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Problem Sheet 1

B2.1: Introduction to Representation Theory

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Throughout this sheet, k denotes a field and G denotes a finite group.

Question 1

Let $g \in GL(V)$ be an element of finite order and suppose that k is algebraically closed. Prove that g is diagonalisable whenever $char(k) = 0$. Does this result also hold for fields of positive characteristic?

Proof. There exists a minimal integer $n \in \mathbb{N}$ such that $g^n = id$. Then $p(x) = x^n - 1 \in k[x]$ annihilates g. The formal derivative of p is $p'(x) = nx^n \in k[x]$. Since $char\, k = 0$, $p'(x) = 0$ if and only if $x = 0$. But $x = 0$ is not a root of p. It follows that p has simple roots only. Since k is algebraically closed, p splits into distinct linear factors in $k[x]$. Let m be the minimal polynomial of g. Then m divided p and hence also splits into distinct factors. We deduce that g is diagonalizable.

The statement is not true for algebraically closed fields of positive characteristic. Let k_p be an algebraically closed fields with $char\, k_p = p$. Consider $A \in GL_2(k_p)$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then we have

$$A^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = I$$

Hence A has finite order. The characteristic polynomial of A is $\chi_A(x) = (x-1)^2$. So 1 is the only eigenvalue of A. If A is diagonalizable, then we must have $A = I$, which is impossible. Hence A is not diagonalizable. □

Question 2

The symmetric group S_n acts on $X := \{x_1, \cdots, x_n\}$ by permuting indices: $\sigma \cdot x_i = x_{\sigma(i)}$ for all $\sigma \in S_n$ and all i. Find all S_n-stable subspaces of the permutation representation $\rho : S_n \to GL(kX)$.

Proof. First we consider the case where $char\, k \nmid n$.

- Following Example 1.20, we observe that $kX = U \oplus V$, where

$$U := \left\{ \sum_{i=1}^n a x_i \in kX : \ a \in k \right\} = \left\langle \sum_{i=1}^n x_i \right\rangle \qquad V := \left\{ \sum_{i=1}^n a_i x_i \in kX : \ \sum_{i=1}^n a_i = 0 \right\}$$

  because every element in kX can be expressed as

$$\sum_{i=1}^n a_i x_i = a \sum_{i=1}^n x_i + \sum_{i=1}^n (a_i - a) x_i, \qquad a := \frac{1}{n} \sum_{i=1}^n a_i$$

  and $U \cap V = \left\{ \sum_i a x_i \in kX : \ na = 0 \right\} = \{0\}$.

- Next we shall show that U and V are S_n-stable:

  The permutation representation $\rho : S_n \to GL(kX)$ is given by:

$$\rho(\sigma)\left(\sum_{i=1}^n a_i x_i\right) := \sum_{i=1}^n a_i \sigma \cdot x_i = \sum_{i=1}^n a_i x_{\sigma(i)}$$

  For $\sum_i a x_i \in U$ and $\sigma \in S_n$, $\rho(\sigma)(\sum_i a x_i) = \sum_i a x_{\sigma(i)} = \sum_i a x_i \in U$. Hence U is S_n-stable.

  For $\sum_i a_i x_i \in V$ and $\sigma \in S_n$, $\rho(\sigma)(\sum_i a_i x_i) = \sum_i a_i x_{\sigma(i)} = \sum_i a_{\sigma^{-1}(i)} x_i$. $\sum_i a_i = 0$ implies that $\sum_i a_{\sigma^{-1}(i)} = 0$. Hence $\rho(\sigma)(\sum_i a_i x_i) \in V$. Hence V is S_n-stable.

- We show that there are no other non-trivial S_n-stable subspaces. We claim that the sub-representation ρ_V is irreducible.

  Note that $\{x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n\}$ is a set of linearly independent vectors in V, because the i-th vector is not in the span of the first $i-1$ vectors for $1 \leqslant i \leqslant n$. Since $\dim U = 1$, $\dim V = n - 1$. So the set is in fact a basis of V.

  Let $W \leqslant V$ be a S_n-stable subspace and $W \neq \{0\}$. Consider $v = \sum_i a_i x_i \in W$. Since $U \cap W = \{0\}$, there exists $i, j \in \{1, ..., n\}$

such that $a_i \neq a_j$. Since W is S_n-stable,

$$v - \rho_V(i\,j)(v) = (a_j - a_i)(x_i - x_j) \in W$$

Hence $x_i - x_j \in W$. Finally,

$$\rho_V((i\,k)(j\;k+1))(x_i - x_j) = x_k - x_{k+1} \in W$$

for all $1 \leqslant k \leqslant n-1$. Hence $W = V$. We deduce that ρ_V is irreducible.

