Path Integral Quantisation of Bosonic Strings

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Abstract

We summarise the path integral quantisation of the Polyakov action for the bosonic strings. The gauge symmetries of the string worldsheet are fixed by the Faddeev–Popov procedure which introduces the unphysical ghost fields. We discuss the covariant quantisation and the conformal field theory of the ghost fields, from which we derive the Weyl anomaly and show that it is cancelled for the critical spacetime dimension $D = 26$. Then we outline the BRST quantisation scheme which identifies the physical Hilbert space as the cohomology class of the BRST charge. We also prove its equivalence to the old covariant quantisation.

Contents

Conventions and Notations

Throughtout the report, we adopt the natural units $c = \hbar = 1$. We adopt the Einstein summation convention, where the greek letters $\mu, \nu, ...$ are summed from 0 to D, and the latin letters $a, b, i, j, ...$ are summed from 1 to 2. In §3.1 only the letters $i, j, k, ...$ are summed from 1 to $\dim \mathfrak{g}$.

The background spacetime is the D-dimensional Minkowski spacetime $(\mathbb{R}^D, \eta)$, where $(\eta_{\mu\nu}) = \mathrm{diag}(-1, +1, \ldots, +1)$. The spacetime coordinates are denoted by $X = (X^\mu)$, with μ going from 0 to $D - 1$. The coordinates on the string worldsheet is denoted by $(\xi^a) = (\xi^1, \xi^2) = (\sigma, \tau)$. The worldsheet metric is denoted by $\gamma = \gamma_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$. Further conventions of complex coordinates on the worldsheet are described in §2.2. In particular, the complex δ-function is normalised as $\delta^2(z, \overline{z}) = \delta(x)\delta(y)/2$. For an operator A on a Hilbert space, we use $A^\dagger$ for its adjoint.

Introdution

Bosonic string theory was developed in the late 1960s, initially as a candidate for the theory of strong interactions. Now it is considered as a prototype of the more general string theory, which is believed by many to be the theory of quantum gravity. Bosonic string theory is not a realistic model, as it contains only bosons in its spectrum, and suffers from the presence of tachyons. However, it provides a framework of string theory and leads to superstring theory after incorporating the supersymmetry. There are three equivalent models for the quantum bosonic strings: old covariant, lightcone, and path integral quantisation. In this report, we investigate the path integral quantisation of bosonic strings in a flat background spacetime.

The path integral quantisation of bosonic strings follows the broader scheme of path integral quantisation of gauge theories, which is studied in quantum field theory. In 1967, Faddeev & Popov [1] developed a gauge fixing procedure for the path integral quantisation of the Yang–Mills theory by introducing the ghost fields, now known as the Faddeev–Popov ghosts. In 1981, Polyakov [2] adopted the procedure to give a path integral quantisation of the bosonic strings. We discuss this method in Section 1, following the treatment in Chapter 3 of Polchinski [3].

The novelty of bosonic strings in the path integral quantisation scheme is the Weyl symmetry of the Polyakov action. This leads to the two-dimensional conformal field theory, which is studied in Belavin, Polyakov & Zamolodchikov [4]. We briefly review some results of conformal field theory in §2.2 and §2.3, including the operator product expansion, which is used to compute the central charge of the string system. The techniques can be found in Chapter 2 of Polchinski [3] and Chapter 4 of Tong [5]. After quantisation, the Weyl invariance of the Polyakov action may be anomalous. We prove in §2.4 that the Weyl anomaly is cancelled if and only if the total central charge of the system with ghosts vanishes. This leads to the critical spacetime dimension $D = 26$ as a consistency condition, which agrees with the results from the old covariant and the lightcone quantisation. In §3.2, we also show that this is necessary for the BRST operator to be nilpotent.

In 1975–1976, Becchi, Rouet, & Stora [6] and Tyutin [7] discovered the BRST symmetry for the non-Abelian gauge theory. In 1979, Kugo & Ojima [8] formulated the BRST operator and associated the physical observables with the BRST cohomology. Kato & Ogawa [9] constructed the BRST operator for bosonic string theory and proved a version of the no ghost theorem. In Section 3 we outline the BRST quantisation scheme and its application to bosonic strings, following Chapter 4 of Polchinski [3] and Section 3.2 of Green, Schwarz & Witten [10]. This improved method does not require a manual choice of gauge, but rather encodes the gauge symmetry in the Lie algebra cohomology. We discuss the homological algebra background in §3.1. The physical Hilbert space of bosonic strings is constructed in §3.3 by BRST quantisation, which is equivalent to the Hilbert space in old covariant quantisation. Finally we will quote the no ghost theorem, referring the proof to Section 4.4 of Polchinski [3]. A more mathematical treatment can be found in Figueroa-O'Farrill [11].

1 Faddeev–Popov Procedure

1.1 Polyakov Path Integral

We begin by recalling the classical theory of bosonic strings. Let Σ be the string worldsheet parametrised by $(\xi^a) = (\sigma, \tau)$. The **Polyakov action** of a bosonic string is given by

$$S_{\mathrm{P}}[\gamma, X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{-\det\gamma}\, \gamma^{ab} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu}, \tag{1.1}$$

where α' is a constant called the Regge slope, $\gamma = \gamma_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$ is a metric on the worldsheet, and (X^μ) are the spacetime coordinates. The Polyakov action (1.1) has the following symmetries:

- Global Poincaré symmetry.

$$X^\mu(\sigma,\tau) \longmapsto \Gamma^\mu{}_\nu X^\nu(\sigma,\tau) + c^\mu; \qquad \gamma_{ab}(\sigma,\tau) \longmapsto \gamma_{ab}(\sigma,\tau).$$

- Local diffeomorphism $(\sigma,\tau) \longmapsto (\widetilde{\sigma},\widetilde{\tau})$.

$$X^\mu(\sigma,\tau) \longmapsto \widetilde{X}(\widetilde{\sigma},\widetilde{\tau}) = X(\sigma,\tau); \qquad \gamma_{ab}(\sigma,\tau) \longmapsto \widetilde{\gamma}_{ab}(\widetilde{\sigma},\widetilde{\tau}) = \frac{\partial\xi^c}{\partial\widetilde{\xi}^a}\frac{\partial\xi^d}{\partial\widetilde{\xi}^b}\gamma_{cd}(\sigma,\tau).$$

- Weyl transformation.

$$X^\mu(\sigma,\tau) \longmapsto X^\mu(\sigma,\tau); \qquad \gamma_{ab}(\sigma,\tau) \longmapsto \mathrm{e}^{2\omega(\sigma,\tau)}\,\gamma_{ab}(\sigma,\tau).$$

By Noether's Theorem, the stress–energy tensor of the string worldsheet is given by

$$T^{ab} = \frac{4\pi}{\sqrt{-\det\gamma}}\frac{\partial S_{\mathrm{P}}}{\partial\gamma_{ab}}. \tag{1.2}$$

where the choice of normalisation constant agrees with Blumenhagen, Lüst & Theisen [12].

Lemma 1.1

The stress-energy tensor is traceless: $T^a{}_a = 0$.

