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Problem Sheet 2

B4.2: Functional Analysis II

12 February, 2020

Question 1

Let X be a Hilbert space and $A \in \mathscr{B}(X)$.

(a) Prove that $\operatorname{Ker} A = (\operatorname{Im} A^*)^\perp$ and $(\operatorname{Ker} A)^\perp = \overline{\operatorname{Im} A^*}$.

(b) Assume that A is a projection, i.e. $A^2 = A$. Show that $\operatorname{Im} A$ is closed. Prove that

$$A = A^* \iff (\operatorname{Im} A)^\perp = \operatorname{Ker} A \iff \|A\| \leqslant 1.$$

Deduce that either $\|A\| = 1$ or $A = 0$ provided that one of the above statements is true.

[*Hint: To prove that* $\|A\| \leqslant 1$ *implies* $A = A^*$, *show that, for every given point in* $\operatorname{Im}(I - A)$, *the origin is the point in* $\operatorname{Im} A$ *which is closest to that given point, and then use Q3 of Sheet 1 to show that* $\operatorname{Im} A$ *and* $\operatorname{Im}(I - A)$ *are orthogonal complementary spaces.*]

Proof. (a) For $x \in \ker A$ and $y \in \operatorname{im} A^*$, let $z \in X$ such that $y = A^* z$. Then $\langle x, y\rangle = \langle x, A^* z\rangle = \langle Ax, z\rangle = 0$. Hence $x \in (\operatorname{im} A^*)^\perp$. Hence $\ker A \subseteq (\operatorname{im} A^*)^\perp$.

Let $x \in (\operatorname{im} A^*)^\perp$. For $y \in X$, we have $\langle x, A^* y\rangle = \langle Ax, y\rangle = 0$. In particular, $\langle Ax, Ax\rangle = \|Ax\|^2 = 0$. Hence $Ax = 0$ and $x \in \ker A$. Hence $(\operatorname{im} A^*)^\perp \subseteq \ker A$.

We deduce that $\ker A = (\operatorname{im} A^*)^\perp$. Now taking orthogonal complements on both sides, we have $(\ker A)^\perp = (\operatorname{im} A^*)^{\perp\perp}$. By Proposition 1.2.8, we have $(\operatorname{im} A^*)^{\perp\perp} = \overline{\operatorname{im} A^*}$. Hence $(\ker A)^\perp = \overline{\operatorname{im} A^*}$.

(b) Consider ~~the~~ a sequence $\{Ax_n\}_{n=0}^\infty$ in $\operatorname{im} A$ such that $Ax_n \to y \in X$ as $n \to \infty$. By continuity of A,

Not unique

$$Ay = \lim_{n\to\infty} A^2 x_n = \lim_{n\to\infty} A x_n = y$$

alternatively note that $\operatorname{Im} A = \underbrace{\ker(I - A)}_{\text{closed}}$.

Hence $y \in \operatorname{im} A$. We deduce that $\operatorname{im} A$ is closed.

- Suppose that $A = A^*$. Then by (a), $\ker A = (\operatorname{im} A^*)^\perp = (\operatorname{im} A)^\perp$.
- Suppose that $\ker A = (\operatorname{im} A)^\perp$. By Projection Theorem, we have $X = \operatorname{im} A \oplus (\operatorname{im} A)^\perp = \operatorname{im} A \oplus \ker A$. For $x \in X$, let $x = y + z$, where $y \in \operatorname{im} A$ and $z \in \ker A$. By Pythagoras' Theorem, $\|x\|^2 = \|y\|^2 + \|z\|^2$. Then we have

$$\|Ax\| = \|A(y+z)\| = \|Ay\| = \|y\| = \sqrt{\|x\|^2 - \|z\|^2} \leqslant \|x\|$$

where we used the fact that $A^2 = A$ implies $A|_{\operatorname{im} A} = \mathrm{id}$. We deduce that $\|A\| \leqslant 1$.