Since $\dim U = 1$, the sub-representation ρ_U is also irreducible.

- We deduce that the S_n-stable subspaces of kX are $\{0\}, U, V, kX$.

Now we consider the case $\operatorname{char} k \mid n$.

- We observe that $U \subseteq V$, because
$$\sum_{i=1}^{n} 1 = n = 0 \implies \sum_{i=1}^{n} x_i \in V \implies U \subseteq V$$
- By the same argument U and V are still S_n-stable, although ρ is no longer completely reducible.
- Let $W \leqslant V$ be a S_n-stable subspace and $W \neq \{0\}$. If $x_i - x_j \in W$ for some $i \neq j$, then by the reasoning above $V \subseteq W$. So either $W = V$ or $W = kX$ since $\dim V = \dim kX - 1$.
- Now assume that W is not V or kX. Then $x_i - x_j \notin W$ for all $i \neq j$. If $v = \sum_i a_i x_i \in W$, then the same reasoning as before leads to $a_i = a_j$ for all indices. Hence $W \subseteq U$. Since $\dim U = 1$, we deduce that $W = U$.
- In conclusion, the S_n-stable subspaces of kX are still $\{0\}, U, V, kX$. □

Question 3

Show that in Example 1.17, the G-stable subspace $\langle v_1 \rangle$ has no G-stable complement in $V = \langle v_1, v_2 \rangle$.

Proof. Suppose that it has a G-stable complement $\langle w \rangle$ for some $w = (a, b)^{\mathrm{T}} \in V \setminus \langle v_1 \rangle$. Then

$$\rho(g^i)(w) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + bi \\ b \end{pmatrix} = \lambda w \implies a + bi = \lambda a \wedge b = \lambda b$$

for some $\lambda \in \mathbb{F}_p$. Since $w \notin \langle v_1 \rangle$, $b \neq 0$. Hence $\lambda = 1$ and $bi = 0$. This is impossible. Hence no such G-stable subspace exists. □

Question 4

Let X be a G-set and suppose that the permutation representation $\rho : G \to \mathrm{GL}(kX)$ is irreducible. Prove that the G-action on X must be transitive. Is the converse true?

Proof. For $x \in X$, we observe that $V = \left\{ \sum_{y \in \mathrm{Orb}(x)} a_y y : \; a_y \in k \right\}$ is a G-stable subspace. This is because

$$\forall\, g \in G \quad \rho(g)\left(\sum_{y \in \mathrm{Orb}(x)} a_y y \right) = \sum_{y \in \mathrm{Orb}(x)} a_y (g \cdot y) \in V \qquad \text{as } g \cdot y \in \mathrm{Orb}(x)$$

If G is not transitive, then $\mathrm{Orb}(x) \neq X$. Then V is a non-trivial G-stable subspace, contradicting that ρ is irreducible.

The converse is not true. Let $\rho : S_n \to \mathrm{GL}(kX)$ be a permutation representation and $\operatorname{char} k = 0$. Then by Question 2 ρ is completely reducible. But S_n acting on $X = \{x_1, \ldots, x_n\}$ is transitive. □

Question 5

For each conjugacy class C in G, define its *conjugacy class sum* to be $\widehat{C} := \sum_{x \in C} x \in kG$. Prove that the conjugacy class sums form a basis for $Z(kG)$.

Proof. First we show that $\widehat{C} \in Z(kG)$. For $g \in G$, we know that $\rho_g : C \to C$, $\rho_g(x) = g^{-1}xg$ is a bijection. For $\sum_{g\in G} a_g g$,

$$\widehat{C}\left(\sum_{g\in G} a_g g\right) = \sum_{x\in C}\sum_{g\in G} a_g xg = \sum_{x\in C}\sum_{g\in G} a_g g\rho_g(x) = \sum_{\rho_g(x)\in C}\sum_{g\in G} a_g g\rho_g(x) = \sum_{x\in C}\sum_{g\in G} a_g gx = \left(\sum_{g\in G} a_g g\right)\widehat{C}$$

Let $G = C_1 \cup \cdots C_k$ be a partition of G into conjugacy classes.

Since different conjugacy classes are disjoint in G, $\{\widehat{C}_1, ..., \widehat{C}_k\}$ are linearly independent. So it remains to show that the conjugacy class sums span $Z(kG)$.