Proof. Consider a variation of the metric by a Weyl transformation $\delta\gamma_{ab} = 2\omega\gamma_{ab}$. This is a symmetry of the action, so $\delta S_{\mathrm{P}} = 0$. Therefore we have

$$0 = \frac{\partial S_{\mathrm{P}}}{\partial\gamma_{ab}}\delta\gamma_{ab} = -\frac{\omega}{2\pi\alpha'}\sqrt{-\det\gamma}\,T^{ab}\gamma_{ab},$$

which implies that $T^a{}_a = T^{ab}\gamma_{ab} = 0$. □

The covariant path integral quantisation of the bosonic strings begins with considering the partition function associated to the Polyakov action.

$$Z = \int \mathscr{D}[\gamma]\,\mathscr{D}[X]\,\mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\gamma,X]}. \tag{1.3}$$

There are redundancies in the metric γ, because different metrics may be related by a gauge transformation and hence correspond to the same physical configuration. Therefore we would like to integrate over a space quotient by the gauge symmetries. Let G be the gauge group of thw worldsheet. We would like to express the Polyakov path integral in the form

$$Z = \int_G \mathrm{d}\zeta\, Z_{\mathrm{phys}}[\widehat{\gamma}], \tag{1.4}$$

where $Z_{\mathrm{phys}}[\widehat{\gamma}]$ fixes the worldsheet metric $\widehat{\gamma}$ and corresponds to the path integral over all physically distinct configurations.

1.2 Faddeev–Popov Ghosts

A gauge transformation $\zeta \in G$ of the worldsheet is a composition of a local diffeomorphism and a Weyl transformation. So

$$\zeta\colon \gamma_{ab}(\xi) \longmapsto \zeta(\gamma)_{ab}(\widetilde{\xi}) = \mathrm{e}^{2\phi(\xi)} \frac{\partial \xi^c}{\partial \widetilde{\xi}^a} \frac{\partial \xi^d}{\partial \widetilde{\xi}^b} \gamma_{cd}(\xi).$$

Let $\mathfrak{g}$ be the Lie algebra of G. An infinitesimal gauge transformation $(\omega, v) \in \mathfrak{g}$ is parametrised by a conformal factor ω and a vector field v^a:

$$\xi^a \longmapsto \xi^a + v^a; \qquad \widehat{\gamma}_{ab} \longmapsto \widehat{\gamma}_{ab} + 2\omega\widehat{\gamma}_{ab} + \widehat{\nabla}_a v_b + \widehat{\nabla}_b v_a.$$

We fix a worldsheet metric $\widehat{\gamma}$, called the **fiducial metric**. The **Faddeev–Popov determinant** $\Delta_{\mathrm{FP}}(\gamma)$ is a functional determinant defined by the following identity:

$$1 = \Delta_{\mathrm{FP}}(\gamma) \int_G \mathrm{d}\zeta\, \delta(\gamma - \zeta(\widehat{\gamma})),$$

where $\mathrm{d}\zeta$ is the Haar measure of the gauge group G and is gauge invariant.

Lemma 1.2

The Faddeev–Popov determinant $\Delta_{\mathrm{FP}}(\gamma)$ is gauge invariant.

Proof. This follows directly from the definition and the gauge invariance of the measure $\mathrm{d}\zeta$:

$$\begin{aligned}\Delta_{\mathrm{FP}}(\zeta(\gamma))^{-1} &= \int_G \mathrm{d}\zeta'\, \delta(\zeta(\gamma) - \zeta'(\widehat{\gamma})) = \int_G \mathrm{d}\zeta'\, \delta\big(\gamma - \zeta^{-1}\zeta'(\widehat{\gamma})\big) \\ &= \int_G \mathrm{d}(\zeta^{-1}\zeta')\, \delta\big(\gamma - \zeta^{-1}\zeta'(\widehat{\gamma})\big) = \Delta_{\mathrm{FP}}(\gamma)^{-1}.\end{aligned}$$

□

Now we insert the identity (1.2) into the Polyakov path integral (1.3):

$$\begin{aligned}Z &= \int_G \mathrm{d}\zeta \int \mathscr{D}[\gamma]\, \mathscr{D}[X]\, \Delta_{\mathrm{FP}}(\gamma)\delta(\gamma - \zeta(\widehat{\gamma}))\, \mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\gamma, X]} \\ &= \int_G \mathrm{d}\zeta \int \mathscr{D}[X]\, \Delta_{\mathrm{FP}}(\zeta(\widehat{\gamma}))\, \mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\zeta(\widehat{\gamma}), X]} \\ &= \int_G \mathrm{d}\zeta \int \mathscr{D}[\zeta(X)]\, \Delta_{\mathrm{FP}}(\zeta(\widehat{\gamma}))\, \mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\zeta(\widehat{\gamma}), \zeta(X)]} \\ &= \int_G \mathrm{d}\zeta \int \mathscr{D}[X]\, \Delta_{\mathrm{FP}}(\widehat{\gamma})\, \mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\widehat{\gamma}, X]}\,.\end{aligned} \tag{1.5}$$

where, in the fourth line, we used the gauge invariance of the Faddeev–Popov determinant. Note that the integrand of (1.5) is independent of the gauge transformation ζ. Comparing (1.5) with (1.4) we obtain the physical partition function:

$$Z_{\mathrm{phys}}[\widehat{\gamma}] = \int \mathscr{D}[X]\, \Delta_{\mathrm{FP}}(\widehat{\gamma})\, \mathrm{e}^{\mathrm{i}S_{\mathrm{P}}[\widehat{\gamma}, X]}\,. \tag{1.6}$$

From now on we drop the subscript and regard (1.6) as the Polyakov path integral we need. Next we would like to evaluate the Faddeev–Popov determinant.

Proposition 1.3. Ghost Fields

With some extra assumption on the topology of the worldsheet, the Faddeev–Popov determinant can be expressed as a *formal* path integral:

$$\Delta_{\mathrm{FP}}(\widehat{\gamma}) = \int \mathscr{D}[b]\, \mathscr{D}[c]\ \mathrm{e}^{\mathrm{i}S_{\mathrm{gh}}[\widehat{\gamma},b,c]}, \tag{1.7}$$

where $b = (b_{ij})$ and $c = (c^i)$ are Grassmann-valued fields, and b is symmetric and traceless, called the **ghost fields**, and $S_{\mathrm{gh}}[\widehat{\gamma}, b, c]$ is the ghost action, given by

$$S_{\mathrm{gh}}[\widehat{\gamma}, b, c] = -\frac{\mathrm{i}}{2\pi} \int_\Sigma \mathrm{d}\sigma\, \mathrm{d}\tau\, \sqrt{-\det \widehat{\gamma}}\, \widehat{\gamma}^{ij} b_{jk} \widehat{\nabla}_i c^k, \tag{1.8}$$

where $\widehat{\nabla}$ is the Levi-Civita connection associated with the fiducial metric $\widehat{\gamma}$.

Proof. We adapt the proof from §3.3 of Polchinski [3].

Let $(\omega, v) \in \mathfrak{g}$ be an infinitesimal gauge transformation:

$$\widehat{\gamma}_{ab} \longmapsto \widehat{\gamma}_{ab} + 2\omega\widehat{\gamma}_{ab} + \widehat{\nabla}_a v_b + \widehat{\nabla}_b v_a = \widehat{\gamma}_{ab} + (2\omega + \nabla_c v^c)\widehat{\gamma}_{ab} + (Pv)_{ab},$$

where

$$(Pv)_{ab} := \widehat{\nabla}_a v_b + \widehat{\nabla}_b v_a - \nabla_c v^c \widehat{\gamma}_{ab}.$$

So P is a operator mapping covector fields to symmetric traceless type-(0,2) tensor fields.

