\left(\|\alpha x + y\|^2 - \|\alpha x - y\|^2\right) = \lim_{n\to\infty} \frac{1}{4}\left(\|q_n x + y\|^2 - \|q_n x - y\|^2\right) = \lim_{n\to\infty} f(q_n x, y) = \lim_{n\to\infty} q_n f(x, y) = \alpha f(x, y)$$

$\square$

(b) and (c) implies the linearity in the first slot. Combining (a) we also obtain the linearity in the second slot: For $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

$$f(x, \alpha y + \beta z) = f(\alpha y + \beta z, x) = \alpha f(y, x) + \beta f(z, x) = \alpha f(x, y) + \beta f(x, z)$$

From the definition of norm we also know that f is positive definite. Hence f defines a inner product on X.

Question 2

Let $A^2(\mathbb{D})$ be the Bergman space of functions which are holomorphic and square integrable on the unit disk $\mathbb{D} \subseteq \mathbb{C}$. Let $f \in A^2(\mathbb{D})$, $0 < s < 1$ and $|z| < s$. Cauchy's integral formula gives

$$r f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{i\theta}\right) r \, d\theta$$

for any $0 < r < 1 - s$

(a) Integrating the above formula in $0 < r < 1 - s$, show that

$$f(z) = \frac{1}{\pi(1-s)^2} \langle f, \chi_{D(z,1-s)} \rangle_{L^2(\mathbb{D})}$$

where $D(z, 1-s)$ is the disk of radius $1 - s$ with the centre z.

(b) Deduce that

$$|f(z)| \leq \frac{\|f\|_{L^2(\mathbb{D})}}{\sqrt{\pi}(1-s)}$$

(c) Deduce that if f_n is a Cauchy sequence in $A^2(\mathbb{D})$ then f_n converges uniformly on compact subsets of $\mathbb{D}$.

(d) Deduce that $A^2(\mathbb{D})$ is closed in $L^2(\mathbb{D})$.

Proof. (a) Integrating the equation above:

$$\int_0^{1-s} r f(z) \, \mathrm{d}r = \frac{1}{2\pi} \int_0^{1-s} \int_0^{2\pi} f(z + r\,\mathrm{e}^{\mathrm{i}\theta}) r \, \mathrm{d}\theta \, \mathrm{d}r$$

The left hand side becomes $\frac{1}{2}\pi(1-s)^2 f(z)$. Since $f \in L^2(\mathbb{D})$, and $\mathbb{D}$ has finite Lebesgue measure, we have $f \in L^1(\mathbb{D})$. By Fubini's Theorem,

$$\frac{1}{2\pi} \int_0^{1-s} \int_0^{2\pi} f(z + r\,\mathrm{e}^{\mathrm{i}\theta}) r \, \mathrm{d}\theta \, \mathrm{d}r = \frac{1}{2\pi} \iint_{D(z,1-s)} f(z) \, \mathrm{d}A = \frac{1}{2\pi} \iint_{\mathbb{D}} f(z) \mathbf{1}_{D(z,1-s)} \, \mathrm{d}A = \langle f, \mathbf{1}_{D(z,1-s)} \rangle_{L^2(\mathbb{D})}$$

Hence

$$f(z) = \frac{1}{\pi(1-s)^2} \langle f, \mathbf{1}_{D(z,1-s)} \rangle_{L^2(\mathbb{D})}$$

(b) By Cauchy-Schwarz inequality,

$$\left| \langle f, \mathbf{1}_{D(z,1-s)} \rangle_{L^2(\mathbb{D})} \right| \leq \|f\|_{L^2(\mathbb{D})} \|\mathbf{1}_{D(z,1-s)}\|_{L^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{D})} \sqrt{m(D(z, 1-s))} = \sqrt{\pi}(1-s) \, \|f\|_{L^2(\mathbb{D})}$$

Hence

$$|f(z)| = \frac{1}{\pi(1-s)^2} \left| \langle f, \mathbf{1}_{D(z,1-s)} \rangle_{L^2(\mathbb{D})} \right| \leq \frac{\|f\|_{L^2(\mathbb{D})}}{\sqrt{\pi}(1-s)}$$

(c) Suppose that $\{f_n\}$ is a Cauchy sequence in $A^2(\mathbb{D})$. Let $K \subseteq \mathbb{D}$ be a compact subset. Choose $s \in (0, 1)$ such that $K \subseteq D(0, s)$.