For $\sum_{g\in G} a_g g \in Z(kG)$:

$$\forall\, h \in G: \quad \sum_{g\in G} a_g g = h^{-1}\left(\sum_{g\in G} a_g g\right)h = \sum_{g\in G} a_g (h^{-1}gh) = \sum_{g\in G} a_g \rho_h(g) = \sum_{g\in G} a_{\rho_h^{-1}(g)}\, g$$

Hence $a_g = a_{\rho_h^{-1}(g)}$ for all $g, h \in G$. As a group action, the orbits of ρ are exactly the conjugacy classes of G. Hence we deduce that $a_g = a_h$ if g and h are in the same conjugacy class. Then

$$\sum_{g\in G} a_g g = \sum_{i=1}^{k}\sum_{g\in C_i} a_i g = \sum_{i=1}^{k} a_i \widehat{C}_i$$

Hence $\{\widehat{C}_1, ..., \widehat{C}_k\}$ spans $Z(kG)$. □

Question 6

Suppose that $A = M_n(k)$ be the ring of $n \times n$ matrices with entries in k and let $V := k^n$ be the natural left A-module of $n \times 1$ column vectors.

(a) Prove that V is a simple A-module.

(b) Prove that A has no nonzero proper two-sided ideals.

(c) Exhibit explicit simple left ideals $L_1, \cdots, L_n$ of A such that $A = L_1 \oplus \cdots \oplus L_n$.

(d) Is the decomposition you found in (iii) unique? Justify your answer.

Proof. (a) Suppose that W is a non-zero sub A-module of V. Suppose that $v \in W\setminus\{0\}$. Then for any $u \in V$, $u \in W$ because

$$u = Tv \quad \text{where } T = \frac{1}{\|v\|}uv^{\mathrm{T}} \in M_n(k)$$

Hence $W = V$. We deduce that V is a simple A-module.

(b) Suppose that J is a non-zero two sided ideal of A. Let $B \in A$ such that the (m, p)-th entry of B is $b_{m,p} \neq 0$. Let $E_{i,j} \in M_n(k)$ be such that the (i, j)-th entry of $E_{i,j}$ is $1 \in k$ and all other entries are 0. Then

$$E_{m,p} = \frac{1}{b_{m,p}}E_{m,m}BE_{p,p} \in J$$

Let $F_i, j \in M_n(k)$ be the elementary matrix that exchanges the i-th and j-th rows. Then we have $E_{1,1} = F_{1,m}E_{m,p}F_{p,1} \in J$. Then

$$I = \sum_{i=1}^{n} E_{i,i} = \sum_{i=1}^{n} F_{1,i}E_{1,1}F_{1,i} \in J$$

We deduce that $J = A$. Hence A has no non-trivial two-sided ideals.

(c) Let $L_i = \langle E_{i,i}\rangle_{\text{left}}$ for each i. For $B \in M_n(k)$, $B = BI = \sum_{i=1}^{n} BE_{i,i} \in \sum_{i=1}^{n} L_i$. Hence $A = \sum_{i=1}^{n} L_i$. On the other hand, suppose that $M \in L_i \cap L_j$ where $i \neq j$. Then there exists $B, C \in M_n(k)$ such that $M = BE_{i,i} = CE_{j,j}$. But the entries of $BE_{i,i}$ are zero except at the i-th column, whereae the entries of $CE_{j,j}$ are zero except at the $jq4$-th column. Hence $M = 0$. $L_i \cap L_j = \{0\}$. we deduce that $A = \bigoplus_{i=1}^{n} L_i$.

(d) The decomposition in (iii) is not unique. Consider a general invertible matrix $P \neq I$. Let $K_i = \langle P^{-1}E_{i,i}P\rangle_{\text{left}}$. P can be chosen such that $K_i \neq L_j$ for any j. We still have $B = BP^{-1}IP = \sum_{i=1}^{n} BP^{-1}E_{i,i}P \in \sum_{i=1}^{n} K_i$ for $B \in M_n(k)$. Hence $A = \sum_{i=1}^{n} K_i$. And

$$M \in K_i \cap K_j \implies \exists B, C \in M_n(k) : M = BP^{-1}E_{i,i}P = CP^{-1}E_{j,j}P \implies BP^{-1}E_{i,i} = CP^{-1}E_{j,j} \implies M = 0 \implies K_i \cap K_j = \{0\}$$

Hence $A = \bigoplus_{i=1}^{n} K_i$. □

Question 7

Let A be k-algebra for some field k and let M be a finite dimensional A-module. A *composition series* for M is a finite ascending chain

$$\{0\} = M_0 < M_1 < M_2 < \cdots < M_n = M$$

such that each subquotient M_k/M_{k-1} is a simple A-module for each $k = 1, \cdots, n$. These subquotients are called *composition factors*. Prove the *Jordan-Hölder Theorem*, which states that if

$$\{0\} = N_0 < N_1 < N_2 < \cdots < N_m = M$$

is another composition series for M, then necessarily $m = n$ and there exists a permutation $\sigma \in S_n$ together with A-module isomorphisms

$$M_k/M_{k-1} \xrightarrow{\cong} N_{\sigma(k)}/N_{\sigma(k)-1} \quad \text{for all} \quad k = 1, \cdots, n$$

Deduce that G has only finitely many irreducible representations, up to isomorphism.

