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Problem Sheet 1

C3.1: Algebraic Topology

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Convention: All spaces are topological spaces. Maps of spaces are always continuous.

Question 1

A map $f : X \to Y$ of spaces is **homotopic** to $g : X \to Y$ if f can be continuously deformed into g, meaning that there exists a map $F : X \times [0,1] \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We write $f \simeq g$.

a) Show that $\simeq$ is an equivalence relation on maps $X \to Y$.

Two spaces X, Y are **homotopic equivalent**, if there exist $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

b) Show that $\simeq$ is an equivalence relation on spaces.

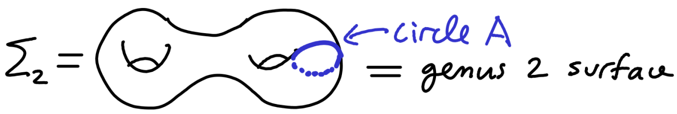
c) Show that point $\simeq \mathbb{R}^n$.

d) Show that the <u>solid</u> torus $\simeq$ circle.

e) Let $A \subseteq X$ be a subspace. We say that "*A can be contracted down to a point in X*", if there exists $H : X \times [0,1] \to X$ with $H_t(A) \subseteq A$ for all $t \in [0,1]$, $H_0 = \mathrm{id}_X$ and $H_1(A) =$ some point in A, where $H_t = H(\cdot, t) : X \to X$. Deduce that $X \simeq X/A$.

(Hint: Let $f : X \to X/A$ be the quotient map. Comstruct a map $Q_t : X/A \to X/A$ such that $f \circ H_t = Q_t \circ f$. Build $g : X/A \to X$ with $g \circ f = H_1$.)

f) Let Σ_2 be a genus 2 surface and A be a circle on Σ_2. Show (by drawing convincing pictures) that $\Sigma_2/A \simeq T^2 \vee S^1$.



g) Prove (using pictures) that $S^n \setminus \text{point} \simeq \mathbb{D}^n$ and $S^n \setminus (k \text{ points}) \simeq \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{k-1 \text{ copies}}$, where $k \geqslant 2$.

Proof. This question is about standard material in B3.5 Topology and Groups.

a)
- Reflectivity:

  $F : X \times [0,1] \to Y$ defined by $F(x,t) = F(x,0) = f(x)$ is a homotopy from f to f.

- Symmetry:

  Suppose that F is a homotopy from f to g. Then $G : X \times [0,1] \to Y$ defined by $G(x,t) = F(x, 1-t)$ is a homotopy from g to f.

- Transitivity:

  Suppose that F is a homotopy from f to g and G is from g to h. Then we define $H : X \times [0,1] \to Y$ be

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leqslant t \leqslant 1/2 \\ G(x, 2t-1), & 1/2 \leqslant t \leqslant 1 \end{cases}$$

  continous by glueing lemma

  H is a homotopy from f to h.

Hence homotopy of continuous maps is an equivalence relation.

b)
- Reflectivity:

  $F : X \times [0,1] \to Y$ with $F(x,t) = x$ is a homotopy from id_X to id_X. It is clear that $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \simeq \mathrm{id}_X$. Hence $X \simeq X$.

- Symmetry:

  It is trivial from definition that $X \simeq Y$ implies that $Y \simeq X$.

- Transitivity:

  Suppose that $X \simeq Y$ and $Y \simeq Z$. There exist $f : X \to Y$, $g : Y \to X$, $h \in Y \to Z$ and $i \in Z \to Y$ such that $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$, $i \circ h \simeq \mathrm{id}_Y$, and $h \circ i \simeq \mathrm{id}_Z$. Then we have $h \circ f : X \to Z$ and $g \circ i : Z \to X$ satisfying

  $$g \circ i \circ h \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X, \qquad h \circ f \circ g \circ i \simeq h \circ \mathrm{id}_Y \circ i = h \circ i \simeq \mathrm{id}_Z$$

  Hence $X \simeq Z$.[1]

Hence homotopic equivalence of topological spaces is an equivalence relation.