Note that for $m, n \in \mathbb{N}$, by part (b) we have

$$\|f_n - f_m\|_{L^\infty(K)} \leq \|f_n - f_m\|_{L^\infty(D(0,s))} = \sup_{z\in D(0,s)} |f_n(z) - f_m(z)| \leq \frac{\|f_n - f_m\|_{L^2(\mathbb{D})}}{\sqrt{\pi}(1-s)}$$

Then $\{f_n\}$ is a Cauchy sequence in the Banach space $(C^0(K), \|\cdot\|_{\sup})$. Hence $\{f_n\}$ converges in $(C^0(K), \|\cdot\|_{\sup})$. In order words, $\{f_n\}$ converges uniformly in K.

(d) Suppose that $\{f_n\}$ is a sequence in $A^2(\mathbb{D})$ such that $f_n \to f$ as $n \to \infty$ for some $f \in L^2(\mathbb{D})$. Then $\{f_n\}$ is a Cauchy sequence. By part (c) f_n converges normally on $\mathbb{D}$. By Weierstrass' Theorem in complex analysis we know that the uniform limit f is holomorphic. f is also square integrable. Hence $f \in A^2(\mathbb{D})$. We deduce that $A^2(\mathbb{D})$ is closed in $L^2(\mathbb{D})$. □

Question 3

Let K be a non-empty convex set of a real Hilbert space X. Suppose that $x \in X$ and $y \in K$. Prove that the following are equivalent:

(1) $\|x - y\| \leq \|x - z\|$ for all $z \in K$

(2) $\langle x - y, z - y\rangle \leq 0$ for all $z \in K$

Proof.

(1) $\Longrightarrow$ (2): Fix $z \in K$. Since K is convex, $tz + (1-t)y \in K$ for $t \in [0,1]$. We have

$$\begin{aligned} &\|x-y\| \leq \|x - (tz + (1-t)y)\| \\ \Longrightarrow\ &\|x-y\|^2 \leq \|(x-y) + t(y-z)\|^2 = \|x-y\|^2 + t^2\|y-z\|^2 - 2t\langle x-y, z-y\rangle \\ \Longrightarrow\ &t^2\|y-z\|^2 - 2t\langle x-y, z-y\rangle \geq 0 \end{aligned}$$

Let $f(t) := t^2\|y-z\|^2 - 2t\langle x-y, z-y\rangle$. Then $f(t) \geq 0$ for $t \in [0,1]$. Note that $f(0) = 0$. Hence we must have

$$f'(0) = -2\langle x-y, z-y\rangle \geq 0$$

which implies that $\langle x-y, z-y\rangle \leq 0$ as required.

(2) $\Longrightarrow$ (1): For all $z \in K$,

$$\|x-z\|^2 = \|(x-y) + (y-z)\|^2 = \|x-y\|^2 + \|y-z\|^2 - 2\langle x-y, z-y\rangle \geq \|x-y\|^2$$

because $\|y-z\| \geq 0$ and $\langle x-y, y-z\rangle \leq 0$. Hence $\|x-y\| \leq \|x-z\|$ for all $z \in K$. □

Question 4

Let Y be a subspace of a Hilbert space X over $\mathbb{C}$ and $\ell : Y \to \mathbb{C}$ be a bounded linear functional on Y.

(a) Using the Riesz representation theorem, show that there is a *unique* extension of ℓ to a bounded linear functional $\tilde{\ell}$ on X with $\|\tilde{\ell}\|_{X^*} = \|\ell\|_{Y^*}$.

(b) By examining the behavior of $\tilde{\ell}$ on the orthogonal complement of Y, reprove (a) without using the Riesz representation theorem.

Proof. (a) By Riesz Representation Theorem, there exists $x_\ell \in Y$ such that for any $y \in Y$, $\ell(y) = \langle y, x_\ell\rangle$ and $\|x_\ell\|_X = \|\ell\|_{Y^*}$. We claim that $\tilde{\ell} : x \mapsto \langle x, x_\ell\rangle$ is a bounded linear functional on X such that $\|\tilde{\ell}\|_{X^*} = \|x_\ell\|_X$.

Note that Y is not assumed closed and so may not be a Hilbert Space. First extend ℓ to $\overline{Y}$.

