Notes on dispersionless KdV equation and inverse scattering problem in the WKB approximation

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0 Introduction

0.1 Background of Inverse Scattering Transform

The method of inverse scattering transform is first developed to solve the initial value problem of the Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \qquad u(x,0) = u_0(x)$$

Later this method is applied to a wider class of non-linear PDEs, which is known as integrable systems. We briefly summarise the inverse scattering transform applying to the KdV equation. For the details please refer to Ablowitz [2].

The one-dimensional scattering problem in quantum mechanics is to find the solution to the time-independent Schrödinger's equation

$$L\psi(x) := \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x)\right)\psi(x) = E\psi(x)$$

subjected to the condition that the potential function $u(x) \to 0$ sufficiently rapid as $|x| \to \infty$. Specifically, $u(x)$ satisfies the integrability condition

$$\int_{\mathbb{R}} (1+|x|)\,|u(x)|\,\mathrm{d}x < \infty \tag{0.1}$$

which ensures that the Hamiltonian L has finitely many discrete positive eigenvalues.[1] The point spectrum of L is $\{\kappa_1^2, \ldots, \kappa_N^2\}$ and the continuous spectrum is $E(k) = -k^2$ for $k \in \mathbb{R}$.

A right-to-left scattering wave function is given by

$$\begin{aligned} \psi(x) &\sim \mathrm{e}^{-\mathrm{i}kx} + r(k)\mathrm{e}^{\mathrm{i}kx}, & x \to +\infty \\ \psi(x) &\sim a(k)\mathrm{e}^{-\mathrm{i}kx}, & x \to -\infty \end{aligned}$$

where $r(k)$ is the reflection coefficient and $a(k)$ is the transmission coefficient. In this way we have the scattering data $S = (\kappa_1, \ldots, \kappa_N; r(k), a(k))$, associated to the potential function $u(x)$.

The problem of finding the scattering data from given potential function is called the direct scattering. The inverse problem of reconstructing the potential from the scattering data is called inverse scattering.

Next we shall show the connection of scattering problem with the KdV equation. We introduce the time t as a parameter of the Hamiltonian operator L:

$$L(t) = -\frac{\partial^2}{\partial x^2} + u(x, t)$$

The eigenvalues of L will depends on t in general. Consider an operator A which governs the time evolution of the wave function:

$$A\psi = \frac{\partial \psi}{\partial t}$$

We have the following theorem:

Theorem 0.1. Lax's Theorem

The eigenvalues of $L(t)$ is independent of t if and only if the operator A satisfies

$$\frac{\mathrm{d}L}{\mathrm{d}t} + [L, A] = 0 \tag{0.2}$$

which is called the Lax equation. The pair (L, A) is called a Lax pair.

Proof. Immediate by differentiating the Schrödinger's equation. □

For the KdV equation, through direct computation we can verify that the operator

$$A := -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}$$

satisfies that Lax equation (0.2) if and only if $u_t = 6uu_x - u_{xxx}$, which is exactly the KdV equation. Therefore we say that the KdV equation arises as the compatibility equation for L and A.

Lax theorem tells us the discrete spectrum of the Hamiltonian operator $L(t)$ is time independent. Furthermore, we shall prove in Proposition 2.5 that the operator A determines the time evolution of the scattering data $r(k, t)$ and $a(k, t)$ by the Lax equation.

Now we can put the steps together to give a scheme for the inverse scattering transform:

1. Given the initial data $u(x, 0)$, we can obtain the scattering data $S(0) = (\kappa_1, \ldots, \kappa_N; r(k, 0), a(k, 0))$ at $t = 0$ by solving the direct scattering problem;

2. Using the Lax equation we can solve the scattering data $S(t) = (\kappa_1, \ldots, \kappa_N; r(k, t), a(k, t))$ at any given time;

3. For each t, we solve the inverse scattering problem to obtain the solution $u(x, t)$ to the KdV equation.

[1] See §2.2 of [2] which explains different types of integrability conditions for $u(x)$ and gives references for the proofs.

0.2 Dispersionless Korteweg-de Vries Equation

We study the initial value problem of Korteweg-de Vries equation in the dispersionless limit. We begin with considering the rescaled KdV equation. (Our choice of the coefficients agrees with Wheeler [8].)

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, \qquad u(x,0) = u_0(x) \tag{0.3}$$

It arises as the compatibility equation for the following Lax pair:

$$L = -\varepsilon^2 \frac{\partial^2}{\partial x^2} + u(x,t), \qquad A = -4\varepsilon^2 \frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}$$

The corresponding equations are

$$L\psi = \lambda^2 \psi, \qquad A\psi = \frac{\partial \psi}{\partial t}, \qquad (\lambda \in \mathbb{C})$$

We make the WKB approximation

$$\psi(x,\lambda,t) = \exp\left(\frac{\mathrm{i}}{\varepsilon} S(x,\lambda,t)\right)$$

for some function $S(x,\lambda,t)$, which is called the action. We substitute the ansatz into the Lax equations and demand that $\varepsilon \to 0$. To the leading order, we obtain the Hamilton-Jacobi equations:

$$\left(\frac{\partial S}{\partial x}\right)^2 = \lambda^2 - u \tag{0.4}$$

$$4\left(\frac{\partial S}{\partial x}\right)^3 + 6u\frac{\partial S}{\partial x} = \frac{\partial S}{\partial t} \tag{0.5}$$

Put $p := \dfrac{\partial S}{\partial x}$. Then S is the Hamilton's prinicipal function generated by the Hamiltonian

$$H(x,p,t) := -4p^3 - 6u(x,t)p$$

The Lax theorem implies that $\lambda^2 = u - p^2$ is a constant of motion along the Hamiltonian flow. We have

$$0 = \frac{\partial \lambda^2}{\partial t} + \{\lambda^2, H\} = \frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} \tag{0.6}$$

which is the equation obtained by taking $\varepsilon \to 0$ formally in the KdV equation, and is hence called the dispersionless KdV (dKdV) equation[2].

1 Method of Characteristics

The initial value problem of the dKdV equation can be solved by the method of characteristics. Following the convention in [2], we assume that the initial data $u_0(x)$ is smooth and satisfies the integrability condition (0.1). We parametrise the data curve as

$$u(s) = u_0(s), \qquad x(s) = s, \qquad t(s) = 0, \qquad \text{at } \tau = 0$$

Then we solve the initial value problems for the system of ODEs

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = -6u, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = 0$$

We obtain the solution $u(x,t)$ in the following implicit form:

$$u = u_0(x + 6ut) \tag{1.7}$$

[2] also known as the inviscid Burgers' equation.