- Suppose that $\|A\| \leqslant 1$. Let $x \in \operatorname{im} A$ and $u \in \operatorname{im}(I - A)$. Note that we have $Au = 0$ and $Ax = x$, since $A^2 = A$. Then

$$\|x\| = \|A(u - x)\| \leqslant \|u - x\|$$

which holds for all $x \in \operatorname{im} A$ and $u \in \operatorname{im}(I - A)$. By Question 3 of Sheet 1, we have $\langle x, u\rangle \leqslant 0$ for all $x \in \operatorname{im} A$ and $u \in \operatorname{im}(I - A)$. But $u \in \operatorname{im}(I - A)$ implies that $-u \in \operatorname{im}(I - A)$. So we also have $\langle x, u\rangle = -\langle x, -u\rangle \geqslant 0$. Hence $\langle x, u\rangle = 0$ for all $x \in \operatorname{im} A$ and $u \in \operatorname{im}(I - A)$.

Finally, for $x, y \in X$,

$$\langle Ax, y\rangle = \langle Ax, y - Ay\rangle + \langle Ax, Ay\rangle = \langle Ax, Ay\rangle = \langle x - Ax, Ay\rangle + \langle Ax, Ay\rangle = \langle x, Ay\rangle = \langle A^* x, y\rangle$$

Hence $A = A^*$.

Now suppose that all of the above conditions holds of the projection $A \in \mathscr{B}(X)$. Suppose that $A \neq 0$. Let $x \in \operatorname{im} A$. Then $Ax = x$. So

$$\|A\| \geqslant \frac{\|Ax\|}{\|x\|} = 1$$

But we also have $\|A\| \leqslant 1$. Hence $\|A\| = 1$. $\square$

Question 2

Let X be a Hilbert space and $U : X \to X$ be a unitary operator.

(a) Show that $\mathrm{Ker}(I-U)=\mathrm{Ker}(I-U^*)$;

(b) Show that $X=\overline{\mathrm{Im}(I-U)}\oplus\mathrm{Ker}(I-U)$;

(c) Show that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx=x$ if $x\in\mathrm{Ker}(I-U)$ and $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx=0$ if $x\in\overline{\mathrm{Im}(I-U)}$;

(d) Deduce that, for each $x\in X$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx=Px$$

where P is the orthogonal projection onto $\mathrm{Ker}(I-U)$.

Proof. First we note that $U\in\mathscr{B}(X)$. This is not explicitly stated in the notes. The proof, however, is similar to the proof that self-adjointness implies boundedness.

Suppose that $\{x_n\}$ is a sequence in X such that $x_n\to x\in X$ and $Ux_n\to z\in X$ as $n\to\infty$. For any $y\in X$, since U is surjective, there exists $w\in X$ such that $y=Uw$. Then

$$\langle z,y\rangle=\left\langle\lim_{n\to\infty}Ux_n,y\right\rangle=\lim_{n\to\infty}\langle Ux_n,y\rangle=\lim_{n\to\infty}\langle Ux_n,Uw\rangle=\lim_{n\to\infty}\langle x_n,w\rangle=\langle x,w\rangle=\langle Ux,Uw\rangle=\langle Ux,y\rangle$$

Hence $z=Ux$. We deduce that the graph of U is closed. By closed graph theorem, U is bounded.

The fact that U is an isometry immediately implies that it is bounded. $\|Ux\|=\|x\|$ by definition.

(a) U is bijective and $U^*=U^{-1}$. We have

$$x\in\ker(I-U)\iff Ux=x\iff U^{-1}x=x\iff U^*x=x\iff x\in\ker(I-U^*)$$

Hence $\ker(I-U)=\ker(I-U^*)$.

(b) From Question 1.(a), we know that $(\ker(I-U^*))^\perp=\overline{\mathrm{im}(I-U)}$ (because $(I-U^*)^*=I-U^{**}=I-U$). Since $\ker(I-U)$ is closed in X, by projection theorem and (a),

$$X=\ker(I-U^*)\oplus(\ker(I-U^*))^\perp=\ker(I-U)\oplus\overline{\mathrm{im}(I-U)}$$

(c) For $x\in\ker(I-U)$, $Ux=x$. Hence

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}x=\lim_{N\to\infty}\frac{N-1}{N}x=x$$

