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Problem Sheet 2

C2.1: Lie Algebras

Good! Review Q.3 and Q.6 to check you can now solve them.
Simplify your solution to Q.5.

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Throughout this sheet we assume that all Lie algebras are over a field k.

Assume throughout the problems that we work over a field k which is algebraically closed of characteristic zero, unless the contrary is explicitly stated.

Question 1

Suppose that $\mathfrak{g}$ is a Lie ara over the complex numbers. Show that $\mathfrak{g}$ is nilpotent if and only if any 2-dimensional subalgebra is Abelian.

Proof. $\Longrightarrow$ Suppose that $\mathfrak{g}$ is nilpotent. Let $\mathfrak{h}$ be a 2-dimensional subalgebra of $\mathfrak{g}$. Then by Lemma 3.18, $\mathfrak{h}$ is also nilpotent. Let $\{x, y\}$ be a basis of $\mathfrak{h}$. We have

as a general rule, don't refer to lecture notes in this way, say "as any subalgebra of a nilpotent Lie algebra is nilpotent".

$$[x, y] = \operatorname{ad} x(y) = ax + by, \quad (\operatorname{ad} x)^2(y) = b \operatorname{ad} x(y)$$

Since $\mathfrak{h}$ is nilpotent, $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{h})$ is a nilpotent operator. Hence we must have $b = 0$. Similarly, that $\operatorname{ad} y$ is nilpotent implies that $a = 0$. So $[x, y] = 0$. $\mathfrak{h}$ is an Abelian subalgebra.

(or $\mathfrak{g}$ nilp $\Rightarrow$ $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, and c.f. sheet 1, if $\mathfrak{z}(\mathfrak{g}) \neq \mathfrak{g}$, $\dim(\mathfrak{z}(\mathfrak{g})) \leq \dim(\mathfrak{g}) - 2$.)

$\Longleftarrow$ Suppose that any 2-dimensional subalgebra of $\mathfrak{g}$ is Abelian. For $x \in \mathfrak{g}$, consider $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{g})$. Since $\mathfrak{g}$ is over an algebraically closed field, $\operatorname{ad} x$ has eigenvalues. For any eigenvector $y \in \mathfrak{g}$ of x, we have

$$[x, y] = \operatorname{ad} x(y) = \lambda_y y$$

Therefore $\{x, y\}$ spans a 2-dimensional subalgebra of $\mathfrak{g}$. By assumption the subalgebra is Abelian. So $\lambda_y = 0$. We thus show that 0 is the unique eigenvalue of $\operatorname{ad} x$. Hence $\operatorname{ad} x$ is nilpotent. By Engel's Theorem, $\mathfrak{g}$ is nilpotent. good. □

Question 2

a) Let k be a field of characteristic 2 and let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Show that $\mathfrak{g}$ is solvable (and even nilpotent) but that the natural two-dimensional representation of $\mathfrak{g}$ is irreducible. Conclude that Lie's theorem is not true in positive characteristic.

b) Let $\mathbb{C}[x]$ denote a polynomial ring in x, and consider the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(\mathbb{C}[x])$ generated by the endomorphisms given by multiplication by x and $\frac{d}{dx}$. Show that $\mathfrak{g}$ is a three dimensional nilpotent Lie algebra, isomorphic to the Heisenberg algebra. Does $\mathfrak{g}$ fix a line in $\mathbb{C}[x]$? Why doesn't this contradict Lie's theorem?

Proof. a) We can write down a basis of $\mathfrak{sl}_2(\mathsf{k})$:

should point out explicitly that $\operatorname{tr}(\operatorname{id}) = 0$ in characteristic 2!

$$\operatorname{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where

$$[x, y] = \operatorname{id}, \qquad [x, \operatorname{id}] = [y, \operatorname{id}] = 0$$

Therefore $\mathfrak{z}(\mathfrak{sl}_2(\mathsf{k})) = \operatorname{span}_{\mathsf{k}}\{\operatorname{id}\}$, and $\mathfrak{sl}_2(\mathsf{k})/\mathfrak{z}(\mathfrak{sl}_2(\mathsf{k})) = \operatorname{span}_{\mathsf{k}}\left\{x + \mathfrak{z}(\mathfrak{sl}_2(\mathsf{k})), y + \mathfrak{z}(\mathfrak{sl}_2(\mathsf{k}))\right\}$, where $[x + \mathfrak{z}(\mathfrak{sl}_2(\mathsf{k})), y + \mathfrak{z}(\mathfrak{sl}_2(\mathsf{k}))] = \mathfrak{z}(\mathfrak{sl}_2(\mathsf{k}))$. Hence $\mathfrak{sl}_2(\mathsf{k})/\mathfrak{z}(\mathfrak{sl}_2(\mathsf{k}))$ is Abelian. By Lemma 3.18, $\mathfrak{sl}_2(\mathsf{k})$ is nilpotent.