For $x_0 \in \mathbb{R}$ where $\mathrm{d}u_0/\mathrm{d}x \neq 0$, by inverse function theorem there exists a neighbourhood $(x_0 - \delta, x_0 + \delta)$ such that the solution takes the form

$$x + 6u(x,t)t = w(u) \tag{1.8}$$

which is the form obtain by Tsarëv in Theorem 10 of [7] for a broader class of systems called semi-Hamiltonian systems. We shall refer (1.8) as the Tsarëv's hodograph formula.

In (1.7), we take the derivative with respect to x:

$$\frac{\partial u}{\partial x} = \frac{u_0'(x+6ut)}{1-6tu_0'(x+6ut)}$$

We observe that at some finite time $t = t_0$, $\partial u/\partial x = \infty$ at some $x \in \mathbb{R}$, where the solution ceases to be single-valued. This is the well-known phenomenon of the breaking wave.

2 Inverse Scattering Transform

2.1 Solving the Action

Next we turn back to the WKB approximation and dKdV equation. We investigate the inverse scattering problem for the action $S(x,\lambda,t)$.

Without loss of generality we assume that $u_0 \geqslant 0$ on $\mathbb{R}$. Equation (0.4) gives

$$p(x,\lambda,t) = \frac{\partial S}{\partial x} = \left(\lambda^2 - u(x,t)\right)^{1/2} \tag{2.9}$$

where we choose the square root with a cut on the interval $\lambda \in \left[-\sqrt{u(x,t)}, \sqrt{u(x,t)}\right] \subseteq \mathbb{R}$ and the branch such that $\operatorname{Im} p \geqslant 0$ for $\lambda \in \mathbb{R}$. We integrate the equation to obtain that

$$S(x,\lambda,t) = \lambda x + \int_{x_0}^{x} \left(\lambda^2 - u(s,t)\right)^{1/2} \mathrm{d}s + f(\lambda,t)$$

for some $x_0 \in \mathbb{R}$ and entire function $f(\lambda,t)$. Given the integrability condition we have

$$\int_{-\infty}^{x_0} \left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s < \infty$$

We can replace $f(\lambda,t)$ by

$$f(\lambda,t) + \lambda x_0 + \int_{-\infty}^{x_0} \left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s$$

So we have

$$S(x,\lambda,t) = \lambda x + \int_{-\infty}^{x} \left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s + f(\lambda,t) \tag{2.10}$$

Next we consider the time evolution of S. Taking the derivative with respect to t:

$$\frac{\partial S}{\partial t}(x,\lambda,t) = \int_{-\infty}^{x} \frac{\partial}{\partial t}\left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s + \frac{\partial f}{\partial t}(\lambda,t)$$

where the differentiation under the integral sign is justified by DCT.

In the limit $x \to \pm\infty$, $p = \lambda + \mathrm{O}(\lambda^{-1})$. By (0.5) we have $\dfrac{\partial S}{\partial t}(x,\lambda,t) \sim 4\lambda^3$. Therefore $\dfrac{\partial f}{\partial t}(\lambda,t) = 4\lambda^3$. By integration we have

$$f(\lambda,t) = g(\lambda) + 4\lambda^3 t$$

for some entire function $g(\lambda)$. The action is now expressed as

$$S(x,\lambda,t) = \lambda x + 4\lambda^3 t + \int_{-\infty}^{x} \left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s + g(\lambda)$$

We have the freedom to choose $g(\lambda)$, as it does not contain physical quantities (x,t). It is safe to set $g(\lambda)=0$, and we obtain that

$$S(x,\lambda,t)=\lambda x+4\lambda^3 t+\int_{-\infty}^{x}\left(\left(\lambda^2-u(s,t)\right)^{1/2}-\lambda\right)\mathrm{d}s \tag{2.11}$$

On the λ-plane, the action $S(x,\lambda,t)$ is holomorphic in $\mathbb{C}\setminus\left[-\sqrt{u(x,t)},\sqrt{u(x,t)}\right]$. From (2.11) we observe that

$$S(x,\lambda,t)=\lambda x+4\lambda^3 t+\mathrm{O}(\lambda^{-1}),\qquad \text{as } |\lambda|\to\infty$$

In this section we only consider the initial curve $u_0(x)$ with one "hump". More specifically, there exists $x_m\in\mathbb{R}$ such that

- $u_m := u_0(x_m)=\sup_{x\in\mathbb{R}} u_0(x)$;
- u_0 is monotonically increasing for $x\in(-\infty,x_m)$;
- u_0 is monotonically decreasing for $x\in(x_m,+\infty)$.
- $u_0\geqslant 0$ for all $x\in\mathbb{R}$.

We know from the solution obtained by the method of characteristics that the above condition also holds for $u(x,t)$ for a finite time interval $t\in(0,t_0)$.

For $\lambda\in\left(-\sqrt{u_m},\sqrt{u_m}\right)$, there exist $x_-(\lambda,t)<x_+(\lambda,t)$ such that $u(x_\pm(\lambda,t),t)=\lambda^2$.

We define the following auxiliary functions, which play the role of scattering data (will be shown in the next subsection):

$$\omega(\lambda,t):=\lambda x_-(\lambda,t)+\int_{-\infty}^{x_-(\lambda,t)}\left(\left(\lambda^2-u(s,t)\right)^{1/2}-\lambda\right)\mathrm{d}s, \tag{2.12}$$

$$\sigma(\lambda,t):=-\lambda x_+(\lambda,t)+\int_{x_+(\lambda,t)}^{+\infty}\left(\left(\lambda^2-u(s,t)\right)^{1/2}-\lambda\right)\mathrm{d}s \tag{2.13}$$

$$\tau(\lambda,t):=\int_{x_-(\lambda,t)}^{x_+(\lambda,t)}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s \tag{2.14}$$

Remark. The integrand $\sqrt{u(s,t)-\lambda^2}$ should be understood as the positive sqaure root of a real function in the usual sense. Note that this is different from the complex square root $\left(\lambda^2-u(s,t)\right)^{1/2}$, which in fact changes sign under the map $\lambda\mapsto-\lambda$.