Here we need the assumption that the gauge group G acts on the space of worldsheet metric *faithfully*, in other words, $\widehat{\gamma} = \zeta(\widehat{\gamma})$ if and only if $\zeta = \mathrm{id}$. As pointed out in Polchinski [3], this is always true locally but not globally if the worldsheet has non-trivial topology. Under this assumption, we evaluate (1.2) near the identity, where we can replace the integral over the gauge group G by an integral over the Lie algebra $\mathfrak{g}$:

$$\Delta_{\mathrm{FP}}(\widehat{\gamma})^{-1} = \int_{\mathfrak{g}} \mathscr{D}[\omega]\, \mathscr{D}[v]\, \delta((2\omega + \nabla_c v^c)\widehat{\gamma}_{ab} + (Pv)_{ab}).$$

We can express the delta functional in the integrand as a path integral:

$$\delta((2\omega+\nabla_c v^c)\widehat{\gamma}_{ab}+(Pv)_{ab}) = \int \mathscr{D}[\beta] \exp\left(2\pi\mathrm{i} \int_\Sigma \mathrm{d}\sigma\, \mathrm{d}\tau\, \sqrt{-\det \widehat{\gamma}}\, \beta^{ab} \left((2\omega + \nabla_c v^c)\widehat{\gamma}_{ab} + (Pv)_{ab}\right)\right),$$

where β is a symmetric type-$(2,0)$ tensor field. Combining the above two equations and perform the integral over the conformal factor $\mathscr{D}[\omega]$, we assume that the dependence will drop out. Formally we have

$$\int \mathscr{D}[\omega] = \mathbf{1}_{\{\beta^{ab}\widehat{\gamma}_{ab}=0\}},$$

which is equivalent to imposing the constraint $\beta^{ab}\widehat{\gamma}_{ab} = 0$, or, β is symmetric and traceless. We have

$$\Delta_{\mathrm{FP}}(\widehat{\gamma})^{-1} = \int_{\{\mathrm{tr}\,\beta=0\}} \mathscr{D}[v]\mathscr{D}[\beta] \exp\left(2\pi\mathrm{i} \int_\Sigma \mathrm{d}\sigma\, \mathrm{d}\tau\, \sqrt{-\det \widehat{\gamma}}\, \beta^{ab}(Pv)_{ab}\right). \tag{1.9}$$

The equation expresses $\Delta_{\mathrm{FP}}(\widehat{\gamma})^{-1}$ as the functional determinant $\det P$. Now by the algebraic properties of Grassmann numbers and Berezin integrals (see Appendix A.2 of Polchinski [3] or §9.5 of Peskin & Schroeder [13]), we can invert (1.9) by replacing β_{ab} and

v^a by the Grassmann-valued fields b_{ab} and c^a. We have

$$\Delta_{\mathrm{FP}}(\widehat{\gamma}) = \int \mathscr{D}[b]\mathscr{D}[c] \exp\left(\frac{1}{2\pi}\int_\Sigma \mathrm{d}\sigma\,\mathrm{d}\tau\,\sqrt{-\det\widehat{\gamma}}\,b_{ij}\widehat{\nabla}^i c^j\right). \qquad \square$$

In light of Proposition 1.3, the Polyakov path integral is given by

$$Z[\widehat{\gamma}] = \int \mathscr{D}[X]\,\mathscr{D}[b]\,\mathscr{D}[c]\,\exp(\mathrm{i}(S_{\mathrm{P}}[\widehat{\gamma}, X] + S_{\mathrm{gh}}[\widehat{\gamma}, b, c])). \tag{1.10}$$

The Faddeev–Popov procedure fix the gauge symmetries of the worldsheet at the expense of introducing the unphysical ghost fields b and c.

Before quantising the ghost fields, we note the following problems in the Faddeev–Popov procedure:

- We assumed that the topology of the worldsheet has the topology such that the gauge group acts on it faithfully.
- We neglected possible Weyl anomaly and drop the integral over the conformal factor. We will discuss this in §2.4.
- We neglected the possible terms in the path integral arising from the global topology (the Euler characteristic $\chi(\Sigma)$, see §3.2 of Polchinski [3]) and boundary conditions.

2 Quantisation of Ghost Fields

2.1 Covariant Quantisation

Now we take a closer look at the ghost action (1.8). We fix the fiducial worldsheet metric to the conformal gauge $\widehat{\gamma}_{ab} = \mathrm{e}^{2\omega}\,\eta_{ab}$ and use the complex coordinates (2.9) and the conventions in §2.2:

$$\widehat{\gamma} = \mathrm{e}^{2\omega}\,\mathrm{d}z \otimes \mathrm{d}\overline{z}.$$

Since b is symmetric and traceless, the only nonzero components are b_{zz} and $b_{\overline{z}\overline{z}}$. Note that the only nonvanishing Christoffel symbols are Γ^z_{zz} and $\Gamma^{\overline{z}}_{\overline{z}\overline{z}}$. Hence $\nabla_{\overline{z}} c^z = \partial_{\overline{z}} c^z$ and $\nabla_z c^{\overline{z}} = \partial_z c^{\overline{z}}$. We have

$$\widehat{\gamma}^{ij} b_{jk} \nabla_i c^k = \mathrm{e}^{-2\omega}(b_{zz}\partial_{\overline{z}} c^z + b_{\overline{z}\overline{z}}\partial_z c^{\overline{z}}).$$

The ghost action (1.8) becomes

$$S_{\mathrm{gh}}[\mathrm{e}^{2\omega}\,\eta, b, c] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2 z\,(b_{zz}\partial_{\overline{z}} c^z + b_{\overline{z}\overline{z}}\partial_z c^{\overline{z}}). \tag{2.1}$$

We see that the ghost action decouples into the sum of two free fields, and is independent of the conformal factor ω. This verifies the classical Weyl invariance of S_{gh}. The non-zero stress-energy tensor components are given by

$$T_{zz} = 2(\partial_z c^z) b_{zz} + c^z \partial_z b_{zz}; \qquad T_{\overline{z}\overline{z}} = 2(\partial_{\overline{z}} c^{\overline{z}}) b_{\overline{z}\overline{z}} + c^{\overline{z}}\partial_{\overline{z}} b_{\overline{z}\overline{z}}. \tag{2.2}$$

The classical equations of motions are given by

$$\partial_{\overline{z}} b_{zz} = \partial_z b_{\overline{z}\overline{z}} = 0, \qquad \partial_{\overline{z}} c^z = \partial_z c^{\overline{z}} = 0,$$

which are supplemented by some boundary conditions. The canoncial quantisation of the ghost fields is done by imposing the anti-commutation relations:

$$\{b_{zz}(\sigma,\tau), c^z(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma'); \qquad \{b_{\overline{zz}}(\sigma,\tau), c^{\overline{z}}(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma').$$