Proof. We use induction on the length of the shortest composition series of M. Base case: Suppose that M has composition series $\{0\} < M$. That is, M is simple. If M has another composition series

$$\{0\} = M_0 < M_1 < \cdots < M_n = M$$

such that $n \geq 2$, then M is not simple. Contradiction.

Induction case: Suppose that M two composition series:

$$\{0\} = M_0 < M_1 < M_2 < \cdots < M_n = M \tag{1}$$
$$\{0\} = N_0 < N_1 < N_2 < \cdots < N_m = M \tag{2}$$

where n is the shortest length of composition series of M. So $m \geq n$. If $M_{n-1} = N_{m-1}$, then $M' = M_{n-1} = N_{m-1}$ is a module with shortest length $n-1$. By induction hypothesis $n-1 = m-1$, and the two composition series for M' is equivalent. Hence $n = m$ and the composition series for M is equivalent.

Now suppose that $M_{n-1} \neq N_{m-1}$. If $M_{n-1} \subsetneq N_{m-1}$, then by third isomorphism theorem

$$\frac{M/M_{n-1}}{N_{m-1}/M_{n-1}} \cong \frac{M}{N_{m-1}}$$

contradicting that M/M_{n-1} is simple. Similarly we cannot have $N_{m-1} \subsetneq M_{n-1}$. Hence $N_{m-1} \subsetneq N_{m-1} + M_{n-1}$. Since M/N_{m-1} is simple, we must have $M = N_{m-1} + M_{n-1}$.

Let $P = M_{n-1} \cap N_{m-1} < M$. By second isomorphism theorem, $M/M_{n-1} \cong N_{m-1}/P$ and $M/N_{m-1} \cong M_{n-1}/P$. Hence N_{m-1}/P and M_{n-1}/P are simple. P has a composition series:

$$\{0\} = P_0 < P_1 < \cdots < P_{p-1} < P_p = P \tag{3}$$

Then

$$\{0\} = P_0 < P_1 < \cdots < P_{p-1} < P_p < M_{n-1} \tag{4}$$
$$\{0\} = M_0 < M_1 < M_2 < \cdots < M_{n-1} \tag{5}$$

are two composition series of M_{n-1} of length $p+1$ and $n-1$ respectively. By induction hypothesis, $p = n-2$ and the composition series (4) and (5) are equivalent. Similarly, the composition series of N_{m-1}

$$\{0\} = P_0 < P_1 < \cdots < P_{p-1} < P_p < N_{m-1} \tag{6}$$
$$\{0\} = N_0 < M_1 < N_2 < \cdots < N_{m-1} \tag{7}$$

are equivalent, and $p = m-2$. We deduce that $m = n$. Finally, the composition series of M

$$\{0\} = P_0 < P_1 < \cdots < P_{p-1} < P_p < M_{n-1} < M \tag{8}$$
$$\{0\} = P_0 < P_1 < \cdots < P_{p-1} < P_p < N_{m-1} < M \tag{9}$$

are equivalent, because $M/M_{n-1} \cong N_{m-1}/P$ and $M/N_{m-1} \cong M_{n-1}/P$. Since (4) is equivalent to (5), then (8) is equivalent to (1). SInce (6) is equivalent to (7), then (9) is equivalent to (2). We conclude that the composition series (1) and (2) are equivalent.

Let M be a simple kG-module. Fix $m \in M\backslash\{0\}$. Let $f_m : kG \to M$ given by $f_m(a) = a \cdot m$. It is clear that f_m is a kG-module homomorphism. By first isomorphism theorem, $kG/\ker f_m \cong \operatorname{im} f_m = M$ because M is simple. Then $\ker f_m$ is a sub kG-module of kG, and hence is a left ideal of kG. Hence kG has a composition series of the form

$$\{0\} < M_1, \cdots < M_n = \ker f_m < A$$

We deduce that M is a composition factor of kG. Since kG has finite length, it has finitely many simple kG-modules up to isomorphism.

Note that there is a bijective correspondence between the ireducible representations of G and simple kG-modules (up to isomorphism). We conclude that G has finitely many irreducible representations. □