c) Let $f : \{*\} \to \mathbb{R}^n$ with $f(*) = 0$, and $g : \mathbb{R}^n \to \{*\}$. ($g$ is unique as singletons are final in Top.) Trivially $g \circ f = \mathrm{id}_{\{*\}}$. Let $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ given by $F(\boldsymbol{x}, t) = \boldsymbol{x}t$. Then $F(\boldsymbol{x}, 0) = \boldsymbol{x}$ and $F(\boldsymbol{x}, 1) = 0 = f \circ g(\boldsymbol{x})$. Hence $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n}$. we conclude that $\mathbb{R}^n \simeq \{*\}$.

fancy

d) We parametrise the solid torus as $S^1 \times \mathbb{D}^2 = \{z, w \colon |z| = 1, |w| \leqslant 1\} \subseteq \mathbb{C}^2$. Then $S^1 \times \mathbb{D}^2$ retracts onto S^1 via $r(z, w) = z$. And S^1 embeds into $\mathbb{D}^2$ via $\iota(z) = (z, 0)$. It is clear that $r \circ \iota = \mathrm{id}_{S^1}$. Moreover, $F : S^1 \times \mathbb{D}^2 \times I \to S^1 \times \mathbb{D}^2$ given by $F(z, w, t) = (z, tw)$ defines a homotopy from $i \circ r$ to $\mathrm{id}_{S^1 \times \mathbb{D}^2}$. Hence $S^1 \times \mathbb{D}^2 \simeq S^1$.

e) Let $f : X \to X/A$ be the quotient map. Consider the composition $f \circ H_t : X \to X/A$. Since $H_t(A) \subseteq A$, then $f \circ H_t$ is constant on A. By the universal property of quotient, there exists a unique morphism $Q_t : X/A \to X/A$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "H_t"'] & X/A \arrow[d, dashed, "\exists! Q_t"] \\
X \arrow[r, two heads, "f"] & X/A
\end{tikzcd}$$

as $H_1(A) = \{a\}$

not required

Suppose that $H_1(A) = a$ for some $a \in A$. We define $g : X/A \to X$ by $g([x]) = H_1(x)$ for $x \in X \setminus A$ and $g(A) = a$. g is well defined, because $[x] = \{x\}$ for $x \in X \setminus A$ in X/A. We have $g \circ f = H_1$ by construction. Since H is a homotopy, we have $g \circ f \simeq H_0 = \mathrm{id}_X$.

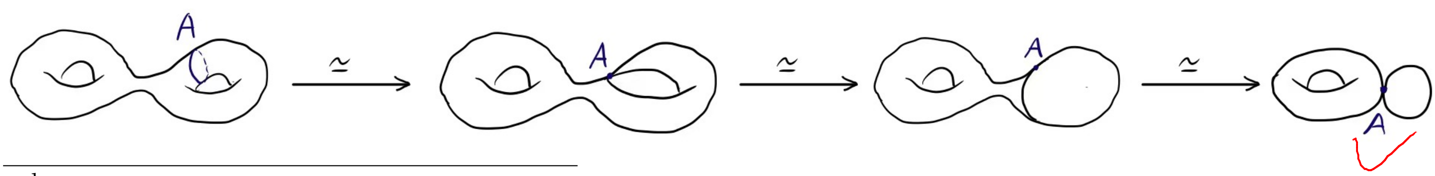
More generally we have the commutative diagram:

$$\begin{tikzcd}
X \times I \arrow[r, two heads, "{(f, \mathrm{id}_I)}"] \arrow[d, "H"'] & X/A \times I \arrow[d, dashed, "\exists! Q"] \\
X \times I \arrow[r, two heads, "{(f, \mathrm{id}_I)}"] & X/A \times I
\end{tikzcd}$$

which proves that Q is a homotopy and $Q_t = Q(-, t)$.

Next we note that for $x \in X \setminus A$, $f \circ g([x]) = f \circ H_1(x) = Q_1 \circ f(x) = Q_1([x])$, and $f \circ g(A) = f(a) = A = Q_1(A)$. We deduce that $f \circ g = Q_1$. Hence $f \circ g \simeq Q_0 = \mathrm{id}_{X/A}$. We conclude that $X \simeq X/A$. good!

f) (Please forgive my bad drawing...) Schematically we have:



[1]We used the following lemma: if $f_0 \simeq f_1$, then $j \circ f_0 \simeq j \circ f_1$ and $f_0 \circ k \simeq f_1 \circ k$.
If F is a homotopy from f_0 to f_1, then $j \circ F$ is a homotopy from $j \circ f_0$ to $j \circ f_1$, and $F \circ (k \times \mathrm{id}_{[0,1]})$ is a homotopy from $f_0 \circ k$ to $f_1 \circ k$.

g) We can embed S^n into $\mathbb{R}^{n+1}$ and set the generalised spherical coordinate system $(\theta_0, \dots, \theta_n)$ on S^n minus two points:

$$x_0 = \cos\theta_1, \quad x_1 = \sin\theta_1 \cos\theta_2, \quad \dots, \quad x_{n-1} = \cos\theta_n \prod_{i=1}^{n-1} \sin\theta_i, \quad x_n = \sin\theta_n \prod_{i=1}^{n-1} \sin\theta_i$$

Let $\boldsymbol{x}_0 = (1, 0, \dots, 0)$. The upper hemisphere without north pole is an atlas:

$$\left\{\boldsymbol{x} \in \mathbb{R}^{n+1} : \theta_1 \in (0, \pi/2]\right\}$$

It has a deformation retraction on to the equator $\theta_1 = \pi/2$ given by

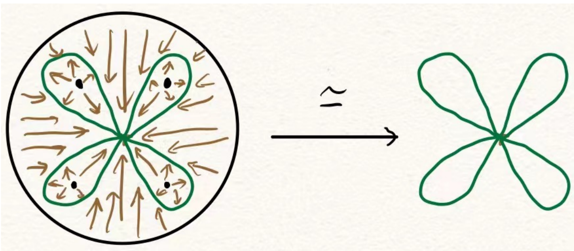
$$F(\theta_1, \dots, \theta_n, t) = \left(\theta_1 + t\left(\frac{\pi}{2} - \theta_1\right), \theta_2, \dots, \theta_n\right)$$

We keep the lower hemisphere fixed. The resulting space is

$$L = \left\{\boldsymbol{x} \in \mathbb{R}^{n+1} : \theta_1 \in [\pi/2, \pi)\right\} \cup \{0\}$$

It is homeomorphic to $\mathbb{D}^n$, which is the image of L under the projection in the 0-th coordinate. We conclude that $S^n \setminus \{*\} \simeq \mathbb{D}^n$. pictures would be fine.

Next we consider $S^n \setminus A$ where $|A| = k \geqslant 2$. The exists an embedding of S^n into $\mathbb{R}^{n+1}$ such that exactly one point in A lies in the upper hemisphere. The same argument proves that $S^n \setminus A \simeq \mathbb{D}^n \setminus B$, where $|B| = k - 1$. It remains to prove that $\mathbb{D}^n \setminus B$ is the wedge sum of $k - 1$ copies of S^1. It is hard to prove rigorously. But we can draw the deformation retract in the diagram when $k = 4$ as an example:



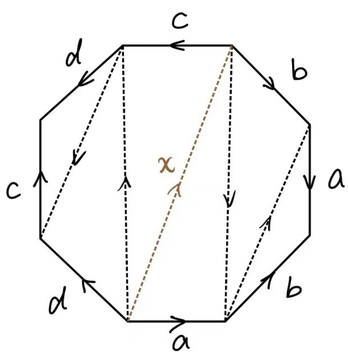
nice

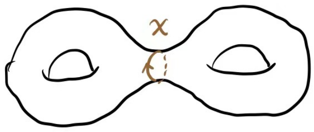
We deduce that $S^n \setminus (k \text{ points}) \simeq \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{k-1 \text{ copies}}$. □

Question 2

Draw an example of a loop in Σ_2 which is non-zero in $\pi_1(\Sigma_2)$ but is zero in $H_1(\Sigma_2)$. (Proof not required.)

Proof. From Part A Topology, Σ_2 has the following fundamental polygon.