Then we can follow the standard procedure by imposing constraints, considering the mode expansions of the ghosts, and extracting the Virasoro generators L_m^{gh} from the mode expansions of the stress-energy tensor. For example, for closed strings the periodic boundary conditions imply the mode expansion:

$$b_{zz}(\sigma,\tau) = \sum_{n\in\mathbb{Z}} b_n \, \mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}, \qquad b_{\overline{zz}}(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \overline{b}_n \, \mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}, \tag{2.3}$$

$$c^z(\sigma,\tau) = \sum_{n\in\mathbb{Z}} c_n \, \mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}, \qquad c^{\overline{z}}(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \overline{c}_n \, \mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}, \tag{2.4}$$

where $c_n = c_{-n}^\dagger$ and $b_n = b_{-n}^\dagger$ by the self-adjointness of b and c. The Fourier coefficients satisfy the anti-commutation relations:

$$\{b_m, c_n\} = \delta_{m,-n}; \qquad \{b_m, b_n\} = \{c_m, c_n\} = 0.$$

The ghost Virasoro generators are given by

$$L_m^{\mathrm{gh}} = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\sigma \; \mathrm{e}^{\mathrm{i}m\sigma}\, T_{zz}\Big|_{\tau=0} = \sum_{n\in\mathbb{Z}} (m-n) \; {:}b_{m+n}c_{-n}{:}$$

where the normal ordering moves the modes c_n, b_n with $n > 0$ to the right, taking care of the sign change from anticommutivity. We obtain the ghost Virasoro algebra

$$[L_m^{\mathrm{gh}}, L_n^{\mathrm{gh}}] = (m-n)L_{m+n}^{\mathrm{gh}} + \frac{1}{12}(-26m^3 + 2m)\delta_{m,-n}, \tag{2.5}$$

which hints that the ghost system has central charge $c_{\mathrm{gh}} = -26$. We refer to §3.5 of Blumenhagen, Lüst & Theisen [12] for the proof.

Recall that the free scalar fields have the Virasoro algebra

$$[L_m^X, L_n^X] = (m-n)L_{m+n}^X + \frac{D}{12}\left(m^3 - m\right)\delta_{m,-n}. \tag{2.6}$$

The total Virasoro algebra of the system with ghosts is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}\left((D-26)m^3 - (D-2-24a)m\right)\delta_{m,-n}. \tag{2.7}$$

where the total Virasoro generator L_m is given by

$$L_m = L_m^X + L_m^{\mathrm{gh}} - a\delta_{m,0}. \tag{2.8}$$

where a is a normal ordering costant. The Virasoro algebra (2.7) reduces to the Witt algebra $[L_m, L_n] = (m-n)L_{m+n}$ if and only if $a = 1$ and $D = 26$.

In the next section we present another equivalent definition of the central charge through operator product expansion, and prove this result in Proposition 2.4 by analysing the ghost conformal field theory.

2.2 Some Conformal Field Theory

A **conformal field theory** (CFT) is a quantum field theory invariant under the conformal transformation of the metric. The study of 2-dimensional CFT is summarised in the paper [4]. We collect some technical results which will be useful for our analysis of the Weyl anomaly.

It is known that every two-dimensional Lorentzian manifold is conformally flat. In this part we fix the worldsheet metric to the conformal gauge and apply a Weyl transformation to make it flat. By a Wick rotation $\tau \mapsto \tau' := \mathrm{i}\tau$, we change from the $(1+1)$ Minkowski metric to the Euclidean metric. We use the complex coordinates on the worldsheet:

$$z := \sigma + \mathrm{i}\tau'; \qquad \overline{z} := \sigma - \mathrm{i}\tau'. \tag{2.9}$$

This is equivalent the lightcone coordinates, but we are complexifying the worldsheet so that $z, \overline{z}$ are treated as independent variables. The convention of the δ-function is $\delta^2(z,\overline{z}) = \frac{1}{2}\delta(\sigma)\delta(\tau)$. The Wirtinger derivatives are defined by

$$\partial = \partial_z := \frac{1}{2}\left(\frac{\partial}{\partial\sigma} - \mathrm{i}\frac{\partial}{\partial\tau}\right); \qquad \overline{\partial} = \partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial\sigma} + \mathrm{i}\frac{\partial}{\partial\tau}\right)$$

The tracelessness of the classical stress-energy tensor implies that $T_{z\overline{z}} = 0$ in the complex coordinates. For *flat* worldsheet metric, the conservation $\partial_a T^{ab} = 0$ implies that $\overline{\partial} T_{zz} = 0$ and $\partial T_{\overline{z}\overline{z}} = 0$. Hence T_{zz} is holomorphic and $T_{\overline{z}\overline{z}}$ is anti-holomorphic.

Let $\{\mathcal{O}_i\}_{i\in I}$ be a family of local operators of the conformal field theory. An **operator product expansion** (OPE) is an operator identity:

$$\mathcal{O}_i(z,\overline{z})\mathcal{O}_j(w,\overline{w}) = \sum_{k\in I} C_{ij}^k(z-w,\overline{z}-\overline{w})\mathcal{O}_k(w,\overline{w}),$$

which hold as operator equations on a Hilbert space. We are primarily interested in the TT OPE.

Lemma 2.1

The operator product expansion of the stress-energy tensor component T_{zz} takes the form

$$T_{zz}(z)T_{ww}(w) = \frac{c/2}{(z-w)^4} + \frac{2T_{ww}(w)}{(z-w)^2} + \frac{\partial T_{ww}(w)}{z-w} + \text{holomorphic part}, \tag{2.10}$$

where the constant c is called the **central charge** of the conformal field theory.

Proof. The general proof of this fact is given in §4.4 of Tong [5]. However, we will only need to compute the OPE in the special cases of Proposition 2.3 and 2.4, which do not rely on the proof of this lemma. □

Remark. The central charge c defined above is the same central charge c in the corresponding Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}.$$

For the proof see §4.5 of Tong [5] or §2.6 of Polchinski [3].

Lemma 2.2

The operator product expansion of $T_{z\overline{z}}$ is given by

$$T_{z\overline{z}}(z,\overline{z})T_{w\overline{w}}(w,\overline{w}) = \frac{c\pi}{6}\partial_z\overline{\partial}_{\overline{w}}\delta^2(z-w,\overline{z}-\overline{w}). \tag{2.11}$$

Proof. First we note the following identity (which can be proven by integrating both sides):

$$\overline{\partial}_{\overline{z}}\partial_z \ln|z-w|^2 = \overline{\partial}_{\overline{z}}\left(\frac{1}{z-w}\right) = 2\pi\delta^2(z-w,\overline{z}-\overline{w}), \tag{2.12}$$

which implies that

$$\overline{\partial}_{\overline{z}}\overline{\partial}_{\overline{w}}\left(\frac{1}{(z-w)^4}\right) = \frac{1}{6}\overline{\partial}_{\overline{z}}\overline{\partial}_{\overline{w}}\partial_z^2\partial_w\left(\frac{1}{z-w}\right) = \frac{\pi}{3}\partial_z^2\partial_w\overline{\partial}_{\overline{w}}\delta^2(z-w,\overline{z}-\overline{w}). \tag{2.13}$$

We consider the OPE $\partial_z T_{z\overline{z}}\partial_w T_{w\overline{w}}$. By conservation we have $\partial T_{z\overline{z}} = -\overline{\partial}T_{zz}$. Therefore by (2.10) and (2.13) we have

$$\begin{aligned}\partial_z T_{z\overline{z}}(z,\overline{z})\partial_w T_{w\overline{w}} &= \overline{\partial}_{\overline{z}}T_{zz}(z,\overline{z})\overline{\partial}_{\overline{w}}T_{ww}(w,\overline{w}) = \overline{\partial}_{\overline{z}}\overline{\partial}_{\overline{w}}\left(\frac{c/2}{(z-w)^4}+\cdots\right)\\ &= \frac{c\pi}{6}\partial_z^2\partial_w\overline{\partial}_{\overline{w}}\delta^2(z-w,\overline{z}-\overline{w}).\end{aligned}$$

The result follows from cancelling the derivatives $\partial_z\partial_w$ from both sides. □

We compute the central charge of free scalar fields using operator product expansion.