see above

The embedding $\mathfrak{sl}_2(\mathsf{k}) \hookrightarrow \mathfrak{gl}_2(\mathsf{k})$ is a natural 2-dimensional representation of $\mathfrak{sl}_2(\mathsf{k})$. If it is reducible, that is, $\mathfrak{sl}_2(\mathsf{k})$ is a direct sum of two 1-dimensional subrepresentations, then id, x, y can be simultaneously diagonalised. But x is clearly not diagonalisable. So the natural 2-dimensional representation is irreducible.

In conclusion, Lie's Theorem does not hold in positive characteristic. good

b) For typographical reason we write ∂_x for $\mathrm{d}/\mathrm{d}x$. For $p \in \mathbb{C}[x]$, we have

$$[x, \partial_x]p(x) = xp'(x) - \frac{\mathrm{d}}{\mathrm{d}x}(xp(x)) = xp'(x) - p(x) - xp'(x) = -p(x)$$

Hence $\mathfrak{g}$ is generated by $x, \partial_x, \mathrm{id}$, with the Lie brackets given by

$$[x, \partial_x] = -\mathrm{id}, \qquad [x, \mathrm{id}] = [\partial_x, \mathrm{id}] = 0$$

In particular $\dim \mathfrak{g} = 3$. It is called the Heisenberg algebra, because of the canonical commutation relations in quantum mechanics:

$$[x, p_x] = \mathrm{i}\hbar\,\mathrm{id}, \qquad [x, \mathrm{id}] = [p_x, \mathrm{id}] = 0, \qquad \text{where } p_x = -\mathrm{i}\hbar\partial_x \text{ in the position representation}$$

why should this exist?

Suppose that $\mathfrak{g}$ fixes a subspace $V \subseteq \mathbb{C}[x]$. Let $p \in V$ be a polynomial of highest degree. But $\deg(xp) > \deg(p)$. So $xp \notin V$, which is a contradiction. Hence $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathbb{C}[x])$ is an irreducible representation of $\mathfrak{g}$.

This does not contradict Lie's Theorem because the representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathbb{C}[x])$ is infinite dimensional. □

this is correct but your proof only shows V has no subrep. with all elements of bounded degree.

Question 3

Let V be a finite dimensional vector space, and let $\mathscr{F}$ be a flag $0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n = V$ of subspaces where $\dim(V_i) = i$. If $\mathfrak{n}_{\mathscr{F}} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_{i-1}\}$ and $\mathfrak{b}_{\mathscr{F}} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_i\}$, then we have seen in lecture that $\mathfrak{n}_{\mathscr{F}}$ is an ideal in $\mathfrak{b}_{\mathscr{F}}$ and so we have an exact sequence,

$$0 \longrightarrow \mathfrak{n}_{\mathscr{F}} \longrightarrow \mathfrak{b}_{\mathscr{F}} \longrightarrow \mathfrak{t} \longrightarrow 0$$

where $\mathfrak{t}$ is defined to be the quotient $\mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}}$. Show that this sequence is split, and that there are infinitely many splitting maps $s : \mathfrak{t} \to \mathfrak{b}_{\mathscr{F}}$.

Proof. Let $\{x_1, \ldots, x_k\}$ be a basis of $\mathfrak{n}_{\mathscr{F}}$ and $\{y_1 + \mathfrak{n}_{\mathscr{F}}, \ldots, y_\ell + \mathfrak{n}_{\mathscr{F}}\}$ be a basis of $\mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}}$.