We give Σ_2 a Δ-complex structure as shown in the diagram. The loop $x = [a, b] \in H_1(\Sigma_2)$ in the diagram is clearly zero. But $x \neq 0 \in \pi_1(\Sigma_2)$, because

$$\pi_1(\Sigma_2) = \langle a, b, c, d \mid [a,b][c,d] \rangle \cong \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$$ perfect. □

Question 3

A **retraction** a space X onto a subspace A is a map $r : X \to X$ with $r(X) = A$ and $r(a) = a$ for all $a \in A$.

a) Show that the Möbius band X retracts onto the equator A.

b) Assume that we have a functor $F : \mathsf{Top} \to \mathsf{Grp}$ such that $F(S^1) = \mathbb{Z}$,

$$F(S^1 \xrightarrow{z^2} S^1) = \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$$

and if $(f : A \to X) \simeq (g : A \to X)$ then $F(f) = F(g)$. (For example $F = H_1$ is the first homology group.)

By considering the maps $A \xrightarrow{i} X \xrightarrow{r} A$, show that $F(i)$ is injective and $F(r)$ is surjective.

Deduce that the Möbius band X does not retract onto the boundary circle $A_2 = \partial X$.

Having seen the funtorial proof, could you rephrase the proof into a topological argument for a Part A Topology undergraduate?

Proof. a) The Möbius band is homeomorphic to

$$X = [0,1]^2 / \big\langle (0, y) \sim (1, 1-y) \big\rangle$$

With the equator A given by the image of $\{(x, y) : x = 1/2\}$ under the quotient map. We see that the retraction of X onto A is induced y the projection $(x, y) \mapsto (1/2, y)$.

b) The functor F maps a commutative diagram in Top to Grp:

needs to be more precise. these are maps between different spaces $i : S^1 \hookrightarrow X$, $S^1 \xrightarrow{z^2} S^1$

you should show $F(X) = \mathbb{Z}$

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "i"] & X \arrow[u, "r"'] \end{tikzcd} \implies \begin{tikzcd} & F(A) \\ F(A) \arrow[ur, equal] \arrow[r, "F(i)"] & F(X) \arrow[u, "F(r)"'] \end{tikzcd}$$

That is, $F(r) \circ F(i) = \mathrm{id}$. Hence $F(i)$ is injective and $F(r)$ is surjective.

Let $i : S^1 \hookrightarrow X$ be the inclusion map. Note that $\operatorname{im} i = \partial X \cong S^1$ and i is which homotopic equivalent to $z^2 : S^1 \to S^1$. (A, A_2)
Hence $F(i) : \mathbb{Z} \to \mathbb{Z}$ is given by $n \mapsto 2n$. You used i before as $i : S^1 \to A$. Now $S^1 \to A_2$ and also as $i : S^1 \to X$.

Suppose that there exists a retraction $r : X \to A_2$. Then $F(r) : \mathbb{Z} \to \mathbb{Z}$ is surjective. By first isomorphism theorem, $\mathbb{Z} \cong \mathbb{Z} / \ker F(r)$. But $\mathbb{Z} / \ker F(r)$ is a finite group unless $\ker F(r) = 0$. So we have $F(r)$ is an isomorphism. As $\operatorname{Aut}(\mathbb{Z}) = \{\mathrm{id}\}$ ($\neq \mathrm{id}$), $F(r) = \mathrm{id}$. Then $F(r) \circ F(i) = F(i) \neq \mathrm{id}$. Contradiction. We deduce that X does not retract onto its boundary. good effort though! Good intuition

To translate this into a topological language, we can simply take $F = \pi_1$, and operate on the fundamental groups. Fundamental groups were in Part B :)

(*My experience is that Part A students should learn category theory as early as possible.*) □
lol

The maps that are homotopic are

$$S^1 \xrightarrow{z^2} A_2 \hookrightarrow X$$

$$S^1 \xrightarrow{z^2} S^1 \xrightarrow{\simeq} A \hookrightarrow X$$

If you draw such diagrams it's also easier to apply F.

See class for the solution which incorporates more details
B

Question 4

Given a functor $F_\bullet : \mathsf{Top} \to \mathsf{GradedAb}$ with

$$F_\bullet(\text{point}) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases}$$

Define $\widetilde{F}_\bullet(X) = \ker f_\bullet$, where $f_\bullet : F_\bullet(X) \to F_\bullet(\text{point})$ is induced by the constant map on X. Prove that $F_\bullet(X) \cong \widetilde{F}_\bullet(X)$ for $\bullet \neq 0$ and $F_0(X) \cong \widetilde{F}_0(X) \oplus \mathbb{Z}$.