Proposition 2.3. Free Scalar Fields CFT

The system with D free scalar fields (X^μ) has central charge $c_X = D$.

Proof. The Polyakov action in the flat Euclidean metric is given by

$$S = \frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\, \partial X^\mu \overline{\partial} X_\mu.$$

We know that the path integral of a total derivative vanishes. Consider the equation:

$$\begin{aligned}0 &= \int \mathscr{D}[X]\frac{\delta}{\delta X^\mu(z,\overline{z})}\left(\mathrm{e}^{-S}\, X^\nu(w,\overline{w})\right)\\ &= \int \mathscr{D}[X]\,\mathrm{e}^{-S}\left(\frac{1}{\pi\alpha'}\partial\overline{\partial}X^\mu(z,\overline{z})X^\nu(w,\overline{w}) + \eta^{\mu\nu}\delta^2(z-w,\overline{z}-\overline{w})\right).\end{aligned}$$

which, together with (2.12), implies that:

$$\left\langle\partial\overline{\partial}X^\mu(z,\overline{z})X^\nu(w,\overline{w})\right\rangle = \pi\alpha'\eta^{\mu\nu}\delta^2(z-w,\overline{z}-\overline{w}) = \frac{\alpha'}{2}\eta^{\mu\nu}\partial\overline{\partial}\ln|z-w|^2.$$

Integrating the above equation, we obtain the expectation

$$\langle X^\mu(z,\overline{z})X^\nu(w,\overline{w})\rangle = \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{2.14}$$

The classical stress-energy tensor is given by

$$T_{zz} = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu, \qquad T_{\overline{z}\overline{z}} = -\frac{1}{\alpha'}\overline{\partial} X^\mu \overline{\partial} X_\mu, \qquad T_{z\overline{z}} = 0.$$

When quantised, we need to put a **normal ordering** on the fields in the expression of T_{zz}:

$$T_{zz} = -\frac{1}{\alpha'} \; :\partial X^\mu \partial X_\mu: \,,$$

where, consistent with the convention in Polchinski [3], the normal ordering is given by

$$\begin{aligned} :X^\mu(z,\overline{z})X^\nu(w,\overline{w}): &:= X^\mu(z,\overline{z})X^\nu(w,\overline{w}) - \langle X^\mu(z,\overline{z})X^\nu(w,\overline{w})\rangle \\ &= X^\mu(z,\overline{z})X^\nu(w,\overline{w}) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \end{aligned} \tag{2.15}$$

The computation of the TT OPE is an application of the Wick's Theorem (as in the QFT, see §4.3 of Peskin & Schroeder [13]). We suppress the dependence on $\overline{z}$.

$$\begin{aligned} \alpha'^2 T_{zz}(z)T_{zz}(w) &= :\partial_z X^\mu(z)\partial_z X_\mu(z): \, :\partial_w X^\mu(w)\partial_w X_\mu(w): \\ &= :\partial_z X^\mu(z)\partial_z X_\mu(z)\partial_w X^\mu(w)\partial_w X_\mu(w): \\ &\quad - 4\cdot\frac{\alpha'}{2}\partial_z\partial_w \ln|z-w|^2 \; :\partial_z X^\mu(z)\partial_w X_\mu(w): \, + 2\eta^\mu{}_\mu\left(-\frac{\alpha'}{2}\partial_z\partial_w\ln|z-w|^2\right)^2 \\ &= \alpha'^2\left(\frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \mathcal{O}((z-w)^{-1})\right). \end{aligned} \tag{2.16}$$

By comparing (2.16) with (2.10), we find that the free scalar fields have central charge $c_X = D$. □

2.3 Central Charge of Ghost Fields

In this part we compute the central charge of the ghost conformal field theory in a similar manner as above.

Proposition 2.4. Ghost Fields CFT

The bc ghost system described by the ghost action (2.1) has central charge $c_{\text{gh}} = -26$.

Proof. For simplicity, we write $b(z) := b_{zz}$, $\overline{b}(\overline{z}) := b_{\overline{z}\overline{z}}$, $c(z) := c^z$, $\overline{c}(\overline{z}) := c^{\overline{z}}$. The ghost action (2.1) is given by

$$S_{\text{gh}} = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2 z\,\left(b\overline{\partial}c + \overline{b}\partial\overline{c}\right).$$

Consider the path integral of the total derivative

$$0 = \int \mathscr{D}[b]\mathscr{D}[c]\frac{\delta}{\delta b(z)}\left(\mathrm{e}^{-S_{\text{gh}}}\, b(w)\right) = \int \mathscr{D}[b]\mathscr{D}[c]\,\mathrm{e}^{-S_{\text{gh}}}\left(-\frac{1}{2\pi}\overline{\partial}c(z)b(w) + \delta^2(z-w,\overline{z}-\overline{w})\right),$$

which, together with (2.12), implies that:

$$\left\langle\overline{\partial}c(z)b(w)\right\rangle = 2\pi\delta^2(z-w,\overline{z}-\overline{w}) = \overline{\partial}\left(\frac{1}{z-w}\right).$$

By integration we obtain the operator product expansion of the ghosts are given by

$$c(z)b(w) = b(z)c(w) = \frac{1}{z-w} + \cdots,$$

in which an extra minus sign is picked up when exchanging the b and c, since these are Grassmann-valued fields. The quantum stress-energy tensor T_{zz} is obtained by the normal ordering of (2.2):

$$T_{zz} = 2 \ {:}\partial c(z) b(z){:} \ + \ {:}c(z)\partial b(z){:}\,,$$

where

$${:}b(z)c(w){:} = b(z)c(w) - \frac{1}{z-w}.$$

Next we can compute the TT OPE:

$$\begin{aligned} T_{zz}(z)T_{ww}(w) = 4 \ &{:}\partial c(z)b(z){:} \ {:}\partial c(w)b(w){:} \ +2 \ {:}\partial c(z)b(z){:} \ {:}c(w)\partial b(w){:} \\ &+ 2 \ {:}c(z)\partial b(z){:} \ {:}\partial c(w)b(w){:} \ + \ {:}c(z)\partial b(z){:} \ {:}c(w)\partial b(w){:} \ . \end{aligned}$$

The computation again follows from Wick's Theorem. Since we need the central charge, we only need to compute the $(z-w)^{-4}$ terms, which come from the total contractions:

$$\begin{aligned} T_{zz}(z)T_{ww}(w) &= -4 \langle b(z)\partial c(w)\rangle \langle \partial c(z) b(w)\rangle - 2 \langle \partial c(z)\partial b(w)\rangle \langle b(z)c(w)\rangle \\ &\quad - 2 \langle \partial b(z)\partial c(w)\rangle \langle c(z)b(w)\rangle - \langle c(z)\partial b(w)\rangle \langle \partial b(z)c(w)\rangle + \mathcal{O}\big((z-w)^{-2}\big) \\ &= \frac{-4\cdot 1}{(z-w)^4} + \frac{-2\cdot 2}{(z-w)^4} + \frac{-2\cdot 2}{(z-w)^4} + \frac{-1}{(z-w)^4} + \mathcal{O}\big((z-w)^{-2}\big) \\ &= \frac{-13}{(z-w)^4} + \mathcal{O}\big((z-w)^{-2}\big). \end{aligned}$$