- By Cauchy-Schwarz inequality,

$$\|\tilde{\ell}\|_{X^*} = \sup_{x\in X\setminus\{0\}} \frac{|\tilde{\ell}(x)|}{\|x\|_X} = \sup_{x\in X\setminus\{0\}} \frac{|\langle x, x_\ell\rangle|}{\|x\|_X} \leq \sup_{x\in X\setminus\{0\}} \frac{\|x\|_X\|x_\ell\|_X}{\|x\|_X} = \|x_\ell\|_X$$

- $\|x_\ell\|_X^2 = \langle x_\ell, x_\ell\rangle = \tilde{\ell}(x_\ell) \leq \|\tilde{\ell}\|_{X^*}\|x_\ell\|_X$. Hence $\|x_\ell\|_X \leq \|\tilde{\ell}\|_{X^*}$.

We deduce that $\|\tilde{\ell}\|_{X^*} = \|x_\ell\|_X = \|\ell\|_{Y^*}$.

To check that $\widetilde{\ell}$ is unique, let $\ell' : X \to \mathbb{C}$ be a bounded linear functional such that $\ell'|_Y = \ell$ and $\|\ell'\|_{X^*} = \|\ell\|_{Y^*}$. Then by Riesz Representation Theorem there exists $x'_\ell \in X$ such that $\ell'(x) = \langle x, x'_\ell \rangle$ for any $x \in X$ and $\|x'_\ell\|_X = \|\ell'\|_{X^*}$.

For $y \in Y$,

$$0 = \ell(y) - \ell(y) = \widetilde{\ell}(y) - \ell'(y) = \langle y, x_\ell - x'_\ell \rangle$$

Hence $x_\ell - x'_\ell \in Y^\perp$. Also note that $x_\ell \in Y$. By Pythagoras' Theorem,

$$\|x'_\ell\|_X^2 = \|x_\ell + (x'_\ell - x_\ell)\|_X^2 = \|x_\ell\|_X^2 + \|x_\ell - x'_\ell\|_X^2 \implies \|x_\ell - x'_\ell\|_X = 0$$

because $\|x_\ell\|_X = \|x'_\ell\|_X$. We deduce that $x_\ell = x'_\ell$ and hence $\ell' = \widetilde{\ell}$.

(b) First, since Y is dense in $\overline{Y}$, ℓ has a unique continuous extension on $\overline{Y}$. So without loss of generality we assume that Y is closed. By Projection Theorem, we have $X = Y \oplus Y^\perp$. We therefore can define $\widetilde{\ell}$ as follows. For $x \in X$, $x = y + y^\perp$ for unique $y \in Y$ and $y^\perp \in Y^\perp$. Let $\widetilde{\ell}(x) = \widetilde{\ell}(y + y^\perp) := \ell(y)$. In this way, we have

$$\|\widetilde{\ell}\|_{X^*} = \sup_{x \in X\setminus\{0\}} \frac{|\widetilde{\ell}(x)|}{\|x\|_X} = \sup_{x \in X\setminus\{0\}} \frac{|\ell \circ P_Y(x)|}{\|x\|_X} \leqslant \sup_{x \in X\setminus\{0\}} \frac{|\ell \circ P_Y(x)|}{\|P_Y(x)\|_X} = \sup_{y \in Y\setminus\{0\}} \frac{|\ell(y)|}{\|y\|_X} = \|\ell\|_{Y^*}$$

where $P_Y : X \to Y$ is the projection operator onto Y, and

$$\|\ell\|_{Y^*} = \sup_{y \in Y\setminus\{0\}} \frac{|\ell(y)|}{\|y\|_X} = \sup_{y \in Y\setminus\{0\}} \frac{|\widetilde{\ell}(y)|}{\|y\|_X} \leqslant \sup_{y \in X\setminus\{0\}} \frac{|\widetilde{\ell}(x)|}{\|x\|_X} = \|\widetilde{\ell}\|_{X^*}$$

Hence $\|\ell\|_{Y^*} = \|\widetilde{\ell}\|_{X^*}$.