Proposition 2.1

1. For $0<\lambda^2<u(x,t)$, the real part of $S(x,\lambda,t)$ is independent of x and t, and we have

$$\operatorname{Re} S(x,\lambda,t)=4\lambda^3 t+\omega(\lambda,t)$$

2. For $u(x,t)<\lambda^2<u_m$ and $x>x_+(\lambda,t)$, the imaginary part of $S(x,\lambda,t)$ is independent of x and t, and we have

$$\operatorname{Im} S(x,\lambda,t)=\tau(\lambda,t)$$

Proof. For real λ we note that $p=\left(\lambda^2-u(s,t)\right)^{1/2}\in\mathbb{R}$ when $\lambda^2<u(x,t)$ and $0<\lambda^2<u(x,t)\in\mathrm{i}\mathbb{R}$ when $\lambda^2>u(x,t)$.

Therefore, for $0<\lambda^2<u(x,t)$, by (2.11) and (2.12),

$$\begin{aligned}S(x,\lambda,t)&=4\lambda^3 t+\omega(\lambda,t)+\lambda(x-x_-(\lambda,t))+\int_{x_-(\lambda,t)}^{x}\left(\left(\lambda^2-u(s,t)\right)^{1/2}-\lambda\right)\mathrm{d}s\\&=4\lambda^3 t+\omega(\lambda,t)+\mathrm{i}\int_{x_-(\lambda,t)}^{x}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s\end{aligned}$$

Hence

$$\operatorname{Re} S(x,\lambda,t) = 4\lambda^3 t + \omega(\lambda,t)$$

For $u(x,t) < \lambda^2 < u_m$ and $x > x_+(\lambda,t)$,

$$\operatorname{Im} S(x,\lambda,t) = \operatorname{Im}\left(\int_{-\infty}^{x} \left(\lambda^2 - u(s,t)\right)^{1/2} \mathrm{d}s\right) = \tau(\lambda,t)$$

For the time-dependence, we note from (0.5) that $p \in \mathbb{R}$ implies that $\partial S/\partial t \in \mathbb{R}$, and that $p \in \mathrm{i}\mathbb{R}$ implies that $\partial S/\partial t \in \mathrm{i}\mathbb{R}$. When $\lambda^2 < u(x,t)$, $p \in \mathrm{i}\mathbb{R}$, and then $\operatorname{Re} S$ is time-independent. When $\lambda^2 > u(x,t)$, $p \in \mathbb{R}$, and then $\operatorname{Im} S$ is time-independent. □

Corollary 2.2

The time-evolution of $\omega(\lambda,t)$ and $\tau(\lambda,t)$ are given by

$$\omega(\lambda,t) = \omega_0(\lambda) - 4\lambda^3 t \tag{2.15}$$

$$\tau(\lambda,t) = \tau_0(\lambda) \tag{2.16}$$

2.2 WKB Method for Direct Scattering

We review the direct scattering problem of the Schrödinger's equation in the WKB approximation, following Section 10.6 in Bender & Orszag [4]. In particular, the reflection and transmission coefficients are given in terms of the real and imaginary parts of the action.

Starting from the Schrödinger's equation

$$-\varepsilon^2 \frac{\partial}{\partial x^2}\psi(x,\lambda,t) + u(x,t)\psi(x,\lambda,t) = \lambda^2\psi(x,\lambda,t)$$

Assuming a left-to-right direct scattering, we can write down the asymptotics of the wave function:

$$\psi(x,\lambda,t) \sim \begin{cases} \mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon} + r(\lambda,t)\,\mathrm{e}^{\mathrm{i}\lambda x/\varepsilon}, & x \to -\infty \\ \rho(\lambda,t)\,\mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon}, & x \to +\infty \end{cases} \tag{2.17}$$

where $r(\lambda,t)$ and $\rho(\lambda,t)$ are the reflection and transmission coefficients respectively.

For $\lambda^2 > u_m$, the whole real line is classically allowed. We have $r(\lambda,t) = 0$ and $\rho(\lambda,t) = 1$ trivially. Now we focus on the case $0 < \lambda^2 < u_m$, where $x \in (x_-(\lambda,t), x_+(\lambda,t))$ is classically forbidden. Our main result is the following:

Proposition 2.3

For $0 < \lambda^2 < u_m$, the reflection and transmission coefficients are given by

$$r(\lambda,t) = \mathrm{i}\exp\left(-\frac{2\mathrm{i}\omega}{\varepsilon}\right) \tag{2.18}$$

$$\rho(\lambda,t) = \exp\left(-\frac{\tau + \mathrm{i}(\omega+\sigma)}{\varepsilon}\right) \tag{2.19}$$

Proof. First we write down the corresponding WKB wave functions in the classically allowed regions:

$$\begin{aligned}\psi_\pm(x,\lambda,t) &= \frac{A_\pm}{(\lambda^2 - u(x,t))^{1/4}} \exp\left(\pm\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-(\lambda,t)} \sqrt{\lambda^2 - u}\,\mathrm{d}s\right) \\ &= \frac{A_\pm\,\mathrm{e}^{\pm\mathrm{i}\lambda x_-(\lambda,t)/\varepsilon}}{(\lambda^2 - u(x,t))^{1/4}} \exp\left(\pm\frac{\mathrm{i}}{\varepsilon}\left(-\lambda x + \int_x^{x_-(\lambda,t)} \left(\sqrt{\lambda^2 - u} - \lambda\right)\mathrm{d}s\right)\right), \qquad x \in (-\infty, x_-(\lambda,t))\end{aligned}$$

$$
\begin{aligned}
\psi_{\pm}(x,\lambda,t) &= \frac{C_{\pm}}{(\lambda^2-u(x,t))^{1/4}}\exp\left(\pm\frac{\mathrm{i}}{\varepsilon}\int_{x_+(\lambda,t)}^{x}\sqrt{\lambda^2-u}\,\mathrm{d}s\right)\\
&= \frac{C_{\pm}\,\mathrm{e}^{\mp\mathrm{i}\lambda x_+(\lambda,t)/\varepsilon}}{(\lambda^2-u(x,t))^{1/4}}\exp\left(\pm\frac{\mathrm{i}}{\varepsilon}\left(\lambda x+\int_{x_+(\lambda,t)}^{x}\left(\sqrt{\lambda^2-u}-\lambda\right)\mathrm{d}s\right)\right), \qquad x\in(x_+(\lambda,t),+\infty)
\end{aligned}
$$

which gives

$$
\psi_{\pm}(x,\lambda,t)\sim\begin{cases} A_{\pm}\lambda^{-1/2}\exp\left(\pm\frac{\mathrm{i}}{\varepsilon}(-\lambda x+\omega(\lambda,t))\right), & x\to-\infty\\ C_{\pm}\lambda^{-1/2}\exp\left(\pm\frac{\mathrm{i}}{\varepsilon}(\lambda x+\sigma(\lambda,t))\right), & x\to+\infty\end{cases} \tag{2.20}
$$