Proof. For $n > 0$, we have $F_n(\text{pt}) = 0$. Hence $\ker f_n = F_n(X)$, and $\widetilde{F}_n(X) = F_n(X)$ by definition.

For $n = 0$, let $g : \widetilde{F}_0(X) = \ker f_0 \hookrightarrow F_0(X)$ be the inclusion map. Note that g is surjective because the constant map $X \to \text{pt}$ is. Then we have a short exact sequence

$$0 \longrightarrow \widetilde{F}_0(X) \xrightarrow{\ g\ } F_0(X) \xrightarrow{\ f_0\ } \mathbb{Z} \longrightarrow 0$$

In addition, the inclusion $\text{pt} \to X$ give rise to a section $s : F_0(\text{pt}) \to F_0(X)$. That is, $f_0 \circ s = \text{id}_{\text{pt}}$. Hence the sequence splits by short five lemma[2]:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{F}_0(X) \arrow[r, "g"] \arrow[d, equal] & F_0(X) \arrow[r, shift left, "f_0"] & \mathbb{Z} \arrow[r] \arrow[l, shift left, "s"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{F}_0(X) \arrow[r, "i"] & \widetilde{F}_0(X) \oplus \mathbb{Z} \arrow[r, "\pi"] \arrow[u, "(g+s)"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Hence $\widetilde{F}_0(X) \oplus \mathbb{Z} \cong F_0(X)$. □

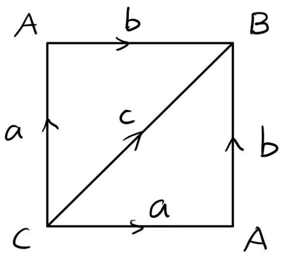
Your footnote explains that there's always a section. Here it's nice you constructed it explicitly.

α

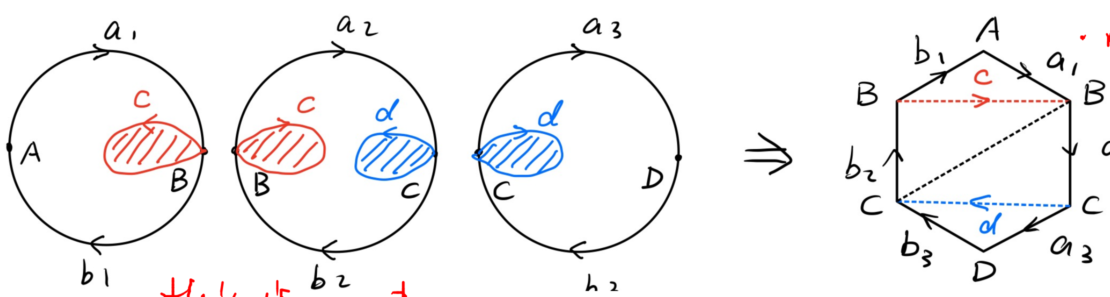
Question 5

Draw a Δ-complex structure on S^2, $\Sigma_2 = T_2 \# T_2$, and $N_3 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

Proof. For S_2 we have the following simple construction:



Σ_2 has been given in Question 2. For N_3:



· why label a_1 & a_2 and not just a?
· need a different subdivision as there's no way to orient the middle edge

this is good.

[2]In Question 5 of Homological Algebra Sheet 1, we have proven that every exact sequence of the form $0 \to A \to B \to \mathbb{Z} \to 0$ splits. This is due to $\mathbb{Z}$ being a projective $\mathbb{Z}$-module.

Perfect. Also there was a lemma called the 'splitting lemma' in AT notes!

β+

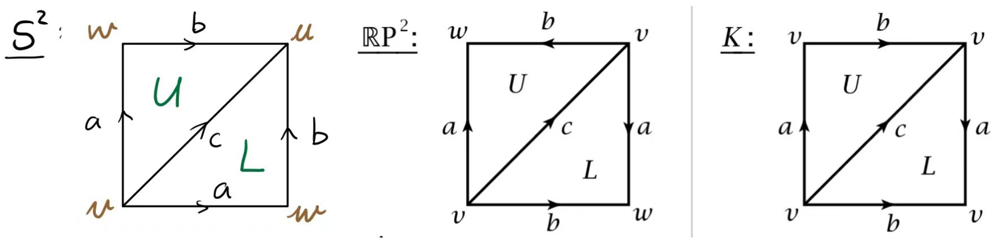
□

Question 6

a) Compute the simplicial homology of S^2, $\mathbb{R}P^2$ and the Klein bottle K.

b) Compute their simplicial homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Proof. The Δ-complex structure on these spaces are given respectively by[3]



- Homology of S^2.