Let $\{x_n\}_{n=0}^\infty\subseteq\mathrm{im}(I-U)$ such that $x_n\to x\in\overline{\mathrm{im}(I-U)}$ as $n\to\infty$. Let $x_n=(I-U)y_n$. Then

$$\frac{1}{N}\sum_{n=1}^{N-1}U^nx_k=\frac{1}{N}\sum_{n=1}^{N-1}U^n(I-U)y_k=\frac{1}{N}(U-U^N)y_k$$

Note that

$$\left\|\frac{1}{N}(U-U^N)y_k\right\|\leqslant\frac{1}{N}\left(\|U\|+\|U\|^N\right)\|y_k\|=\frac{2}{N}\|y_k\|\to 0$$

as $N\to\infty$. Hence

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx_k=\lim_{N\to\infty}\frac{1}{N}(U-U^N)y_k=0$$

Therefore

$$\left\|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx-\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx_k\right\|\leqslant\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}\left\|U^n(x-x_k)\right\|=\lim_{N\to\infty}\frac{N-1}{N}\|x-x_k\|=\|x-x_k\|\to 0$$

as $k\to\infty$. We deduce that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}U^nx=0$$

(d) Let $x \in \overline{\operatorname{im}(I-U)}$ and $y \in \ker(I-U) = \ker(I-U^*)$. Let $\{z_n\} \subseteq X$ such that $(I-U)z_n \to x$ as $n \to \infty$. Then

$$\langle x, y\rangle = \lim_{n\to\infty} \langle (I-U)z_n, y\rangle = \lim_{n\to\infty} \langle z_n, (I-U^*)y\rangle = 0$$

Hence $\overline{\operatorname{im}(I-U)} \subseteq (\ker(I-U))^{\perp}$. By (b) we have in fact $\overline{\operatorname{im}(I-U)} = (\ker(I-U))^{\perp}$. Now the result of (c) shows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} U^n = P$$

where P is the orthogonal projection onto $\ker(I-U)$. ✓ □

Question 3

Let X be a Hilbert space and let $T \in \mathscr{B}(X)$.

(a) Prove that $\operatorname{Ker} TT^* = \operatorname{Ker} T^* = (\operatorname{Im} T)^{\perp}$.

(b) Assume that T is normal, i.e. $T^*T = TT^*$. Prove that $\overline{\operatorname{Im} T} = \overline{\operatorname{Im} T^*}$.

(c) Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in X$.

Proof. (a) From Question 1.(a) we know that $\ker T^* = (\operatorname{im} T)^{\perp}$ ✓ (since $T^{**} = T$). It is clear that $\ker T^* \subseteq \ker(TT^*)$ ✓. For $x \in \ker(TT^*)$,

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle = 0$$

Hence $x \in \ker T^*$ ✓. Hence $\ker(TT^*) \subseteq \ker T^*$. We conclude that

$$\ker(TT^*) = \ker T^* = (\operatorname{im} T)^{\perp} ✓$$

(b) Similar to (a) we can prove that $\ker(T^*T) = \ker T$ ✓ Since T is normal, we have $\ker T = \ker T^*$ ✓. By Question 1.(a), we have

$$\overline{\operatorname{im} T^*} = (\ker T)^{\perp} = (\ker T^*)^{\perp} = \overline{\operatorname{im} T} ✓$$

(c) We have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2$$

Hence $\|Tx\| = \|T^*x\|$ for all $x \in X$. ✓ What about $\|Tx\| = \|T^*x\| \Rightarrow T^*T = TT^*$? □

Question 4

Let M be a complete metric space and, for each $n \in \mathbb{N}$, let A_n be a nowhere dense subset of M and G_n be a dense open subset of M. Show that $\cap_{n\in\mathbb{N}} G_n$ is not contained in $\cup_{n\in\mathbb{N}} A_n$.

Deduce that $\mathbb{Q}$ is not the intersection of a countable number of open subsets of $\mathbb{R}$.

Proof. The statements are true only if $M \neq \varnothing$.

First we claim that $X \subseteq M$ is nowhere dense if and only if $M \setminus X$ contains a dense open subset of M. Let $\operatorname{int}(X)$ denotes the interior of X.