The exact sequence of Lie algebras

true, but better to explain how to split it. Also need a splitting as

$$0 \longrightarrow \mathfrak{n}_{\mathscr{F}} \longrightarrow \mathfrak{b}_{\mathscr{F}} \longrightarrow \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}} \longrightarrow 0$$

is an exact sequence of free $\Bbbk$-modules, which splits because free modules are projective. Then $\mathfrak{b}_{\mathscr{F}} \cong \mathfrak{n}_{\mathscr{F}} \oplus \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}}$ as $\Bbbk$-vector spaces. In particular, $s\colon \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}} \to \mathfrak{b}_{\mathscr{F}}$ induced by $s(y_i + \mathfrak{n}_{\mathscr{F}}) = y_i$ is a splitting map, and $\{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$ is a basis of $\mathfrak{b}_{\mathscr{F}}$.

homomorphism of Lie algebras. An arbitrary splitting of $b_{\mathscr{F}} \to b_{\mathscr{F}}/n_{\mathscr{F}}$ as a map of vector spaces will not do

We need to verify that s is a Lie algebra homomorphism. This is clear, as

$$s([y_i + \mathfrak{n}_{\mathscr{F}}, y_j + \mathfrak{n}_{\mathscr{F}}]) = s([y_i, y_j] + \mathfrak{n}_{\mathscr{F}}) = [y_i, y_j] = [s(y_i + \mathfrak{n}_{\mathscr{F}}), s(y_j + \mathfrak{n}_{\mathscr{F}})]$$

satisfy this – need the $\{y_i\}$ to commute.

Therefore $s\colon \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}} \to \mathfrak{b}_{\mathscr{F}}$ indeed induces a splitting of the short exact sequence of Lie algebras.

Moreover, for $a \in \Bbbk \setminus \{0, 1\}$, the map $s_a\colon \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}} \to \mathfrak{b}_{\mathscr{F}}$ given by $s_a(y_1 + \mathfrak{n}_{\mathscr{F}}) = ay_1$ and $s_a(y_i + \mathfrak{n}_{\mathscr{F}}) = y_i$ for $2 \leqslant i \leqslant \ell$ is also a section of $\mathfrak{b}_{\mathscr{F}} \twoheadrightarrow \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}}$. And $s \neq s_a$. Since $\Bbbk$ is algebraically closed, in particular it is infinite. We deduce that there are infinitely many sections $s\colon \mathfrak{b}_{\mathscr{F}}/\mathfrak{n}_{\mathscr{F}} \to \mathfrak{b}_{\mathscr{F}}$. □

Question 4

Suppose that V is a finite dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic zero, and $x, y \in \mathrm{End}_{\Bbbk}(V)$. Suppose that x and y commute with $z = [x, y] = xy - yx$. Show that z is nilpotent.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V such that $V_i = \mathrm{span}\{e_1, e_2, \ldots, e_i\}$. Then if $L_i = \Bbbk \cdot e_i$ and $\mathfrak{t} = \bigoplus_{i=1}^{n} \mathrm{End}(L_i) \subseteq \mathfrak{gl}(V)$ splits $b_{\mathscr{F}} \to b_{\mathscr{F}}/n_{\mathscr{F}}$ in that $q|_{\mathfrak{t}}$ is an isomorphism, so $s = (q|_{\mathfrak{t}})^{-1}$ is a splitting.

Proof. If $z = 0$, then z is trivially nilpotent. Suppose that $z \neq 0$. Then x, y, z are linearly independent. Then $\mathfrak{g} := \operatorname{span}_{\mathsf{k}}\{x, y, z\}$ is the 3-dimensional Heisenberg algebra. In particular $\mathfrak{g}$ is nilpotent. True, but say more.

Since k is algebraically closed. z has eigenvalues. Let λ_z be an eigenvalue of z, and W be the corresponding eigenspace. For $v \in W$, $z(v) = \lambda_z v \in W$, and

$$zx(v) = xz(v) + [z, x](v) = \lambda_z x(v), \qquad zy(v) = yz(v) + [z, y](v) = \lambda_z y(v)$$

Hence $x(v), y(v) \in W$. W affords a subrepresentation of $\mathfrak{g} \hookrightarrow \mathfrak{gl}_{\mathsf{k}}(V)$. By Lie's Theorem, there exists $w \in W \setminus \{0\}$ such that

$$x(w) = \lambda_x w, \qquad y(w) = \lambda_y w, \qquad z(w) = \lambda_z w$$

You don't need this – just consider $\operatorname{tr}_W(z) = \dim(W).\lambda$.