By comparing (2.17) and (2.20) we obtain that

$$
\begin{aligned}
A_+\lambda^{-1/2}\,\mathrm{e}^{\mathrm{i}\omega(\lambda,t)/\varepsilon} &= 1 & A_-\lambda^{-1/2}\,\mathrm{e}^{-\mathrm{i}\omega(\lambda,t)/\varepsilon} &= r(\lambda,t)\\
C_+\lambda^{-1/2}\,\mathrm{e}^{\mathrm{i}\sigma(\lambda,t)/\varepsilon} &= 0 & C_-\lambda^{-1/2}\,\mathrm{e}^{-\mathrm{i}\sigma(\lambda,t)/\varepsilon} &= \rho(\lambda,t)
\end{aligned}
$$

Then the WKB wave functions are given by

$$
\psi(x,\lambda,t)=\begin{cases}\left(\frac{\lambda^2}{\lambda^2-u}\right)^{1/4}\left(\mathrm{e}^{-\mathrm{i}\omega/\varepsilon}\exp\left(\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right)\right.\\ \qquad\left.+r\,\mathrm{e}^{\mathrm{i}\omega/\varepsilon}\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right)\right), & x\in(-\infty,x_-)\\ \left(\frac{\lambda^2}{\lambda^2-u}\right)^{1/4}\rho\,\mathrm{e}^{\mathrm{i}\sigma/\varepsilon}\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_{x_+}^{x}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right), & x\in(x_+,+\infty)\end{cases}
$$

Next we consider the classically forbidden region. At the turning points $x_\pm(\lambda,t)$, the WKB connection formulae are given by

$$
\begin{aligned}
\left(\frac{\lambda^2}{\lambda^2-u}\right)^{1/4}\exp\left(\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right) &\sim \left(\frac{\lambda^2}{u-\lambda^2}\right)^{1/4}\exp\left(\frac{\pi\mathrm{i}}{4}-\frac{1}{\varepsilon}\int_{x_-}^{x}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s\right)\\
\left(\frac{\lambda^2}{\lambda^2-u}\right)^{1/4}\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_{x_+}^{x}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right) &\sim \left(\frac{\lambda^2}{u-\lambda^2}\right)^{1/4}\exp\left(\frac{\pi\mathrm{i}}{4}+\frac{1}{\varepsilon}\int_{x}^{x_+}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s\right)
\end{aligned}
$$

They allow us to express $\psi(x,\lambda,t)$ for $x<x_+$ in terms of the transmission coefficient $\rho(\lambda,t)$:

$$
\begin{aligned}
\psi(x,\lambda,t) = \left(\frac{\lambda^2}{\lambda^2-u}\right)^{1/4}\rho\,\mathrm{e}^{\mathrm{i}\omega/\varepsilon}\exp\left(\frac{1}{\varepsilon}\int_{x_-}^{x_+}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s\right)&\left(\exp\left(\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right)\right.\\
&\left.+\,\mathrm{i}\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_x^{x_-}\sqrt{\lambda^2-u(s,t)}\,\mathrm{d}s\right)\right)
\end{aligned}
$$

By comparing coefficients we obtain that

$$
\begin{aligned}
r(\lambda,t) &= \mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\omega/\varepsilon}\\
\rho(\lambda,t) &= \exp\left(-\frac{1}{\varepsilon}\int_{x_-}^{x_+}\sqrt{u(s,t)-\lambda^2}\,\mathrm{d}s\right)\mathrm{e}^{-\mathrm{i}(\omega+\sigma)/\varepsilon}
\end{aligned}
$$

□

Therefore we associate the real part of S with the reflection coefficient, and the imaginary part of S with the transmission coefficient. They are effective for different ranges of x.

Corollary 2.4

The reflection and transmission coefficients are related to the action by

$$
r(\lambda,t)=\mathrm{i}\,\mathrm{e}^{8\mathrm{i}\lambda^3 t}\exp(-2\mathrm{i}\,\mathrm{Re}\,S(x,\lambda,t)/\varepsilon), \qquad 0<\lambda^2<u(x,t) \tag{2.21}
$$

$$
|\rho(\lambda,t)|=\exp(-\mathrm{Im}\,S(x,\lambda,t)/\varepsilon), \qquad u(x,t)<\lambda^2<u_m,\ x>x_+(\lambda,t) \tag{2.22}
$$

The time-evolution of the scattering data can in fact be determined for the full KdV equation. The following is a standard result from Ablowitz [2] and Dunajski [9].

Proposition 2.5

The time-evolution of $r(\lambda, t)$ and $\rho(\lambda, t)$ are given by

$$r(\lambda, t) = r_0(\lambda)\,\mathrm{e}^{8\mathrm{i}\lambda^3 t/\varepsilon} \tag{2.23}$$

$$\rho(\lambda, t) = \rho_0(\lambda) \tag{2.24}$$

Proof. By Lax theorem, $L\psi = \lambda^2\psi$ implies that

$$L\left(\frac{\partial\varphi}{\partial t} - A\varphi\right) = \lambda^2\left(\frac{\partial\varphi}{\partial t} - A\varphi\right)$$

where

$$\varphi(x,\lambda,t) := \frac{\psi(x,\lambda,t)}{\rho(\lambda,t)} \sim \begin{cases} \dfrac{1}{\rho(\lambda,t)}\,\mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon} + \dfrac{r(\lambda,t)}{\rho(\lambda,t)}\,\mathrm{e}^{\mathrm{i}\lambda x/\varepsilon}, & x\to-\infty \\ \mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon}, & x\to+\infty \end{cases}$$

Therefore we have

$$\frac{\partial\varphi}{\partial t} - A\varphi \sim \begin{cases} \left(\dfrac{\partial\rho^{-1}}{\partial t} + \dfrac{4\mathrm{i}\lambda^3}{\varepsilon}\right)\rho^{-1}\,\mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon} + \left(\dfrac{\partial(r\rho^{-1})}{\partial t} - \dfrac{4\mathrm{i}\lambda^3}{\varepsilon}\right) r\rho^{-1}\,\mathrm{e}^{\mathrm{i}\lambda x/\varepsilon}, & x\to-\infty \\ \dfrac{4\mathrm{i}\lambda^3}{\varepsilon}\,\mathrm{e}^{-\mathrm{i}\lambda x/\varepsilon}, & x\to+\infty \end{cases}$$