Suppose that X is nowhere dense. Then $\operatorname{int}(\overline{X}) = \varnothing$. Hence $M = M \setminus \operatorname{int}(\overline{X}) = \overline{M \setminus \overline{X}}$. We see that $M \setminus X$ contains $M \setminus \overline{X}$, which is a ~~nowhere dense set~~ Dense open subset. Conversely, suppose that $M \setminus X$ contains a dense open set U. Then $U \subseteq \operatorname{int}(M \setminus X) = M \setminus \overline{X}$, and hence $M = \overline{U} = \overline{M \setminus \overline{X}} = M \setminus \operatorname{int}(\overline{X})$. Hence X is nowhere dense. ✓

Since each G_n is dense open, $M \setminus G_n$ is nowhere dense. In particular, $M \setminus \bigcap_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} (M \setminus G_n)$ ✓ is a countable union of nowhere dense sets. Suppose that $\bigcap_{n\in\mathbb{N}} G_n \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Then

$$M = \bigcup_{n\in\mathbb{N}} A_n \cup \left(M \setminus \bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} A_n \cup \left(M \setminus \bigcap_{n\in\mathbb{N}} G_n\right) = \bigcup_{n\in\mathbb{N}} A_n \cup \bigcup_{n\in\mathbb{N}} (M \setminus G_n) ✓$$

is a countable union of nowhere dense sets, contradicting Baire Category Theorem. Hence $\bigcap_{n\in\mathbb{N}} G_n \not\subseteq \bigcup_{n\in\mathbb{N}} A_n$. ✓

Suppose that $\mathbb{Q}$ is a G_δ set. Then $\mathbb{R}\setminus\mathbb{Q}$ is a F_σ set. But $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ is also a F_σ set. Therefore $\mathbb{R} = \mathbb{Q}\cup(\mathbb{R}\setminus\mathbb{Q})$ is a countable union of closed sets, each of which is either contained in $\mathbb{Q}$ or in $\mathbb{R}\setminus\mathbb{Q}$. By Baire Category Theorem, at least one of the closed sets has non-empty interior. But both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ have empty interior. Contradiction. Hence $\mathbb{Q}$ is not a G_δ set. □

Question 5

In this question, all sequence spaces are real.

(a) Consider a double sequence $(a_{n,j})$ such that for every fixed n, the sequence $(a_{n,j})_{j=1}^\infty$ belongs to c_0. Suppose that

$$\sup_n \sum_j a_{n,j}b_j < \infty \text{ for every } b = (b_j) \in \ell^1$$

Show that $\sup_{n,j}|a_{n,j}| < \infty$.

(b) Suppose that (a_j) is a scalar sequence such that $\sum_j a_j b_j$ converges for all $b = (b_j) \in c_0$. Prove that $\sum_j |a_j|$ converges.

[*Hint: Consider the sequences T_n with entries $T_n(j) = a_j$ if $j \leqslant n$ and $T_n(j) = 0$ if $j > n$. Use the principle of uniform boundedness to show that (T_n) is bounded in ℓ^1.*]

(c) Let $2 < p < \infty$ and let $(c_{m,n})$ be a double sequence such that, for every fixed m,

$$\sum_n c_{m,n}a_nb_n \text{ converges for every } a = (a_n), b = (b_n) \in \ell^p$$

and

$$\sup_m \sum_n c_{m,n}a_nb_n < \infty \text{ for every } a = (a_n), b = (b_n) \in \ell^p$$

Prove that, for $q = \frac{p}{p-2}$,

$$\sup_m \sum_n |c_{m,n}|^q < \infty$$

Proof. (a) For each $n \in \mathbb{N}$, we define $T_n : \ell^1 \to \mathbb{R}$ by

$$T_n\left(\{b_j\}_{j=0}^\infty\right) = \sum_{j=0}^\infty a_{n,j}b_j$$

Since $\{b_j\}_{j=0}^\infty \in \ell^1$, $\sum_j |b_j| < \infty$. Since $\{a_{n,j}\}_{j=0}^\infty \in c_0$, it is bounded by some constant $M_n > 0$. Hence

$$\left|T_n(\{b_j\})\right| \leqslant \sum_{j=0}^\infty |a_{n,j}b_j| \leqslant M_n \sum_{j=0}^\infty |b_j| < \infty$$

Therefore T_n is well-defined and bounded by M_n. In particular $T_n \in \mathscr{B}(\ell^1, \mathbb{R})$.