Comparing the equation with (2.10), we deduce that the ghost fields have central charge $c_{\text{gh}} = -26$. □

2.4 Weyl Anomaly

We have seen that the classical stress-energy tensor is traceless for a Weyl invariant theory. This property does not survive in the quantum theory in general, which is a phenomenon known as the **Weyl anomaly**. From another perspective, from (1.10) we note that the Polyakov path integral $Z[\widehat{\gamma}]$ *a priori* depends on the choice of the fiducial metric $\widehat{\gamma}$. It can be shown that the variation of $Z[\widehat{\gamma}]$ in the fiducial metric is proportional to the expectation $\langle T^a{}_a\rangle$. For a well-defined quantuam theory, $\langle T^a{}_a\rangle$ must vanish.

Proposition 2.5. Weyl Anomaly

The vaccum expectation of the trace of the stress-energy tensor T satisfies

$$\langle T^a{}_a\rangle = -\frac{c}{12}R \tag{2.17}$$

where c is the central charge of the conformal field theory, and R is the scalar curvature of the string worldsheet.

Proof. We adapt the proof from §4.4.2 of Tong [5].

$\langle T^a{}_a\rangle$ is invariant under diffeomorphisms and Poincaré transformations, so it depends on the geometry of the worldsheet metric. It should also vanish in the flat metric. By dimensional reason, it should be a linear combination of the contractions of some curvature tensors on the worldsheet. But we know that the 2-dimensional Riemann curvature R_{abcd} has exactly one independent component, related to the scalar curvature R:

$$R_{abcd} = \frac{1}{2}R(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}).$$

Therefore $\langle T^a{}_a \rangle = a_1 R$ for some constant a_1. Now we fix the worldsheet to the conformal gauge $\gamma_{ab} = \mathrm{e}^{2\omega}\,\eta_{ab}$. The Christoffel symbols and the scalar curvature are given by

$$\Gamma^c_{ab} = \delta^c_a \partial_b \omega + \delta^c_b \partial_a \omega - \eta_{ab}\eta^{cd}\partial_d \omega; \qquad R = -2\,\mathrm{e}^{-2\omega}\,\eta^{ab}\partial_a\partial_b\omega.$$

Let $S = S[\phi]$ be the action for some field ϕ (the specfic form is unimportant). Consider the variation of the flat metric by an infinitesimal Weyl transformation: $\delta\gamma_{ab} = 2\omega\eta_{ab}$. The variation of $\langle T^a{}_a \rangle$ is given by

$$\begin{aligned}
\delta \langle T^a{}_a(\xi) \rangle &= \delta \int \mathscr{D}[\phi]\,\mathrm{e}^{\mathrm{i}S}\,T^a{}_a(\xi) \\
&= \int \mathscr{D}[\phi]\,\mathrm{e}^{\mathrm{i}S}\,T^a{}_a(\xi) \int_\Sigma \mathrm{d}^2\xi' \sqrt{-\det\gamma}\frac{\partial S}{\partial \gamma_{bc}}\delta\gamma_{bc} \\
&= -\frac{1}{2\pi}\int \mathscr{D}[\phi]\,\mathrm{e}^{\mathrm{i}S}\,T^a{}_a(\xi) \int_\Sigma \mathrm{d}^2\xi'\,\omega(\xi')T^b{}_b(\xi').
\end{aligned}$$

In complex coordinates, we have $T^a{}_a(\xi)T^b{}_b(\xi') = 16T_{z\overline{z}}(z,\overline{z})T_{w\overline{w}}(w,\overline{w})$. By Lemma 2.2, we have

$$T^a{}_a(\xi)T^b{}_b(\xi') = \frac{16c\pi}{6}\partial_z\overline{\partial}_{\overline{w}}\delta^2(z-w,\overline{z}-\overline{w}) = -\frac{c\pi}{3}\partial_a\partial^a\delta^2(\xi-\xi').$$

Therefore

$$\begin{aligned}
\delta \langle T^a{}_a(\xi) \rangle &= \frac{c}{6}\int \mathscr{D}[\phi]\,\mathrm{e}^{\mathrm{i}S}\,\mathrm{d}^2\xi'\,\omega(\xi')\partial_a\partial^a\delta^2(\xi-\xi') \\
&= \frac{c}{6}\int \mathscr{D}[\phi]\,\mathrm{e}^{\mathrm{i}S}\,\partial_a\partial^a\omega(\xi) \\
&= -\frac{c}{12}\,\mathrm{e}^{2\omega(\xi)}\,\delta R(\xi) \sim -\frac{c}{12}\delta R,
\end{aligned}$$

where in the second line, we used Green's second identity and neglected any possible boundary term. Hence we have $a_1 = -\dfrac{c}{12}$ and $\langle T^a{}_a \rangle = -\dfrac{c}{12}R$. □

By Proposition 2.5 we require that the conformal field theory of the worldsheet must have the central charge $c = c_X + c_{\mathrm{gh}} = 0$, where c_X is the central charge contributed from the free scalar fields X^μ, and c_{gh} is the central charge contributed from the ghost fields. Combining Proposition 2.3 and 2.4, we deduce that the spacetime dimension is given by $D = 26$. This result agrees with the critical spacetime dimension obtained by the old covariant quantisation and the lightcone quantisation.

3 BRST Quantisation

So far we have quantised the Polyakov path integral with the physical fields X and ghost fields b and c. We need to identify the physical states from the spectrum, which will be done by the BRST quantisation. We introduce the BRST charge Q_{B} as a nilpotent Hermitian operator, where the physical states are identified as the equivalence classes in the BRST cohomology $\mathcal{H}_{\mathrm{B}}$.

3.1 Lie Algebra and BRST Cohomology

In this part we outline the homological algebra behind the BRST construction, following the notes [14] by Figueroa-O'Farrill. This is also an expansion on the mathematical background mentioned in §3.2.1 of Green, Schwarz & Witten [10].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let M be a $\mathfrak{g}$-module. Let $C^p(\mathfrak{g};M) := \bigwedge^p \mathfrak{g}^\vee \otimes M$ be the space of p-forms on $\mathfrak{g}$ with values in M. We define a differential $\delta\colon C^p(\mathfrak{g};M) \to C^{p+1}(\mathfrak{g};M)$ by

- $\delta m(x) = x(m)$ for $x \in \mathfrak{g}$ and $m \in M$;
- $(\delta\alpha)(x,y) = -\alpha([x,y])$ for $mx, y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^\vee$;
- extend it to $\bigwedge^\bullet \mathfrak{g}^\vee$ by $\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \delta\beta$;
- extend it to $\bigwedge^\bullet \mathfrak{g}^\vee \otimes M$ by $\delta(\omega \otimes m) = \delta\omega \otimes m + (-1)^{|\omega|}\omega \wedge \delta m$.