  We list the generators for the chain groups:

  - $C_0: \{u, v, w\}$;
  - $C_1: \{a, b, c\}$;
  - $C_2: \{U, L\}$.
  - $C_3 = C_4 = \cdots = 0$.

  The boundary maps are given by

  - $\partial_1: a \mapsto w - v,\ b \mapsto u - w,\ c \mapsto u - v$;
  - $\partial_2: U \mapsto a + b - c,\ L \mapsto a + b - c$.
  - $\partial_3 = \partial_4 = \cdots = 0$.

  For $R = \mathbb{Z}$, the homology groups are given by

  $$H_0(S^2) = \frac{\langle u, v, w\rangle}{\langle w - v, u - w, u - v\rangle} \cong \mathbb{Z}, \quad H_1(S^2) = \frac{\langle a + b - c\rangle}{\langle a + b - c\rangle} \cong 0, \quad H_2(S^2) = \frac{\langle U - L\rangle}{\{0\}} \cong \mathbb{Z}, \quad H_3(S^2) = \cdots = 0$$

  For $R = \mathbb{Z}/2\mathbb{Z}$, the same expression holds as we replace $\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$:

  $$H_0(S^2; \mathbb{Z}/2) = \mathbb{Z}/2, \quad H_1(S^2; \mathbb{Z}/2) = 0, \quad H_2(S^2; \mathbb{Z}/2) = \mathbb{Z}/2, \quad H_3(S^2) = \cdots = 0$$

- Homology of $\mathbb{R}P^2$:

  We list the generators for the chain groups:

  - $C_0: \{v, w\}$;
  - $C_1: \{a, b, c\}$;
  - $C_2: \{U, L\}$.
  - $C_3 = C_4 = \cdots = 0$.

  The boundary maps are given by

  - $\partial_1: a \mapsto w - v,\ b \mapsto w - v,\ c \mapsto 0$;
  - $\partial_2: U \mapsto a - b - c,\ L \mapsto a - b + c$.
  - $\partial_3 = \partial_4 = \cdots = 0$.

[3] Allen Hatcher, *Algebraic Topology* pp. 102.

The zeroth homology group is given by $H_0(\mathbb{R}P^2) = \dfrac{\langle v, w\rangle}{\langle w - v\rangle} \cong \mathbb{Z}$.

If $R = \mathbb{Z}$, the first homology group

$$H_1(\mathbb{R}P^2) = \frac{\langle a-b, c\rangle}{\langle a-b-c, a-b+c\rangle} \cong \frac{\langle \alpha, \beta\rangle}{\langle 2\alpha, \alpha-\beta\rangle} \cong \frac{\langle \alpha\rangle}{\langle 2\alpha\rangle} \cong \mathbb{Z}/2$$

If we use coefficients in $\mathbb{Z}/2$, then

$$H_1(\mathbb{R}P^2;\mathbb{Z}/2) \cong \frac{\langle \alpha, \beta\rangle}{\langle 2\alpha, 2\beta, \alpha-\beta\rangle} \cong \frac{\langle \alpha, \beta\rangle}{\langle \alpha-\beta\rangle} \cong \langle \alpha\rangle \cong \mathbb{Z}/2$$

Next, if $R = \mathbb{Z}$, then $a-b+c$ and $a-b+c$ are linearly independent, and hence $\ker \partial_2 = 0$. We have $H_2(\mathbb{R}P^2) = 0$. For $R = \mathbb{Z}/2$, $\ker \partial_2 = U - L$. Hence $H_2(\mathbb{R}P^2;\mathbb{Z}/2) = \dfrac{\langle U-L\rangle}{\{0\}} = \mathbb{Z}/2$. perfect

- Homology of K:

  We list the generators for the chain groups:

  - C_0: $\{v\}$;
  - C_1: $\{a, b, c\}$;
  - C_2: $\{U, L\}$.
  - $C_3 = C_4 = \cdots = 0$.