By hypothesis, we know that

$$\sup_{n\in\mathbb{N}} T_n(\{b_j\}) = \sup_{n\in\mathbb{N}} |T_n(\{b_j\})| < \infty$$

for each $\{b_j\} \in \ell^1$. By the uniform boundedness principle, $\{T_n : n \in \mathbb{N}\}$ is uniformly bounded in $\mathscr{B}(\ell^1, \mathbb{R})$ by some constant $M > 0$. Hence for any $\{b_k\}_{k=0}^\infty \in \ell^1$, we have

$$\left|\sum_{k=0}^\infty a_{n,k}b_k\right| \leqslant M \sum_{k=0}^\infty |b_k|$$

Take $b_k = \delta_{j,k}$. We deduce that

$$\left|a_{n,j}\right| \leqslant M$$

which holds for all $n, j \in \mathbb{N}$. Therefore $\sup_{n,j}|a_{n,j}| \leqslant M < \infty$.

(The result can also be directly deduced from the uniformly boundedness of $\{T_n\}$ and the isometric isomorphism $(\ell^1)^* = \mathscr{B}(\ell_1, \mathbb{R}) \cong \ell^\infty$.)

(b) For each $n \in \mathbb{N}$, define $T_n : c_0 \to \mathbb{R}$ by

$$T_n(\{b_j\}) = \sum_{j=0}^{n} a_j b_j$$

Then for each $\{b_j\} \in c_0$,

$$|T_n(\{b_j\})| = \left|\sum_{j=0}^{n} a_j b_j\right| \leqslant (n+1) \sup_{0\leqslant j\leqslant n} |a_j b_j| \leqslant (n+1) \sup_{0\leqslant j\leqslant n} |a_j| \cdot \|b\|_\infty$$

Hence each T_n is bounded. $\{T_n\} \subseteq c_0^*$.

For each fixed $\{b_j\} \in c_0$,

$$\sup_{n\in\mathbb{N}} |T_n(\{b_j\})| = \sup_{n\in\mathbb{N}} \left|\sum_{j=0}^{n} a_j b_j\right| < \infty$$

because $\sum_j a_j b_j$ converges. Note that c_0 is a Banach space (being the closed subspace of ℓ^∞). By the uniform boundedness principle, $\{T_n\}$ is uniformly bounded in c_0^* by some constant $M > 0$.

We know from Functional Analysis I that there exists an isometric isomorphism $\ell^1 \cong c_0^*$, given by

$$\{x_n\} \mapsto \left(\{y_n\} \mapsto \sum_{n=0}^{\infty} x_n y_n\right)$$

If we identify the two spaces, we have $T_n \in \ell^1$ such that $T_n(j) = a_j \mathbf{1}_{\{j\leqslant n\}}$. The uniform boundedness gives

$$\sup_{n\in\mathbb{N}} \|T_n\| = \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} |a_j| = \sum_{j=0}^{\infty} |a_j| < \infty$$

Therefore $\{a_n\} \in \ell^1$.

(c) **This part is essentially (a) + (b) + Hölder's inequality.**

For $(d_n) \in \ell^{p/2}$, take $(a_n), (b_n) \in \ell^p$ such that

$$a_n = \sqrt{|d_n|}, \qquad b_n = \begin{cases} \sqrt{|d_n|}, & d_n \geqslant 0 \\ -\sqrt{|d_n|}, & d_n < 0 \end{cases}$$

Then $d_n = a_n b_n$. Therefore, for every fixed m,

$$\sum_n c_{m,n} d_n \text{ converges for every } d = (d_n) \in \ell^{p/2}$$

and

$$\sup_m \sum_n c_{m,n} d_n < \infty \text{ for every } d = (d_n) \in \ell^{p/2}$$