Note that

$$\xi(x,\lambda,t) := \frac{\partial\varphi}{\partial t} - A\varphi - \frac{4\mathrm{i}\lambda^3}{\varepsilon} \in \ker L, \qquad \lim_{x\to\infty}\xi(x,\lambda,t) = 0$$

We must have $\xi = 0$. We therefore use the asymptotics at $x\to+\infty$ to deduce that

$$\frac{\partial\rho^{-1}}{\partial t} + \frac{4\mathrm{i}\lambda^3}{\varepsilon} = \frac{4\mathrm{i}\lambda^3}{\varepsilon}, \qquad \frac{\partial(r\rho^{-1})}{\partial t} - \frac{4\mathrm{i}\lambda^3}{\varepsilon} = \frac{4\mathrm{i}\lambda^3}{\varepsilon}$$

The time-dependence for the scattering data is given by

$$r(\lambda,t) = r(\lambda,0)\,\mathrm{e}^{8\mathrm{i}\lambda^3 t/\varepsilon}, \qquad \rho(\lambda,t) = \rho(\lambda,0)$$ □

Remark. Combining the results of Proposition 2.5 and Proposition 2.3, we can also deduce that time-dependence of $\omega(\lambda,t)$, $\sigma(\lambda,t)$ and $\tau(\lambda,t)$ as in Corollary 2.2.

2.3 Inverse Scattering

In Geogjaev [5], an inversion formula for $u(x,t)$ in terms of the real part of $S(x,\lambda,t)$ is discovered by solving the corresponding Riemann-Hilbert problem.

Recall that

$$\omega(\lambda,t) := \lambda x_-(\lambda,t) + \int_{-\infty}^{x_-(\lambda,t)} \left(\left(\lambda^2 - u(s,t)\right)^{1/2} - \lambda\right) \mathrm{d}s \tag{2.12}$$

For convenience we put

$$\Sigma(x,\lambda,t) := S(x,\lambda,t) - 4\lambda^3 t$$

Therefore we have $\omega(\lambda,t) = \operatorname{Re}\Sigma(x,\lambda,t)$ for $\lambda \in \left(-\sqrt{u(x,t)}, \sqrt{u(x,t)}\right) \subseteq \mathbb{R}$.

Now the problem has become a Riemann-Hilbert problem on the λ-plane. The following proof follows from Section 7.3.2 of Ablowitz & Fokas [3] (see also Section 1.4 of Wheeler [8]). For convenience we suppress the (x,t)-dependence.

Proposition 2.6. The Scalar Riemann-Hilbert Problem

Suppose that Σ is holomorphic on $\mathbb{C} \setminus \left[-\sqrt{u}, \sqrt{u}\right]$ satisfying $\operatorname{Re}\Sigma_\pm(\lambda) = \omega(\lambda)$ for $\lambda \in \left(-\sqrt{u}, \sqrt{u}\right)$, where ω is Hölder continuous on $\left(-\sqrt{u}, \sqrt{u}\right)$. Then we have

$$\Sigma(\lambda) = \left(u - \lambda^2\right)^{1/2} \left(\frac{1}{\pi \mathrm{i}} \int_{-\sqrt{u}}^{\sqrt{u}} \frac{\omega(\zeta)}{\sqrt{u - \zeta^2}(\zeta - z)} \, \mathrm{d}\zeta + H(\lambda) \right)$$

where $H(\lambda)$ is holomorphic in $\mathbb{C} \setminus \left\{-\sqrt{u}, \sqrt{u}\right\}$. $H(\lambda)$ can be uniquely determined by specifying the principal parts of the Laurent series of $\Sigma(\lambda)$ at $\lambda = \pm\sqrt{u}$ and $\lambda = \infty$.

Remark. Note that $(u - \lambda^2)^{1/2}$ is a holomorphic branch of the square root function on the complex analysis, whereas $\sqrt{u - \zeta^2}$ in the integrand should be understood in the usual sense of taking the positive sqaure root of a positive number.

Lemma 2.7. Plemelj Formulae

Let L be a simple, smooth, locally connected curve and let $\varphi(\tau)$ be Hölder continuous on L. Then the Cauchy-type integral

$$\Phi(\lambda) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{\varphi(\lambda)}{\tau - \lambda}$$

has the limiting values $\Phi_\pm(\xi)$ as λ approaches L from the left and the right, respectively, and ξ is not an endpoint of L. These limits are given by

$$\Phi_\pm(\xi) = \pm\frac{1}{2}\varphi(\xi) + \frac{1}{2\pi \mathrm{i}} ⨍_L \frac{\varphi(\tau)}{\tau - \xi} \, \mathrm{d}\tau$$

where the principal value integrals are defined by

$$⨍_L \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi} = \lim_{\varepsilon \to 0} \int_{L \setminus B(\xi, \varepsilon)} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi}$$

Proof. See Lemma 7.2.1 of Ablowitz & Fokas [3]. □

Proof of Proposition 2.6.

We seek a solution of the form of a Cauchy-type integral:

$$\Sigma(\lambda) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{f(\zeta)}{\zeta - \lambda} \, \mathrm{d}\lambda$$

By Plemelj formulae, we have

$$\Sigma_\pm(\xi) = \pm\frac{1}{2} f(\xi) + \frac{1}{2\pi \mathrm{i}} ⨍_{-\sqrt{u}}^{\sqrt{u}} \frac{f(\zeta)}{\zeta - \xi} \, \mathrm{d}\zeta$$

for $\xi \in \left(-\sqrt{u}, \sqrt{u}\right)$. Since $\operatorname{Re}\Sigma_+(\xi) = \operatorname{Re}\Sigma_-(\xi)$, we have

$$0 = \operatorname{Re}(\Sigma_+(\xi) - \Sigma_-(\xi)) = \operatorname{Re} f(\xi)$$

Hence f is purely imaginary, and we have $\operatorname{Im}\Sigma_+(\xi) = -\operatorname{Im}\Sigma_-(\xi)$. Therefore we have the problem expressed in the standard form:

$$\Sigma_+(\xi) = -\Sigma_-(\xi) + 2\omega(\xi)$$

First we consider the corresponding homogeneous problem

$$T_+(\xi) = -T_-(\xi)$$

Taking logarithm:

$$\log T_+(\xi) = \log T_-(\xi) + \pi\mathrm{i}$$

Hence by Plemelj formulae,

$$T(\lambda) = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\pi\mathrm{i}}{\zeta-\lambda}\,\mathrm{d}\lambda + h(\lambda)\right) = \mathrm{e}^{\widetilde{h}(\lambda)}\left(u-\lambda^2\right)^{1/2}$$

where $\widetilde{h}(\lambda)$ is an arbitrary entire function. We take $\widetilde{h}=0$ and return to the inhomogeneous problem:

$$\frac{\Sigma_+(\xi)}{T_+(\xi)} = \frac{\Sigma_-(\xi)}{T_-(\xi)} + \frac{2\omega(\xi)}{T_+(\xi)}$$