It follows from the Jacobi identity that $\delta^2 = 0$. Therefore $(C^\bullet(\mathfrak{g};M), \delta)$ is a complex, called the **Chevalley–Eilenburg complex**. The corresponding cohomology $\mathrm{H}^\bullet(\mathfrak{g};M)$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with value in M.

We define two graded maps on $\bigwedge^\bullet \mathfrak{g}^\vee$. For $\alpha \in \mathfrak{g}^\vee$, we define $\varepsilon(\alpha)\colon \bigwedge^p \mathfrak{g}^\vee \to \bigwedge^{p+1} \mathfrak{g}^\vee$ by $\varepsilon(\alpha)\omega := \alpha \wedge \omega$. For $x \in \mathfrak{g}$, we define $\iota(x)\colon \bigwedge^p \mathfrak{g}^\vee \to \bigwedge^{p-1} \mathfrak{g}^\vee$ by $\iota(x)\alpha := \alpha(x)$ for $\alpha \in \mathfrak{g}^\vee$ and extend it to $\bigwedge^\bullet \mathfrak{g}^\vee$ by $\iota(x)(\alpha \wedge \beta) = \iota(x)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota(x)\beta$. Then

$$\varepsilon(\alpha) \circ \iota(x) + \iota(x) \circ \varepsilon(\alpha) = \alpha(x)\,\mathrm{id}\,. \tag{3.1}$$

Let (K_i) be a basis of $\mathfrak{g}$ and (α^i) be the corresponding dual basis of $\mathfrak{g}^\vee$. The Chevalley–Eilenburg differential is given explicitly by

$$\delta = \varepsilon(\alpha^i) \circ K_i - \frac{1}{2}\varepsilon(\alpha^i) \circ \varepsilon(\alpha^j) \circ \iota([K_i, K_j]) = c^i \left(K_i - \frac{1}{2} f_{ij}{}^k c^j b_k \right). \tag{3.2}$$

where we have introduced the **ghosts** $c^i := \varepsilon(\alpha^i)$, the **anti-ghosts** $b_i := \iota(K_i)$, and the structure constants $f_{ij}{}^k$ such that $[K_i, K_j] = f_{ij}{}^k K_k$. The identity (3.1) implies that the ghosts satisfy the anti-commutation relation:

$$\{c^i, b_j\} = \delta^i_j.$$

Now we consider $\mathfrak{g}$ as the gauge algebra of a physical system M. The grading on the Chevalley–Eilenburg complex is called the **ghost number**. We focus on the ghost number zero states. The zeroth Lie algebra cohomology $\mathrm{H}^0(\mathfrak{g};M) = M^{\mathfrak{g}}$, the $\mathfrak{g}$-invariant submodule of M. We can identify this with the gauge invariant states without ghosts, which are the physical states of the system. In the physics terminology, $Q_\mathrm{B} := \delta$ is called the **BRST operator**, and $\mathcal{H}_\mathrm{B} := \mathrm{H}^0(\mathfrak{g};M)$ is called the **BRST cohomology**. After quantisation, $\mathcal{H}_\mathrm{B}$ will be the physical Hilbert space.

3.2 BRST Quantisation of Bosonic Strings

For bosonic strings, the gauge algebra $\mathfrak{g}$ is realised as the infinite-dimensional Virasoro algebra (2.6). We attempt to carry out the BRST quantisation scheme in the previous section. The ghosts and anti-ghosts are the Fourier coefficients c_n and b_n in (2.3). The BRST operator (3.2) with normal ordering is given by

$$\begin{aligned} Q_\mathrm{B} &= \sum_{m\in\mathbb{Z}} \left(c_{-m} L^X_m - \frac{1}{2} \sum_{n=-\infty}^{\infty} (m-n) \ {:}c_{-m} c_{-n} b_{m+n}{:} \right) - a c_0 \\ &= \sum_{m\in\mathbb{Z}} {:}\left(L^X_m + \frac{1}{2} L^{\mathrm{gh}}_m - a\delta_{m,0} \right) c_m{:}\ . \end{aligned} \tag{3.3}$$

where we used the expression (2.5) for the ghost Virasoro generators. We quote the full BRST symmetry of the system without proof:

$$[Q_{\mathrm{B}}, X^\mu] = (c^z\partial_z + c^{\bar z}\partial_{\bar z})X^\mu, \qquad \{Q_{\mathrm{B}}, b_{zz}\} = T^{\mathrm{tot}}_{zz}, \qquad \{Q_{\mathrm{B}}, b_{\bar z\bar z}\} = T^{\mathrm{tot}}_{\bar z\bar z};$$
$$\{Q_{\mathrm{B}}, c^z\} = c^z\partial_z c^z, \qquad \{Q_{\mathrm{B}}, c^{\bar z}\} = c^{\bar z}\partial_{\bar z} c^{\bar z}.$$

where $T^{\mathrm{tot}} = T^X + T^{\mathrm{gh}}$. In terms of the Fourier modes we have

$$\{Q_{\mathrm{B}}, c_m\} = -\sum_{n\in\mathbb{Z}}(2m+n)c_{-n}c_{m+n}, \qquad \{Q_{\mathrm{B}}, b_m\} = L_m = L^X_m + L^{\mathrm{gh}}_m - a\delta_{m,0}. \tag{3.4}$$

In the infinite-dimensional case, the condition that $Q_{\mathrm{B}}^2 = 0$ maybe anomalous. In fact we have the following result.

Proposition 3.1. BRST Anomaly

The BRST operator satisfies $Q_{\mathrm{B}}^2 = 0$ if and only if the Virasoro algebra (2.7) reduces to the Witt algebra $[L_m, L_n] = (m-n)L_{m+n}$, which means that $a = 1$ and the spacetime dimension $D = 26$.

Proof. We adapt the proof from §3.2.1 of Green, Schwarz & Witten [10].

A direct computation from (3.3) shows that

$$Q_{\mathrm{B}}^2 = \frac{1}{2}\{Q_{\mathrm{B}}, Q_{\mathrm{B}}\} = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}\left([L_m, L_n] - (m-n)L_{m+n}\right)c_{-m}c_{-n}.$$

Therefore $[L_m, L_n] = (m-n)L_{m+n}$ implies that $Q_{\mathrm{B}}^2 = 0$. Conversely, if $Q_{\mathrm{B}}^2 = 0$, then by (3.4) we have

$$[L_m, Q_{\mathrm{B}}] = [\{Q_{\mathrm{B}}, b_m\}, Q_{\mathrm{B}}] = 0.$$

By the Jacobi identity we have

$$[L_m, L_n] = [L_m, \{Q_{\mathrm{B}}, b_n\}] = \{Q_{\mathrm{B}}, [L_m, b_n]\} = (m-n)\{Q, b_{m+n}\} = (m-n)L_{m+n}. \qquad \square$$

For finite-dimensional $\mathfrak{g}$, we can define a ghost number operator $N = \sum_{i=1}^{\dim\mathfrak{g}} c^i b_i$ whose eigenvalues are the ghost numbers of the corresponding eigenstates. For the infinite-dimensional case, we need to put a normal ordering. Our choice is

$$N := \frac{1}{2}(c_0 b_0 + b_0 c_0) + \sum_{m=1}^{\infty}(c_{-m}b_m + b_{-m}c_m) \tag{3.5}$$

and *define* the ghost number of an eigenstate of N to be the associated eigenvalue (so is not necessarily an integer).