To check that $\widetilde{\ell}$ is unique, let $\ell' : X \to \mathbb{C}$ be a bounded linear functional such that $\ell'|_Y = \ell$ and $\|\ell'\|_{X^*} = \|\ell\|_{Y^*}$. (*I did not finish this part.*) To be discussed in class. Show that necessarily $\ell'(Y^\perp) = \{0\}$. □

Question 5

For each of the cases below, determine –– in any order –– (i) the orthogonal complement of Y in X, (ii) if Y is dense in X, and (iii) if Y is closed in X. Here all spaces are over the real.

(a) $X = L^2(-1,1)$, $Y = \left\{ f \in X : \int_{-1}^{1} f(x)dx = 0 \right\}$.

(b) $X = \ell^2$, $Y = \{(a_n) \in X : a_2 = a_4 = \ldots = 0\}$.

(c) $X = L^2(0,1)$, $Y = C[0,1]$.

In (a) and (b) you may find it useful to rewrite the identities defining the space Y as an orthogonal relation e.g. $a_2 = 0$ means $\langle a, e_2 \rangle = 0$.

Proof. (a) $Y = \left\{ f \in X : \int_{-1}^{1} f(x)\,\mathrm{d}x = 0 \right\} = \{ f \in X : \langle f, \mathbf{1}_{(-1,1)} \rangle = 0 \} = \operatorname{span}\{\mathbf{1}_{(-1,1)}\}^\perp$

It is clear that $\operatorname{span}\{\mathbf{1}_{(-1,1)}\}$ is a closed subset of X. By Corollary 1.2.6 we have $Y^\perp = \operatorname{span}\{\mathbf{1}_{(-1,1)}\}^{\perp\perp} = \operatorname{span}\{\mathbf{1}_{(-1,1)}\}$. By Proposition 1.2.3.(i), Y is also closed in X. Since $Y \neq X$, Y is not dense in X.

Y is not dense in X. Consider $g = \mathbf{1}_{(-1,1)} \in Y^\perp$. For any sequence $\{f_n\} \in Y$, by Pythagoras' Theorem,

$$\|f_n - g\|_2^2 = \|f_n\|_2^2 + \|g\|_2^2 \geqslant \|g\|_2^2 = \int_{-1}^{1} \mathrm{d}x = 2$$

It is impossible that $\|f_n - g\|_2 \to 0$ as $n \to \infty$. So $g \notin \overline{Y}$.

(b) $Y = \{\{a_n\} \in X : a_2 = a_4 = \cdots = 0\} = \{\{a_n\} \in X : \forall k \in \mathbb{Z}_+ \ \langle \{a_n\}, e_{2k} \rangle = 0\} = \operatorname{span}\{e_{2k} : k \in \mathbb{Z}_+\}^\perp$

By Proposition 1.2.3.(i), Y is closed in X. Since $Y \neq X$, Y is not dense in X.

By Proposition 1.2.3 and Corollary 1.2.6,

$$Y^\perp = \operatorname{span}\{e_{2k} : k \in \mathbb{Z}_+\}^{\perp\perp} = \overline{\operatorname{span}\{e_{2k} : k \in \mathbb{Z}_+\}}^{\perp\perp} = \overline{\operatorname{span}\{e_{2k} : k \in \mathbb{Z}_+\}} = \left\{ \sum_{k=1}^{\infty} a_k e_{2k} : \sum_{k=1}^{\infty} a_k^2 < \infty \right\}$$

(c) We claim that Y is dense in X. It follows from the (much stronger) fact that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, where $1 \leqslant p < \infty$, for any open subset $\Omega \in \mathbb{R}^n$, which is Lemma 2.9 in B4.3 Distribution Theory.

Since $\overline{Y} = X \neq Y$, Y is not closed in X. Finally,

$$Y^\perp = \overline{Y}^\perp = X^\perp = \{0\}$$

□

Question 6

Let Y be the set of all $g \in L^2(-\pi,\pi)$ such that $g(t-\pi) = g(t)$ for almost all $t \in (0,\pi)$. Show that Y is a closed subspace of $L^2(-\pi,\pi)$ and identify $Y^\perp$. Assume that $f \in L^2(-\pi,\pi)$ and supposed $f = g + g^\perp$ where $g \in Y$ and $g^\perp \in Y^\perp$. Find g and $g^\perp$.

Calculate

$$d := \inf\{\|h-g\|_{L^2(-\pi,\pi)} : g \in Y\}$$

where $h(t) = t$ and specify the element g at which the infimum is attained.