Clearly ω/T_+ is Hölder continuous on $\left(-\sqrt{u},\sqrt{u}\right)$, with integrable singularities at the endpoints. By Plemelj formulae we have

$$\frac{\Sigma(\lambda)}{T(\lambda)} = \frac{1}{2\pi\mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{2\omega(\lambda)}{T_+(\lambda)(\zeta-\lambda)}\,\mathrm{d}\lambda + H(\lambda)$$

where $H(\lambda)$ is holomorphic in $\mathbb{C}\setminus\left\{-\sqrt{u},\sqrt{u}\right\}$. Therefore we obtain the solution

$$\Sigma(\lambda) = \left(u-\lambda^2\right)^{1/2}\left(\frac{1}{\pi\mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\omega(\zeta)}{\sqrt{u-\zeta^2}(\zeta-\lambda)}\,\mathrm{d}\zeta + H(\lambda)\right)$$

Let $\widetilde{\Sigma}$ be another solution to the problem which has the same principal parts of the Laurent series as S at $\lambda = \pm\sqrt{u}$ and $\lambda = \infty$. the let $\Omega := \Sigma - \widetilde{\Sigma}$. Then $\dfrac{\Sigma-\widetilde{\Sigma}}{T}$ satisfies the homogeneous problem $\Omega_+(\xi) = \Omega_-(\xi)$ for $\xi \in \left(-\sqrt{u},\sqrt{u}\right)$.

By Cauchy's integral formula, Ω satisfies the condition if and only if Ω is holomorphic in $\mathbb{C}\setminus\left\{-\sqrt{u},\sqrt{u}\right\}$. But by assumption, Ω has removable singularities at $\lambda = \pm\sqrt{u}$, and is bounded as $\lambda\to\infty$. Hence by Liouville's Theorem Ω is constant. □

Proposition 2.8. Cauchy Integral Representation of the Action

The action can be expressed by $\omega(\lambda,t)$ via the following Cauchy-type integral formula:

$$S(x,\lambda,t) = 4\lambda^3 t + x\left(\lambda^2 - u(x,t)\right)^{1/2} + \frac{1}{\pi\mathrm{i}}\int_{-\sqrt{u(x,t)}}^{\sqrt{u(x,t)}} \frac{\omega(\zeta,t)}{\zeta-\lambda}\frac{\left(u(x,t)-\lambda^2\right)^{1/2}}{\sqrt{u-\zeta^2}}\,\mathrm{d}\zeta \tag{2.25}$$

Proof. We apply the previous proposition to our problem. It remains to analyse the limits $\lambda\to\pm\sqrt{u}$ and $|\lambda|\to\infty$ in order to determine $H(\lambda)$.

In the limits $\lambda,\xi\to\pm\sqrt{u}$, we note that $\omega(\xi)$ is bounded. Therefore by Lemma 7.2.2 of Ablowitz & Fokas [3] the integral

$$\frac{1}{\pi\mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\omega(\zeta)}{\sqrt{u-\zeta^2}(\zeta-z)}\,\mathrm{d}\zeta = \mathrm{O}\left((\xi\pm\sqrt{u})^{-1/2}\right)$$

But since $\sqrt{u-\lambda^2} = \mathrm{O}\left((\lambda\pm\sqrt{u})^{1/2}\right)$, the boundedness of the solution at the endpoints is assured. As $|\lambda|\to\infty$, we have

$$\Sigma(\lambda) = \lambda x + \mathrm{O}(\lambda^{-1})$$

Then we have $H(\lambda) = \mathrm{O}(1)$ as $|\lambda|\to\infty$. Therefore H is in fact entire and bounded, and hence is constant by Liouville's Theorem. We have

$$H(\lambda) = x\mathrm{i}$$

Then $\Sigma(x,\lambda,t)$ is given by

$$\Sigma(x,\lambda,t) = x\left(\lambda^2 - u(x,t)\right)^{1/2} + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u(x,t)}}^{\sqrt{u(x,t)}} \frac{\omega(\zeta,t)}{\zeta-\lambda}\frac{\left(u(x,t)-\lambda^2\right)^{1/2}}{\sqrt{u-\zeta^2}}\,\mathrm{d}\zeta$$

The result follows. □

Theorem 2.9. Tsarëv's hodograph Formula

The left half of solution $u(x,t)$ to the initial value problem of the dKdV equation is generated by the implicit equation:

$$x + 6ut = \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\mathrm{d}\omega_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}} \tag{2.26}$$

Proof. From (2.11) we have

$$\mu(x,\lambda,t) := \frac{\partial S}{\partial \lambda} = x + 12\lambda^2 t + \int_{-\infty}^{x}\left(\frac{\lambda}{\left(\lambda^2 - u(s,t)\right)^{1/2}} - 1\right)\mathrm{d}s \tag{2.27}$$

Since the integral

$$\int_{-\infty}^{x}\left(\sqrt{\frac{u(x,t)}{u(x,t)-u(s,t)}} - 1\right)\mathrm{d}s < \infty$$

μ is bounded at $\lambda = \pm\sqrt{u(x,t)}$. From (2.25) we find that

$$\mu(x,\lambda,t) = 12\lambda^2 t + \frac{\lambda}{\sqrt{\lambda^2-u}}\left(x + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\omega(\zeta,t)}{\sqrt{\zeta^2-u}(\zeta-\lambda)}\,\mathrm{d}\zeta\right) + \sqrt{\lambda^2-u}\frac{\partial}{\partial\lambda}\left(x + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\omega(\zeta,t)}{\sqrt{\zeta^2-u}(\zeta-\lambda)}\,\mathrm{d}\zeta\right)$$

For μ to be bounded at $\lambda = \pm\sqrt{u(x,t)}$, we must have

$$x + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\omega(\zeta,t)}{\sqrt{\zeta^2-u}(\zeta-\sqrt{u})}\,\mathrm{d}\zeta = 0, \qquad x + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\omega(\zeta,t)}{\sqrt{\zeta^2-u}(\zeta+\sqrt{u})}\,\mathrm{d}\zeta = 0$$