3.3 Physical Hilbert Space

Now we address the problem of identifying the physical states. First we consider the ghost ground states. For $m = 0$, the Hamiltonian $H = L_0$ commutes with b_0 and c_0, which satisfies $b_0^2 = c_0^2 = 0$ and $\{c_0, b_0\} = 1$. Therefore we can introduce the degenerate states $|\uparrow\rangle$ and $|\downarrow\rangle$ such that:

$$c_0|\uparrow\rangle = 0, \qquad b_0|\downarrow\rangle = 0; \qquad b_0|\uparrow\rangle = |\downarrow\rangle, \qquad c_0|\downarrow\rangle = |\uparrow\rangle.$$

With the ordering convention in (3.5), we note that $|\uparrow\rangle$ and $|\downarrow\rangle$ have ghost numbers $+1/2$ and $-1/2$ respectively. We define the **BRST physical states** to be the states with ghost number $-1/2$ which are annihilated by c_n and b_n for $n > 0$. The reason for choosing $|\downarrow\rangle$ but not $|\uparrow\rangle$ will be clear after proving Proposition 3.2.

Before preceding, we recall the construction of the physical Hilbert space $\mathcal{H}_{\text{OCQ}}$ in the old covariant quantisation. A state $|\chi\rangle$ is said to be

- **OCQ-physical**, if $(L_m^X - a\delta_{m,0})\,|\chi\rangle = 0$ for all $m \geqslant 0$;
- **spurious**, if $|\chi\rangle = \sum_{m=1}^{\infty} L_{-m}^X\,|\chi_m\rangle$ for some $|\chi_m\rangle$;
- **null**, if $|\chi\rangle$ is both OCQ-physical and spurious.

We define the OCQ Hilbert space $\mathcal{H}_{\text{OCQ}}$ to be the set of OCQ physical states quotient by the set of null states. We have the following important result:

Proposition 3.2. BRST–OCQ Equivalence

The BRST cohomology $\mathcal{H}_{\text{B}}$ is equivalent to the physical Hilbert space $\mathcal{H}_{\text{CQ}}$ from old covariant quantisation.

Proof. We adapt the proof from §4.4 of Polchinski [3].

The proof has two parts. First, we need to prove that a state is OCQ-physical if and only if it is a Q_{B}-cocycle with ghost number $-1/2$, i.e. $Q_{\text{B}}\,|\chi\rangle = 0$. Second, we need to prove that a state is null if and only if it is a Q_{B}-coboundary, i.e. $|\chi\rangle = Q_{\text{B}}\,|\lambda\rangle$ for some state λ.

Step 1. Consider the state $|\chi\rangle = |\psi\rangle_X \otimes |\downarrow\rangle$. We have

$$Q_{\text{B}}\,|\chi\rangle = \left(c_0(L_0^X - 1) + \sum_{m=1}^{\infty} c_{-m} L_m^X\right)|\chi\rangle\,. \tag{3.6}$$

By separating the different ghost numbers, $Q_{\text{B}}\,|\chi\rangle = 0$ holds if and only if

$$(L_0^X - 1)\,|\psi\rangle_X = 0, \qquad L_n^X\,|\psi\rangle_X = 0 \text{ for } n > 0, \tag{3.7}$$

which means that $|\chi\rangle$ is a OCQ physical state with $a = 1$.

Step 2. Suppose that $|\chi\rangle$ is a OCQ null state. Then it is a Q_{B}-cocycle and $\langle\chi|\chi\rangle = 0$. By the no ghost theorem 3.3, $|\chi\rangle$ is a Q_{B}-coboundary.

Conversely, suppose that $|\chi\rangle = Q_{\text{B}}\,|\lambda\rangle$. Then $|\lambda\rangle$ has ghost number $-3/2$. We can expand it as

$$|\lambda\rangle = \sum_{n=1}^{\infty} b_{-n}\,(|\lambda_n\rangle_X \otimes |\downarrow\rangle) + (\text{more ghost excitations}).$$

Therefore we have

$$\begin{aligned} |\chi\rangle &= \left(c_0(L_0^X - 1) + \sum_{m\in\mathbb{Z}} c_{-m} L_m^X\right)|\lambda\rangle \\ &= \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_m L_{-m}^X b_{-n}\,(|\lambda_n\rangle_X \otimes |\downarrow\rangle) = \sum_{m=1}^{\infty} L_{-m}^X\,|\lambda_m\rangle_X \otimes |\downarrow\rangle\,. \end{aligned}$$

Hence $|\chi\rangle$ is a OCQ null state. □

Remark. For Step 1 of the proof, we also find that physics states cannot be built on $|\uparrow\rangle$, since the condition $(L_0^X - 1)\,|\phi\rangle_X = 0$ would be missing.

We conclude this part by a famous result, the no ghost theorem for BRST quantisation.

Theorem 3.3. BRST No Ghost Theorem

The inner product $\langle - | - \rangle$ induced from the string spectrum is positive definite on the BRST cohomology $\mathcal{H}_{\mathrm{B}}$. More explicitly, for states with ghost number $-1/2$, the Q_{B}-cocyles have positive norms and the Q_{B}-coboundaries have zero norms.

Proof. We omit the proof due to length limit of the report. This is a major theorem proven in Kato & Ogawa [9]. Also see §4.4 of Polchinski [3]. A more homological approach to the theorem is presented in Chapter 5 of Figueroa-O'Farrill [11]. □

Conclusion

In this report we have discussed the Faddeev–Popov procedure of the path integral quantisation and the BRST quantisation of the bosonic strings. The central idea is to represent the worldsheet gauge symmetries by the ghost fields b and c. In the Faddeev–Popov procedure, the ghosts are the result of inverting the Faddeev–Popov determinant, which arises from fixing the gauge to some fiducial metric. In the BRST quantisation, the ghosts are encoded in the Chevalley–Eilenburg complex of the gauge Lie algebra.

In §2, we analysed the mode expansion, the Virasoro algebra, and conformal field theory of the ghost fields. In this process we have shown that the Weyl anomaly and the Virasoro anomaly are cancelled for the spacetime dimension $D = 26$, which agrees with the old covariant quantisation and the lightcone quantisation.

In §3, we identified the physical Hilbert space as the BRST cohomology with ghost number $-1/2$ and proved that it is equivalent to the physical Hilbert space in the old covariant quantisation. The no ghost theorem for BRST quantisation is quoted without proof. However, we note that the proof of the theorem is homological algebra in nature, which reflects the vanishing of the BRST cohomology in all higher ghost numbers (see Figueroa-O'Farrill [11]).

The use of conformal field theory is inevitable in this formalism, whence we include a subsection to build up the necessary background. Due to length limit the discussion is kept at the minimum. For example, the equivalence between two definitions of the central charge is not proven. This can be found in §4.5 of Tong [5]. Also there are different notions of normal ordering appeared in §2.1, §2.2, and §2.3 which have subtleties we did not discuss.

There are certain aspects of the path integral quantisation scheme that we did not elaborate on. For example, the worldsheet topology may obstruct us from globally fixing the gauge to arbitrary fiducial metric. This leads to the study of the moduli spaces of Riemann surfaces, which is discussed in Chapter 5 of Polchinski [3] and §3.3 of Green, Schwarz & Witten [10].

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