Let $S_{k,m} : \ell^{p/2} \to \mathbb{R}$ defined by

$$S_{k,m}(\{d_n\}) := \sum_{n=0}^{k} c_{m,n} d_n$$

Note that for $q = p/(p-2)$, $(p/2)^{-1} + q^{-1} = 1$. By Hölder's inequality,

$$\left|S_{k,m}(\{d_n\})\right| \leqslant \sum_{n=0}^{k} |c_{m,n} d_n| \leqslant \left(\sum_{n=0}^{k} |c_{m,n}|^q\right)^{\frac{1}{q}} \left(\sum_{n=0}^{k} |d_n|^{p/2}\right)^{\frac{2}{p}} \leqslant \|d\|_{p/2} \left(\sum_{n=0}^{k} |c_{m,n}|^q\right)^{\frac{1}{q}}$$

Hence $S_{k,m}$ is bounded. $\{S_{k,m}\} \subseteq (\ell^{p/2})^* \cong \ell^q$. In addition, we also have

$$\boxed{\sup_{k\in\mathbb{N}} \left|S_{k,m}(\{d_n\})\right|} \leqslant \|d\|_{p/2} \left(\sum_{n=0}^{\infty} |c_{m,n}|^q\right)^{\frac{1}{q}}$$

How do you know the sum converges?

Hence by uniform boundedness principle,

$$\sup_{k\in\mathbb{N}} \|S_{k,m}\| < K_m$$

for some $K_m > 0$. Given the identification $(\ell^{p/2})^* = \ell^q$, we have

$$\sum_{n=0}^{\infty} |c_{m,n}|^q = \sup_{k\in\mathbb{N}} \sum_{n=0}^{k} |c_{m,n}|^q = \sup_{k\in\mathbb{N}} \|S_{k,m}\| < K_m$$

Hence $\{c_{m,n}\}_{n=0}^{\infty} \in \ell^q$ for each $m \in \mathbb{N}$.

Next, define $T_m : \ell^{p/2} \to \mathbb{R}$ by

$$T_m(\{d_n\}) := \sum_{n=0}^{\infty} c_{m,n} d_n$$

By Hölder's inequality again,

$$|T_m(\{d_n\})| \leqslant \sum_{n=0}^{\infty} |c_{m,n} d_n| \leqslant \sum_{n=0}^{\infty} |c_{m,n}|^q \, \|\{d_n\}\|_{p/2}$$

Hence T_m is bounded. $T_m \in (\ell^{p/2})^* = \ell^q$. Fix $\{d_n\} \in \ell^{p/2}$. By hypothesis we have

$$\sup_{m\in\mathbb{N}} \sum_{n=0}^{\infty} c_{m,n} d_n = \sup_{m\in\mathbb{N}} \sum_{n=0}^{\infty} |c_{m,n} d_n| = \sup_{m\in\mathbb{N}} |T_m(\{d_n\})| < \infty$$

By the uniform boundedness principle,

$$\sup_{m\in\mathbb{N}} \|T_m\| < \infty$$

Hence

$$\sup_{m\in\mathbb{N}} \sum_{n=0}^{\infty} |c_{m,n}|^q = \sup_{m\in\mathbb{N}} \|T_m\|^q < \infty$$

□

Question 6

(a) Let X be a real Banach space, Y and Z be real normed vector spaces, and $B : X \times Y \to Z$ be bilinear (i.e., linear in each variable). Suppose that for each $x \in X$ and $y \in Y$, the linear maps $B^x : Y \to Z$ and $B_y : X \to Z$ defined

$$B^x(y) = B(x,y) = B_y(x)$$

are continuous. Use the principle of uniform boundedness to prove that there exists a constant K such that $\|B(x,y)\| \leqslant K\|x\|\|y\|$ for all $x \in X$ and $y \in Y$. Deduce that B is continuous.

(b) Let X and Y both be the subspace of $L^1(0,1)$ consisting of polynomials, $Z = \mathbb{R}$ and

$$B(f,g) = \int_0^1 f g \, \mathrm{d}t$$

Show that the bilinear form B is continuous in each variables but it is not continuous.

[To put things in perspective, please note that even on $\mathbb{R}^2$, for nonlinear functions, separate continuity does not imply joint continuity. A standard example is the function $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq 0$ and $f(0,0) = 0$.]

Proof. (a) We know that a linear map between two normed vector spaces is continuous if and only if it is bounded.