We can add these two equations up and do integration by parts. Note that the integral in fact diverges and we can only calculate its principal value.

$$\begin{aligned}
0 &= x + \frac{1}{\pi \mathrm{i}}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\lambda\omega(\lambda,t)}{(\lambda^2-u)^{3/2}}\,\mathrm{d}\lambda = x + \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\lambda\omega(\lambda,t)}{(u-\lambda^2)^{3/2}}\,\mathrm{d}\lambda \\
&= x + \frac{1}{\pi}\left(\frac{\omega(\lambda,t)}{\sqrt{u-\lambda^2}}\bigg|_{\lambda=-\sqrt{u}}^{\sqrt{u}} - \int_{-\sqrt{u}}^{\sqrt{u}}\frac{\partial\omega}{\partial\lambda}\frac{1}{\sqrt{u-\lambda^2}}\,\mathrm{d}\lambda\right) \\
&= x - \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\partial\omega}{\partial\lambda}\frac{1}{\sqrt{u-\lambda^2}}\,\mathrm{d}\lambda
\end{aligned} \tag{2.28}$$

We substitute the time dependence (2.2) into the above integral, and use the formula

$$\int_{-\sqrt{u}}^{\sqrt{u}}\frac{\lambda^2}{\sqrt{u-\lambda^2}}\,\mathrm{d}\lambda = \frac{\pi}{2}u$$

We obtain the Tsarëv's hodograph formula for dKdV:

$$x + 6ut = \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\mathrm{d}\omega_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}}$$ □

Given the initial curve u_0, the RHS is a well-defined function of u. We need to compare it with the result obtained from the method of characteristics and determine the range of x of which the hodograph formula is valid.

Remark. We remark that $\omega_0(\lambda)$ depends on $x_-(\lambda)$, which only uses the information of u_0 in the range $x < x_m$. So we do not expect (2.26) to be valid for $x > x_m$. This is because we are doing a left-to-right scattering. If we instead consider the right-to-left scattering, we will have a $\sigma_0(\lambda)$ instead in the hodograph formula:

$$x + 6ut = -\frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\mathrm{d}\sigma_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}}$$

$\sigma_0(\lambda)$ contains the information of u_0 for $x > x_m$. The two patches of the solution correspond to the two regions in which $u_0(x)$ is invertible.

2.4 Verification of Hodograph Formula

Let v_0 be the inverse of u_0 when $x < x_m$. By comparing (1.8) and (2.26), we note that

$$w(u) := \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{\mathrm{d}\omega_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}} = \frac{2}{\pi}\int_{0}^{\sqrt{u}} \frac{\mathrm{d}\omega_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}} \tag{2.29}$$

should be equal to $v_0(u)$. We present a direct proof of this by evaluating the integral explicitly.

By definition $x_-(\lambda, 0) = v_0(\lambda^2)$. From (2.12) we have

$$\omega_0(\lambda) = \lambda v_0(\lambda^2) + \int_{-\infty}^{v_0(\lambda^2)} \left(\sqrt{\lambda^2 - u_0(s)} - \lambda\right)\mathrm{d}s \tag{2.30}$$

Taking derivative:

$$\begin{aligned}
\frac{\mathrm{d}\omega_0}{\mathrm{d}\lambda} &= v_0(\lambda^2) + 2\lambda^2 v_0'(\lambda^2) + \int_{-\infty}^{v_0(\lambda^2)} \frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-u_0(s)}-\lambda\right)\mathrm{d}s + 2\lambda v_0'(\lambda^2)\left(\sqrt{\lambda^2-u_0(v_0(\lambda^2))}-\lambda\right)\\
&= v_0(\lambda^2) + \int_{-\infty}^{v_0(\lambda^2)}\left(\frac{\lambda}{\sqrt{\lambda^2-u_0(s)}}-1\right)\mathrm{d}s\\
&= \lim_{\delta\to-\infty}\left(\int_{\delta}^{v_0(\lambda^2)} \frac{\lambda}{\sqrt{\lambda^2-u_0(s)}}\mathrm{d}s + \delta\right)
\end{aligned}$$

Substituting the expression above into $w(u)$:

$$\begin{aligned}
\frac{\pi}{2}w(u) &= \int_0^{\sqrt{u}} \lim_{\delta\to-\infty}\left(\int_{\delta}^{v_0(\lambda^2)} f(\lambda,s)\,\mathrm{d}s + \frac{\delta}{\sqrt{u-\lambda^2}}\right)\mathrm{d}\lambda\\
&= \lim_{\delta\to-\infty}\left(\int_0^{\sqrt{u}}\int_{\delta}^{v_0(\lambda^2)} f(\lambda,s)\,\mathrm{d}s\,\mathrm{d}\lambda + \delta\int_0^{\sqrt{u}}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}}\right) \qquad \text{(MCT)}\\
&= \lim_{\delta\to-\infty}\left(\int_{\delta}^{v_0(u)}\int_{\sqrt{u_0(s)}}^{\sqrt{u}} f(\lambda,s)\,\mathrm{d}\lambda\,\mathrm{d}s - \int_{-\infty}^{\delta}\int_0^{\sqrt{u_0(s)}} f(\lambda,s)\,\mathrm{d}\lambda\,\mathrm{d}s + \delta\int_0^{\sqrt{u}}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}}\right)
\end{aligned}$$

where $f(\lambda,s) := \dfrac{\lambda}{\sqrt{(u-\lambda^2)(\lambda^2-u_0(s))}}$. From elementary calculus we know that

$$\int_0^{\sqrt{u}}\frac{\mathrm{d}\lambda}{\sqrt{u-\lambda^2}} = \frac{\pi}{2}, \qquad \int_{\sqrt{u_0(s)}}^{\sqrt{u}} \frac{\lambda\mathrm{d}\lambda}{\sqrt{(u-\lambda^2)(\lambda^2-u_0(s))}} = \frac{1}{2}\int_{u_0(s)}^{u}\frac{\mathrm{d}\xi}{\sqrt{(u-\xi)(\xi-u_0(s))}} = \frac{\pi}{2}$$