Proof. First, it is clear that Y is a linear subspace od $L^2(-\pi,\pi)$. To show closedness, suppose that $\{g_n\}$ is a sequence in Y such that $g_n \to g$ as $n \to \infty$ for some $g \in L^2(-\pi,\pi)$. For each g_n, let

$$A_n := \{t \in (0,\pi) : g_n(t-\pi) \neq g_n(t)\}$$

Each A_n is null. Hence $\bigcup_{n=0}^{\infty} A_n$ is also null. For $t \in (0,\pi) \setminus \bigcup_{n=0}^{\infty} A_n$, $g_n(t-\pi) = g_n(t)$ for each $n \in \mathbb{N}$. Hence $g(t-\pi) = \lim_{n\to\infty} g_n(t-\pi) = \lim_{n\to\infty} g_n(t) = g(t)$. It follows that $g \in Y$. We deduce that Y is closed.

Use the general fact that L^2 convergent sequences have an a.e. convergent subsequence.

How do you know that the pointwise limit exists?

Suppose that $h \in Y^\perp$. For any $g \in Y$, we have

$$0 = \int_{-\pi}^{\pi} g(x)h(x)\,dx = \int_0^\pi g(x)h(x) + g(x-\pi)h(x-\pi)\,dx = \int_0^\pi g(x)(h(x)+h(x-\pi))\,dx$$

The integral is zero for any $g \in L^2(0,\pi)$. By the Fundamental Lemma of the Calculus of Variation (Lemma 3.18 in B4.3 Distribution Theory), we deduce that $h(x-\pi) + h(x) = 0$ almost everywhere on $(0,\pi)$. That is,

$$Y^\perp = \{h \in L^2(-\pi,\pi) :\ h(x-\pi) = -h(x) \text{ a.e. on } t \in (0,\pi)\}$$

Look up Typewriter Function.

For $f \in L^2(-\pi,\pi)$, we have

$$\forall\, x \in (0,\pi) \quad f(x) = g(x) + g^\perp(x), \quad f(x-\pi) = g(x-\pi) + g^\perp(x-\pi) = g(x) - g^\perp(x)$$

Hence we can set

$$g(x) = \frac{f(x)+f(x-\pi)}{2}\mathbf{1}_{(0,\pi)} + \frac{f(x)+f(x+\pi)}{2}\mathbf{1}_{(-\pi,0)}, \qquad g^\perp(x) = \frac{f(x)-f(x-\pi)}{2}\mathbf{1}_{(0,\pi)} + \frac{f(x)-f(x+\pi)}{2}\mathbf{1}_{(-\pi,0)}$$

where $g \in Y$ and $g^\perp \in Y^\perp$.

Let $P_Y : X \to Y$ be the projection operator onto Y. By Pythagoras' Theorem, for $f \in X$ and $g \in Y$,

$$\|f-g\|^2 = \|f - P_Y(f)\|^2 + \|P_Y(f) - g\|^2 \leqslant \|f - P_Y(f)\|^2$$

Hence $\inf_{g\in Y} \|f-g\|$ is obtained if and only if $g = P_Y(f)$.

For $h(t) = t$, we have

$$P_Y(h)(t) = \frac{h(x)+h(x-\pi)}{2}\mathbf{1}_{(0,\pi)} + \frac{h(x)+h(x+\pi)}{2}\mathbf{1}_{(-\pi,0)} = \left(x - \frac{\pi}{2}\right)\mathbf{1}_{(0,\pi)} + \left(x + \frac{\pi}{2}\right)\mathbf{1}_{(-\pi,0)}$$

and

$$(h - P_Y(h))(t) = \frac{h(x)-h(x-\pi)}{2}\mathbf{1}_{(0,\pi)} + \frac{h(x)-h(x+\pi)}{2}\mathbf{1}_{(-\pi,0)} = \frac{\pi}{2}\mathbf{1}_{(0,\pi)} - \frac{\pi}{2}\mathbf{1}_{(-\pi,0)}$$

The minimal distance from h to Y is given by

$$d = \|h - P_Y(h)\|_2 = \sqrt{\int_{-\pi}^{\pi} \left(\frac{\pi}{2}\right)^2 \mathrm{d}x} = \frac{\sqrt{2}}{2}\pi^{3/2} \qquad \square$$