But this implies that

$$\lambda_z w = z(w) = [x, y](w) = (\lambda_x \lambda_y - \lambda_y \lambda_x)(w) = 0$$

Hence $\lambda_z = 0$. 0 is the only eigenvalue of z. We conclude that z is nilpotent. □

Question 5

Recall an element x of a Lie algebra $\mathfrak{g}$ is said to be regular if

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} : \exists n > 0, \operatorname{ad}(x)^n(y) = 0\}$$

has minimal possible dimension. Recall further that if V is a k-vector space and $x \in \mathfrak{gl}(V)$, then we may decompose V into the generalised eigenspaces of x, that is, $V = \bigoplus_\lambda V_\lambda$, where

$$V_\lambda = \{v \in V : \exists n \in \mathbb{N}, (x - \lambda)^n(v) = 0\}$$

We define $x_s \in \mathfrak{gl}(V)$ to be the linear map given by $x_s(v) = \lambda \cdot v$ for $v \in V_\lambda$. It is called the semisimple part of x. Clearly it is a diagonalisable linear map.

i) Let $x_n = x - x_s$. Check that x_n and x_s commute and that x_n is nilpotent.

ii) Show that $x \in \mathfrak{gl}(V)$ is regular if and only if x_s is regular.

iii) When is a semisimple (i.e. diagonalisable) element of $\mathfrak{gl}(V)$ regular?

iv) Exhibit a Cartan subalgebra of $\mathfrak{g}(V)$, and describe the set of all regular elements of $\mathfrak{gl}(V)$.

[Hint: For iii) pick a suitable basis of V to identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n$.]

Proof. i) This is a revision of linear algebra.

For $v \in V_\lambda$,

$$[x_s, x_n](v) = [x_s, x - x_s](v) = [x_s, x](v) = x_s x(v) - x x_s(v) = \lambda x(v) - \lambda x(v) = 0$$

Since $V = \bigoplus_\lambda V_\lambda$, we have $[x_s, x_n] = 0$ on V.

To show that x_n is nilpotent, it suffices to show that $x_n|_{V_\lambda}$ is nilpotent for each V_λ. We note that there is an ascending chain of subspaces of V_λ:

$$0 = \ker(x - \lambda)^0 \leqslant \ker(x - \lambda)^1 \leqslant \ker(x - \lambda)^2 \leqslant \cdots$$

Since V_λ has finite dimension, the chain eventually stablises. From the definition of V_λ we infer that there exists $N > 0$ such that $V_\lambda = \ker(x - \lambda)^N$. Note that $x_s|_{V_\lambda} = \lambda \operatorname{id}$. For $v \in V_\lambda$,

$$x_n^N(v) = (x - x_s)^N(v) = (x - \lambda)^N(v) = 0$$

Directly from the definition, if x_s is the semisimple part of x, then the generalized eigenspace V_λ of x is the λ-eigenspace of x_s, so $\mathfrak{g}_{0,x} = \mathfrak{g}_{0,x_s}$ if $\mathrm{ad}(x_s) = \mathrm{ad}(x)_s$, and this gives another approach.

Hence $x_n|_{V_\lambda}$ is nilpotent.

ii) Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of x. Since $[x_s, x_n] = 0$, then $[\mathrm{ad}\, x_s, \mathrm{ad}\, x_n] = 0$. Also, x_n is nilpotent implies that $\mathrm{ad}\, x_n$ is nilpotent. Let $N \in \mathbb{N}$ such that $(\mathrm{ad}\, x_n)^N = 0$.

For sufficiently large $m \in \mathbb{N}$, we have

$$(\mathrm{ad}\, x)^m = (\mathrm{ad}\, x_s + \mathrm{ad}\, x_n)^m = \sum_{k=0}^{m} \binom{m}{k} (\mathrm{ad}\, x_n)^k (\mathrm{ad}\, x_s)^{m-k} = (\mathrm{ad}\, x_s)^{m-N+1} \sum_{k=0}^{N-1} \binom{m}{k} (\mathrm{ad}\, x_n)^k (\mathrm{ad}\, x_s)^{N-k-1}$$

If $y \in \mathfrak{g}_{0,x_s}$, then $(\mathrm{ad}\, x_s)^{m-N+1}(y) = 0$ for sufficiently large m. The above equation implies that $(\mathrm{ad}\, x)^m(y) = 0$. Hence $y \in \mathfrak{g}_{0,x}$. We have $\mathfrak{g}_{0,x_s} \subseteq \mathfrak{g}_{0,x}$. The other direction of inclusion comes from noting that

$$(\mathrm{ad}\, x_s)^m = (\mathrm{ad}\, x - \mathrm{ad}\, x_n)^m = \sum_{k=0}^{m} \binom{m}{k} (-\mathrm{ad}\, x_n)^k (\mathrm{ad}\, x)^{m-k} = (\mathrm{ad}\, x)^{m-N+1} \sum_{k=0}^{N-1} \binom{m}{k} (-\mathrm{ad}\, x_n)^k (\mathrm{ad}\, x)^{N-k-1}$$

So we have $\mathfrak{g}_{0,x} = \mathfrak{g}_{0,x_s}$. In particular, x is regular if and only if x_s is regular.

iii) If $x \in \mathfrak{gl}(V)$ is semisimple, we claim that x is regular if and only if the eigenvalues of x are distinct.