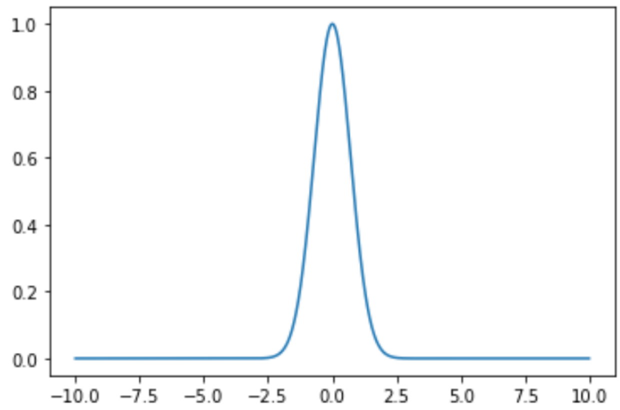
Therefore we have

$$\begin{aligned}
\frac{\pi}{2}w(u) &= \lim_{\delta\to-\infty}\left(\frac{\pi}{2}(v_0(u)-\delta) - \int_{-\infty}^{\delta}\int_0^{\sqrt{u_0(s)}} f(\lambda,s)\,\mathrm{d}\lambda\,\mathrm{d}s + \frac{\pi}{2}\delta\right)\\
&= \frac{\pi}{2}v_0(u) + \lim_{\delta\to-\infty}\int_{-\infty}^{\delta}\int_0^{\sqrt{u_0(s)}} f(\lambda,s)\,\mathrm{d}\lambda\,\mathrm{d}s
\end{aligned}$$

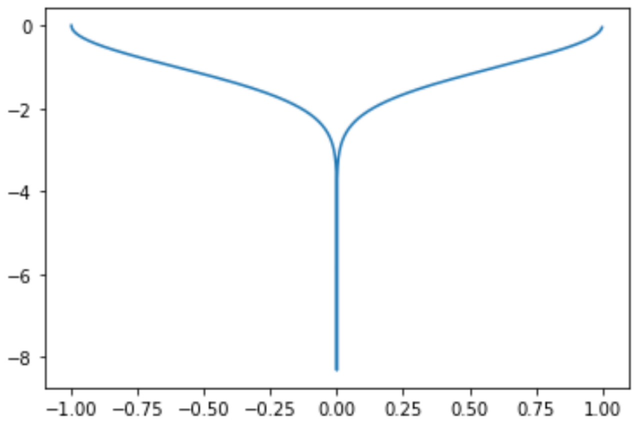
$$= \frac{\pi}{2} v_0(u)$$

Hence we have $w(u) = v_0(u)$ as claimed.

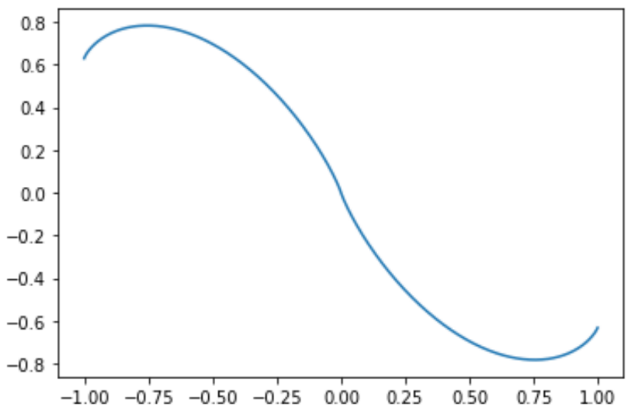
We perform a numerical simulation to compare v_0 and w with the test initial curve $u_0(x) = e^{-x^2}$. The plots are shown below.



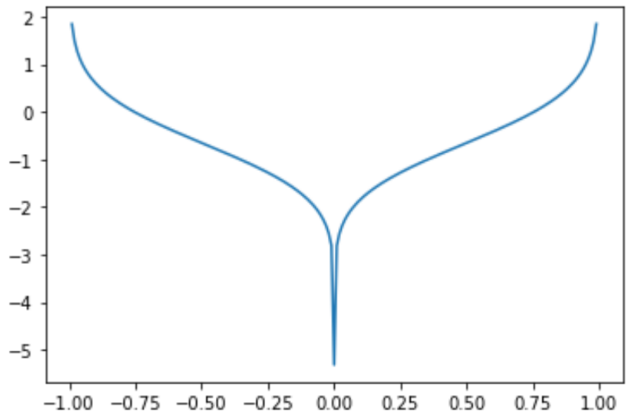
(a) Plot of $u_0(x) = \exp(-x^2)$



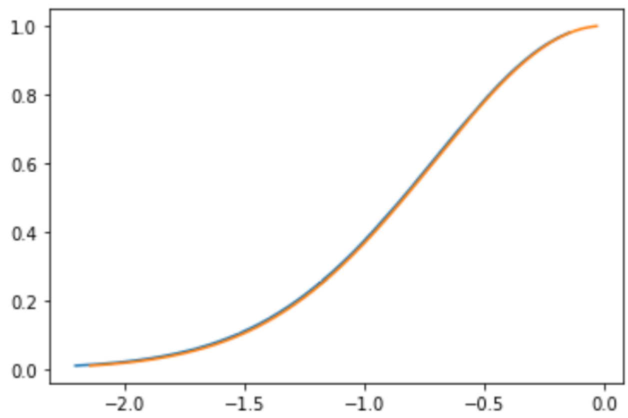
(b) Plot of $x_-(\lambda)$



(c) Plot of $\omega_0(\lambda)$



(d) Plot of $\frac{d\omega_0}{d\lambda}(\lambda)$



(e) Plot of $u_0(x)$ (orange line) and $w^{-1}(x)$ (blue line)

References

[1] Mason, L., and Stone, W. *Inverse Scattering for the Schrödinger Equation in the WKB Approximation.* Draft. 2019.

[2] Ablowitz, M. J., and Clarkson, P. A. *Solitons, nonlinear evolution equations and inverse scattering.* Vol. 149. Cambridge University Press, 1991.

[3] Ablowitz, M. J., and Fokas, A. S. *Complex variables: introduction and applications.* Cambridge University Press, 2003.

[4] Bender, C. M., and Orszag, S. A. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[5] Geogjaev, V. V. "The quasiclassical limit of the inverse scattering problem method." *Singular Limits of Dispersive Waves.* Springer, Boston, MA, 1994. 53-59.

[6] Gibbons, J., and Kodama, Y. "Solving dispersionless Lax equations." *Singular limits of dispersive waves.* Springer, Boston, MA, 1994. 61-66.

[7] Tsarëv, S. P. "The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method." *Mathematics of the USSR-Izvestiya* 37.2 (1991): 397.

[8] Wheeler, M. *An Introduction To Riemann-Hilbert Problems And Their Applications.* Diss. PhD thesis, The University of Melbourne, Department of Mathematics and Statistics, 2005.

[9] Dunajski, M. *Solitons, instantons, and twistors.* Vol. 19. Oxford University Press, 2010.