First we fix $x \in X$. Since $B^x : Y \to Z$ is continuous, there exists $K_x > 0$ such that for each $y \in Y$,

$$\|B^x(y)\|_Z \leqslant K_x \|y\|_Y$$

Consider the family of operators $\{B_y : y \in Y, \|y\| = 1\} \subseteq \mathscr{B}(X,Z)$. For each fixed $x \in X$, we have

$$\|B_y(x)\|_Z = \|B^x(y)\|_Z \leqslant K_x \|y\|_Y = K_x$$

Since X is a Banach space, by the uniform boundedness principle, there exists $K > 0$ such that $\|B_y\| < K$ for each $y \in Y$ with $\|y\| = 1$.

Now, for $x \in X$ and $y \in Y$,

$$\|B(x,y)\|_Z = \|y\|_Y \left\| B\left(x, \frac{y}{\|y\|_Y}\right) \right\| = \|y\|_Y \left\| B_{y/\|y\|_Y}(x) \right\| \leqslant K \|x\|_X \|y\|_Y$$

Finally, we equip $X \times Y$ with the norm $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$. Fix $(x,y) \in X \times Y$. For $\varepsilon > 0$, for $(x',y') \in X \times Y$ such that $\|(x,y)-(x',y')\| < \varepsilon$,

$$\begin{aligned}
\|B(x,y) - B(x',y')\|_Z &= \|B(x,y) - B(x',y) + B(x',y) - B(x',y')\|_Z \\
&= \|B(x-x',y) + B(x',y-y')\|_Z \\
&= \|B(x-x',y) + B(x,y-y') + B(x'-x,y-y')\|_Z \\
&\leqslant \|B(x-x',y)\|_Z + \|B(x,y-y')\|_Z + \|B(x'-x,y-y')\|_Z \\
&\leqslant K\|x-x'\|_X \|y\|_Y + K\|x\|_X \|y-y'\|_Y + K\|x-x'\|_X \|y-y'\|_Y \\
&\leqslant K\left(\|x\|_X + \|y\|_Y + \varepsilon\right)\varepsilon
\end{aligned}$$

Hence B is continuous at (x,y). We deduce that B is continuous on $X \times Y$.

(b) Fix $g \in Y$. Since g is a polynomial on $(0,1)$, it is continuous on $[0,1]$ and hence bounded on $(0,1)$. That is, $\|g\|_\infty < \infty$. For $\{f_n\} \subseteq X$ such that $f_n \to f \in X$ in L^1 norm as $n \to \infty$, by Hölder's inequality,

$$\left\| \int_0^1 (f_n - f) g \,\mathrm{d}t \right\| \leqslant \|f_n - f\|_1 \|g\|_\infty \to 0$$

as $n \to \infty$. Hence $f \mapsto \int_0^1 f g \,\mathrm{d}t$ is continuous. Similarly, $g \mapsto \int_0^1 f g \,\mathrm{d}t$ is also continuous.

Suppose that B is continuous. We claim that B is bounded in the sense of (a).

Since B is continuous at $(0,0)$, there exists $\delta > 0$ such that for $(x,y) \in X \times Y$, $\|(x,y)\| < \delta$ implies that $|B(x,y)| \leqslant 1$. Now for all $x \in X$ and $y \in Y$,

$$|B(x,y)| = \frac{\|x\|}{\delta}\frac{\|y\|}{\delta}\left| B\left(\frac{\delta}{\|x\|}x, \frac{\delta}{\|y\|}y\right)\right| \leqslant \delta^{-2}\|x\|\|y\|$$

which proves the claim.

Suppose that $f_n = (n+1)x^n \in X = Y$. Then

$$\|f_n\|_1 = \left| \int_0^1 (n+1)x^n \,\mathrm{d}x \right| = 1$$

But

$$|B(f_n,f_n)| = \left| \int_0^1 (n+1)^2 x^{2n} \,\mathrm{d}x \right| = \frac{(n+1)^2}{2n+1} \to \infty$$

as $n \to \infty$. Therefore there is no constant $M > 0$ such that $|B(f_n,f_n)| \leqslant K\|f_n\|_1^2$ for all $n \in \mathbb{N}$. This is a contradiction. we conclude that B is not continuous. $\square$