  The boundary maps are given by

  - ∂_1: $a, b, c \mapsto 0$;
  - ∂_2: $U \mapsto a+b-c$, $L \mapsto a-b+c$.
  - $\partial_3 = \partial_4 = \cdots = 0$.

  The zeroth homology group is given by $H_0(K) = \dfrac{\langle v\rangle}{\{0\}} \cong \mathbb{Z}$.

  If $R = \mathbb{Z}$, the first homology group

  $$H_1(K) = \frac{\langle a, b, c\rangle}{\langle a+b-c, a-b+c\rangle} \cong \frac{\langle \alpha, \beta, \gamma\rangle}{\langle \alpha-\beta, \alpha+\beta\rangle} \cong \mathbb{Z}\oplus\mathbb{Z}/2$$

  If $R = \mathbb{Z}/2$, then $a+b-c = a-b+c$, and hence

  $$H_1(K;\mathbb{Z}/2) \cong \frac{\langle \alpha, \beta, \gamma\rangle}{\langle \alpha-\beta\rangle} \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$$

  For the second homology group, the calculation is identical to that of $\mathbb{R}P^2$. We have $H_2(K) = 0$ and $H_2(K;\mathbb{Z}/2) = \mathbb{Z}/2$.

We can summarise the results in the following table:

Homology	S^2		$\mathbb{R}P^2$		K	
	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$
H_0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$
H_1	0		$\mathbb{Z}/2$		$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
H_2	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
H_3	0					
$\vdots$	0					

□

Question 7

Prove that Δ^n and $\mathbb{D}^n$ are homeomorphic.

Proof. The standard n-simplex is given by

$$\Delta^n = \left\{ (x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i = 1,\ x_0, \ldots, x_n \geqslant 0 \right\} \subseteq \mathbb{R}^{n+1}$$

The projection $\pi : (x_0, \ldots, x_n) \mapsto (0, x_1, \ldots, x_n)$ restricting on Δ^n is a bijection onto its image. It is clear that π is conntinuous and the inverse is also continuous. Hence $\Delta^n \cong \pi(\Delta^n) \subseteq \mathbb{R}^n$, where

$$\pi(\Delta^n) = \left\{ (0, \ldots, x_n) : 0 \leqslant \sum_{i=1}^{n} x_i \leqslant 1,\ x_0, \ldots, x_n \geqslant 0 \right\} \subseteq \mathbb{R}^n$$

Let $X = \pi(\Delta^n) - \frac{1}{n+1}(1, \ldots, 1)$, so that $X \cong \pi(\Delta^n)$, $0 \in \operatorname{int}(X)$, and $X \subseteq \mathbb{D}^n$. ✓

For $p \in S^{n-1}$, we define $f(p) = tp$, where $t := \sup\{s \geqslant 0 : sp \in X\}$. Then $\operatorname{im} f \subseteq \partial X$. We claim that f is continuous.

Suppose that f is not continuous. There exists a sequence $\{p_n\} \subseteq S^{n-1}$ such that $p_n \to p$ and $f(p_n) \not\to f(p)$. By compactness of ∂X, after extracting a subsequence we may assume that $f(p_n) \to \alpha$ for some $\alpha \in \partial X$. We have that 0, $f(p)$, α, and p are colinear. Let $\alpha = up$ and $f(p) = vp$ for some $u, v \in (0, 1]$, and $v > u$. Let S be the orthogonal complement of p. Let K be a cone with vertex $f(p)$ and base $S \cap B(0, \varepsilon)$ for some sufficiently small $\varepsilon > 0$. Since X is convex, $K \subseteq X$. But by the construction, α has a neighbourhood entirely contained in K, contradicting that $\alpha \in \partial X$. We deduce that f is continuous. In particular, we have also shown that the line segment joining 0 and p intersect ∂X in a unique point $f(p)$. So f is invertible. The same argument shows that f^{-1} is also continuous.

Finally, For $p \in S^{n-1}$ and $t \in [0, 1]$, we define $\varphi : \mathbb{D}^n \to X$ by $\varphi(tp) = tf(p)$. Then φ is continuous with a continuous inverse $\psi : tq \mapsto tf^{-1}(q)$ for $q \in \partial X$. Hence $\mathbb{D}^n \cong X \cong \Delta^n$. $\square$

very detailed! α
quite complicated too.