Let $\{v_1, \ldots, v_n\}$ be a basis of V with respect to which $x = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Let E_{ij} be the matrix with $(E_{ij})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$. For $y = \sum_{i,j} a_{ij} E_{ij}$, we have

$$(\mathrm{ad}\, x(y))_{\mu\nu} = [x, y]_{\mu\nu} = \sum_\rho (x_{\mu\rho} y_{\rho\nu} - y_{\mu\rho} x_{\rho\nu}) = \sum_{i,j,\rho} (\lambda_\mu \delta_{\mu\rho} a_{ij} \delta_{i\rho} \delta_{j\nu} - \lambda_\rho \delta_{\rho\nu} a_{ij} \delta_{i\mu} \delta_{i\rho}) = a_{\mu\nu}(\lambda_\mu - \lambda_\nu)$$

if x is semisimple so is $\mathrm{ad}(x)$, hence you need not consider powers, you just need to find $\ker(\mathrm{ad}(x))$

Hence $\mathrm{ad}\, x(y) = \sum_{i,j} a_{ij}(\lambda_i - \lambda_j) E_{ij}$. Inductively we have $(\mathrm{ad}\, x)^m(y) = \sum_{i,j} a_{ij}(\lambda_i - \lambda_j)^m E_{ij}$.

We find that, if y is diagonal, that is, $a_{ij} = \xi_i \delta_{ij}$, then $(\mathrm{ad}\, x)^m(y) = 0$ for $m \geqslant 1$. Hence $\mathfrak{g}_{0,x}$ contains all the diagonal matrices. The dimension $\dim \mathfrak{g}_{0,x} \geqslant n = \dim V$.

$\Longleftarrow$ Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of x are distinct. For $y \in \mathfrak{g}_{0,x}$, we have $a_{ij}(\lambda_i - \lambda_j)^m = 0$ for all $1 \leqslant i, j \leqslant n$. As $\lambda_i \neq \lambda_j$, $a_{ij} = 0$ for $i \neq j$. Hence y is diagonal. We thus have shown that $\mathfrak{g}_{0,x}$ is exactly the set of diagonal matrices. It has minimal dimension equal to n. Hence x is regular.

$\Longrightarrow$ Suppose without loss of generality that $\lambda_1 = \lambda_2$. Then $\mathrm{ad}\, x(E_{12}) = 0$. Hence $E_{12} \in \mathfrak{g}_{0,x}$. Since $\mathfrak{g}_{0,x}$ contains non-diagonal matrices, it is not of minimal dimension. Hence x is not regular.

iv) By Lemma 4.7, if x is regular, then $\mathfrak{g}_{0,x}$ is a Cartan subalgebra of $\mathfrak{gl}(V)$.

Let $x \in \mathfrak{gl}(V)$ be a regular element. By (ii) x_s is regular. By (iii) x_s has distinct eigenvalues (i.e. each eigenvalue has algebraic multiplicity 1). But this implies that each generalised eigenspace of x has dimension 1. So in fact $x = x_s$. We conclude that the regular element in $\mathfrak{gl}(V)$ are exactly the elements with distinct eigenvalues.

if $x = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then $\dim(\ker(\mathrm{ad}(x))) = \#\{(i,j) : \lambda_i = \lambda_j\} = \#S$
$\geq n$ since $\{(i,i) : 1 \leq i \leq n\}$
$\subseteq S$. ☐

Question 6

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that the Killing form on $\mathfrak{g}$ is identically zero.

Proof. We assume that $\mathfrak{g}$ is finite dimensional. Let $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathsf{k}$ be the Killing form of $\mathfrak{g}$. Let $x, y \in \mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, so is $\bar{\mathfrak{g}}$. Hence $\mathrm{ad}\, x$, $\mathrm{ad}\, y$ are nilpotent. $(\mathrm{ad}\, x) \circ (\mathrm{ad}\, y)$ is also nilpotent. The unique eigenvalue of it is 0. Since the field k is algebraically closed, the characteristic polynomials of $\mathrm{ad}\, x, \mathrm{ad}\, y \in \mathfrak{gl}(\mathfrak{g})$ splits over k. And the trace of $(\mathrm{ad}\, x) \circ (\mathrm{ad}\, y)$ is just the sum of eigenvalues counting multiplicity.

False: $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$$\kappa(x, y) = \mathrm{tr}\big((\mathrm{ad}\, x) \circ (\mathrm{ad}\, y)\big) = 0$$

$\mathfrak{g}$ nilpotent $\Rightarrow$ $\mathfrak{g}$ is filtered by ideals $\mathfrak{z}_0 = 0 \subseteq \mathfrak{z}_1 \subseteq \mathfrak{z}_2 \subseteq \ldots \subseteq \mathfrak{z}_k = \mathfrak{g}$, and $\mathrm{ad}(x)(\mathfrak{z}_i) \subseteq \mathfrak{z}_{i-1}$

– set $\mathfrak{z}_1 = \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}_k$ = preimage of $\mathfrak{z}(\mathfrak{g}/\mathfrak{z}_{k-1})$ in $\mathfrak{g}$.
($\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ if $\mathfrak{g}$ is nilp. and non-zero.)

We conclude that $\kappa = 0$ for a nilpotent Lie algebra. $\square$

Question 7

Let k be a field and let $\mathfrak{s}_\mathsf{k}$ be the 3-dimensional k-Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$). Show that $\mathfrak{s}_\mathsf{k}$ is a simple Lie algebra. Show that $\mathfrak{s}_\mathbb{R}$ is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ but that $\mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$.

[Hint: Consider characteristic polynomials.]

Proof. (*The more popular way of expressing the structure constant is* $[e_i, e_j] = \sum_{k=0}^{2} \varepsilon_{ijk} e_k$.)

Let I be a non-zero ideal of $\mathfrak{s}_\mathsf{k}$. Let $a_0e_0 + a_1e_1 + a_2e_2 \in I \setminus \{0\}$. We have

$$[e_1, [e_0, a_0e_0 + a_1e_1 + a_2e_2]] = [e_1, a_1e_2 - a_2e_1] = a_1e_0 \in I, \qquad [e_2, [e_0, a_0e_0 + a_1e_1 + a_2e_2]] = [e_2, a_1e_2 - a_2e_1] = a_2e_0 \in I$$

If either $a_1 \neq 0$ or $a_2 \neq 0$, then $e_0 \in I$. If $a_1 = a_2 = 0$, then $a_0 \neq 0$ and hence $e_0 \in I$. So we will have $e_0 \in I$ anyway. By symmetry we also have $e_1, e_2 \in I$. Hence $I = \mathfrak{s}_\mathsf{k}$. $\mathfrak{s}_\mathsf{k}$ is a simple Lie algebra.

Let $\mathsf{k} = \mathbb{C}$. We define the ladder operators $e_\pm = e_1 \pm \mathrm{i}e_2$. Let $e_0' = 2\mathrm{i}e_0$. Then we have

$$[e_0', e_\pm] = 2\mathrm{i}[e_0, e_1 \pm \mathrm{i}e_2] = \pm 2(e_1 \pm \mathrm{i}e_2) = \pm 2e_\pm, \qquad [e_-, e_+] = [e_1 - \mathrm{i}e_2, e_1 + \mathrm{i}e_2] = 2\mathrm{i}e_0 = e_0'$$

We have an Lie algebra isomorphism $\varphi\colon \mathfrak{s}_\mathbb{C} \to \mathfrak{sl}_2(\mathbb{C})$ given by

$$\varphi(e_0') = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varphi(e_+) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \varphi(e_-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $\mathsf{k} = \mathbb{R}$. We compute the Killing form of $\mathfrak{s}_\mathbb{R}$. The matrix of $\operatorname{ad} e_0$, $\operatorname{ad} e_1$ and $\operatorname{ad} e_2$ are given by

$$\operatorname{ad} e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \operatorname{ad} e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad} e_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

With some tedious computation we find that the Killing form $\kappa(e_i, e_j) = -\delta_{ij}$. In particular it is negative-definite.

Let $x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$. We note that

$$\kappa(x, x) = \operatorname{tr} \mathrm{id}_2 = 2 > 0$$

So $\mathfrak{s}_\mathbb{R}$ and $\mathfrak{sl}_2(\mathbb{R})$ have different Killing form over $\mathbb{R}$. They cannot be isomorphic as real Lie algebras. $\square$