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Problem Sheet 1

Scattering Theory

B4: Subatomic Physics

30 October, 2020

92% + 1 missing question
great!

Question 1

In their 1909 paper (Roy. Soc. Proc. A, vol. 82, p. 495,1909) Geiger & Marsden reported on their studies of backscattering of α-particles from the decay of ^{214} Bi from thin metal foils and noted "...that of the incident α-particles about 1 in 8000 was reflected..." from a thin platinum foil. The number of reflected α-particles was estimated from the count rate on a screen covering only a small part of the backward solid angle but "...it was assumed that they [the scattered α-particles] were distributed uniformly round a half sphere with the middle of the reflector [the metal foil] as centre." The observation of a large fraction of backscattered α-particles from a thin foil led Rutherford to suggest his scattering formula and his model for the atom two years later.

(a) Integrate the differential cross-section in the Rutherford scattering formula over the backward hemisphere to find the cross-section for backscattering with $\theta > 90°$ and hence predict the ratio of of N_b/N_{in}, where N_{in} is the number of incoming and N_b is the number of backscattered α-particles. Compare your result with Geiger & Marsden's observations assuming the foil was 1μm thick.

(b) Estimate the number of atoms which the α-particle has to pass when it penetrates the foil.

(c) The paper is unclear about the rate of α particles hitting the foil, but you can estimate it using the assumption that one scattered α-particle per second was detected on a screen with an area of 1mm^2 at a distance of 1cm from the foil at a scattering angle of 120°.

(d) Later measurements of the differential cross-section with much improved angular resolution showed no deviation from the Rutherford scattering formula up to a scattering angle of 140°. From this observation estimate an upper limit for the radius of the platinum nucleus.

[$\rho_{Pt} = 21.5$ g/cm^3, atomic mass 195.1 u, $E_\alpha = 5.6$ MeV and you might find useful that $e^2/(4\pi\epsilon_0) = 1.44$ MeV fm]

Solution. (a) Integrating the differential cross-section to get the cross-section is a circular argument. The relation we obtain by conservation of angular momentum and conservation of energy is

$$b = \frac{Zze^2}{4\pi\epsilon_0 m v_0^2}\cot\frac{\theta}{2} = \frac{Zze^2}{8\pi\epsilon_0 E_\alpha}\cot\frac{\theta}{2}$$

where b is the impact parameter, E_α is the kinetic energy of the α-particle, and θ is the scattering angle. Then

$$b(\theta = \pi/2) = \frac{Zze^2}{8\pi\epsilon_0 E_\alpha} = \frac{78\cdot 2}{2\cdot 5.6\text{ MeV}}\cdot 1.44\text{ MeV fm} = 20.1\text{ fm}$$

The cross-section

$$\sigma(\theta > \pi/2) = \pi b^2(\theta = \pi/2) = 1264\text{ fm}^2$$

Next we estimate the radius of platinum atom. We have

$$\rho_{Pt} V = \mu\frac{V}{V_{Pt} N_A}$$

where $\mu = 195.1$ g/mol is the molar mass, V_{Pt} is the volume of a platinum atom, and N_A is the Avogadro number. The radius of the platinum atom is estimated by

$$74\%\cdot V_{Pt} = \frac{4}{3}\pi r_{Pt}^3$$

(74% comes from the atomic packing factor of the face-centered cubic.)

Plugging in the numbers we obtain

$$r_{Pt} \sim \left(\frac{3\mu}{4\pi\rho_{Pt} N_A}\right)^{1/3} = 1.38\cdot 10^5\text{ fm}$$

Then the ratio of scattered particles of a single platinum atom is given by

$$\left(\frac{b(\theta = \pi/2)}{r_{Pt}}\right)^2 = 2.10\cdot 10^{-8}$$

We also have to estimate the number of atoms which the α-particle has to pass when it penetrates the foil. This is given by

$$N_d = \frac{1\mu\mathrm{m}}{\sqrt{2}r_{\mathrm{Pt}}} = 4615 \quad \approx$$

Then the ratio of scattered particles of the foil is given by

$$\frac{N_{\mathrm{b}}}{N_{\mathrm{in}}} \sim N_d \left(\frac{b(\theta=\pi/2)}{r_{\mathrm{Pt}}}\right)^2 = 9.68\cdot 10^{-5} \simeq \frac{1}{10302}$$ 2+

This prediction is very closed to the experimental result. 2

(b) See part (a).

(c) The differential cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Zze^2}{4\pi\varepsilon_0 E_\alpha}\right)^2 \frac{1}{16\sin^4(\theta/2)} \implies \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\theta=2\pi/3} = 178.7\ \mathrm{fm}^2$$

The solid angle density rate at $\theta = 2\pi/3$:

$$n(2\pi/3) = 1\cdot\frac{4\pi\cdot 1\ \mathrm{cm}^2}{1\ \mathrm{mm}^2} = 1257$$

Hence the rate of α-particle is 2

$$j_{\mathrm{in}} = n\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{-1} = 7.0\ \mathrm{fm}^{-2}$$

(d) We take

8/8

$$b(\theta=140^\circ) = \frac{Zze^2}{8\pi\varepsilon_0 E_\alpha}\cot(70^\circ) = 7.3\ \mathrm{fm}$$ 2

as the estiamte of an upper bound of the nucleus radius. □

Question 2

Electron scattering on hydrogen atoms:

(a) Calculate the Coulomb potential of a hydrogen atom (a bound system of one proton and one shell electron). Throughout the problem you can assume that the proton has infinite mass and is at the origin. The hydrogen atom is in the ground state (the wavefunction of the hydrogen atom is given by $\Psi_{100}(\boldsymbol{r}_e) = \left(\pi a_0^3\right)^{-1/2} e^{-r_e/a_0}$, with the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/\left(me^2\right)$). The potential energy due to the shell electron can then be found from

$$V_{\mathrm{shell}}(\boldsymbol{r}) = \frac{e^2}{4\pi\epsilon_0}\int \frac{|\Psi_{100}(\boldsymbol{r}_e)|^2}{|\boldsymbol{r}_e - \boldsymbol{r}|}\mathrm{d}^3 r_e$$

[*Hint: Use* $|\boldsymbol{r}_e - \boldsymbol{r}| = \sqrt{r^2 + r_e^2 - 2rr_e\cos\theta}$ *and evaluate the integral in spherical polar coordinates.*]

Sketch the potential. What is the behaviour of the potential for the hydrogen atom for $r\to\infty$? Why is this relevant? [Solution: $V_{\mathrm{H}}(r) = -e^2/(4\pi\epsilon_0)\, e^{-2r/a_0}\left(a_0^{-1} + r^{-1}\right)$.]

(b) Use the equation for the differential cross-section from first order perturbation theory (first order Born approximation) which we derived in the lectures to find the differential cross-section as a function of the scattering angle θ. Plot your results for different values of ka_0 where k is the wavenumber of the incoming electron. How would the differential cross-section for scattering on anti-hydrogen differ in first order Born approximation?

(c) What will be the θ-dependence for low energy scattering ($ka_0 \ll 1$)? Which value does the cross-section tend to for $ka_0 \ll 1$? Is this the value you would expect?

(d) What is the lowest value for the energy of the incoming electron for which you expect the first order approximation we are using here to be valid?

Solution. (a) The potential energy due to shell electron

$$V_{\mathrm{shell}}(r) = \frac{e^2}{4\pi\epsilon_0}\int\frac{|\Psi_{100}(\boldsymbol{r}_e)|^2}{|\boldsymbol{r}_e - \boldsymbol{r}|}\mathrm{d}^3 r_e = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\pi a_0^3}\int_{\mathbb{R}^3}\frac{e^{-2r_e/a_0}}{\sqrt{r^2 + r_e^2 - 2rr_e\cos\theta}}\mathrm{d}\boldsymbol{r}_e$$

$$= \frac{e^2}{2\pi\varepsilon_0 a_0^3}\int_0^\infty \mathrm{d}r_e\, r_e^2\,\mathrm{e}^{-2r_e/a_0}\int_0^\pi \mathrm{d}\theta\,\frac{\sin\theta}{\sqrt{r^2+r_e^2-2rr_e\cos\theta}}$$

where

$$\int_0^\pi \frac{\sin\theta}{\sqrt{r^2+r_e^2-2rr_e\cos\theta}}\,\mathrm{d}\theta = \frac{1}{2rr_e}\int_{\theta=0}^{\pi}\frac{1}{\sqrt{s}}\,\mathrm{d}s = \frac{1}{rr_e}\sqrt{r^2+r_e^2-2rr_e\cos\theta}\,\Big|_{\theta=0}^{\pi}$$
$$= \frac{1}{rr_e}\left(\sqrt{r^2+r_e^2+2rr_e}-\sqrt{r^2+r_e^2-2rr_e}\right) \quad \checkmark$$
$$= \frac{1}{rr_e}\left(|r+r_e|-|r-r_e|\right) = \begin{cases} 2/r_e, & r<r_e \\ 2/r, & r\geqslant r_e\end{cases}$$

Then

$$V_{\text{shell}}(r) = \frac{e^2}{\pi\varepsilon_0 a_0^3}\left(\int_0^r \frac{r_e^2}{r}\,\mathrm{e}^{-2r_e/a_0}\,\mathrm{d}r_e + \int_r^\infty r_e\,\mathrm{e}^{-2r_e/a_0}\,\mathrm{d}r_e\right)$$
$$= \frac{e^2}{\pi\varepsilon_0 a_0^3}\left(\int_0^r \frac{a_0^3}{r}t^2\,\mathrm{e}^{-2t}\,\mathrm{d}t + \int_r^\infty a_0^2 t\,\mathrm{e}^{-2t}\,\mathrm{d}t\right)$$
$$= \frac{e^2}{\pi\varepsilon_0 a_0^3}\left(\frac{a_0^3}{r}\left(-\frac{1}{4}\mathrm{e}^{-2t}(2t^2+2t+1)\right)_0^{r/a_0} + a_0^2\left(-\frac{1}{4}\mathrm{e}^{-2t}(2t+1)\right)_{r/a_0}^{\infty}\right)$$
$$= \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1-\mathrm{e}^{2r/a_0}\left(\frac{2r^2}{a_0^2}+\frac{2r}{a_0}+1\right)\right)+\frac{1}{a_0}\mathrm{e}^{-2r/a_0}\left(\frac{2r}{a_0}+1\right)\right)$$
$$= \frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0}\mathrm{e}^{-2r/a_0}\left(\frac{1}{r}+\frac{1}{a_0}\right)$$

The total potential due to the hydrogen atom is

$$V_H(r) = V_{\text{shell}}(r) + V_{\text{nucleus}}(r) = \frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0}\mathrm{e}^{-2r/a_0}\left(\frac{1}{r}+\frac{1}{a_0}\right) - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{e^2}{4\pi\varepsilon_0}\mathrm{e}^{-2r/a_0}\left(\frac{1}{r}+\frac{1}{a_0}\right) \quad \checkmark$$

Note that $V_H(r) = O(\exp(-2r/a_0))$ as $r\to\infty$. This is an exponential decay, which is a "localized potential" according to the lecture notes, as $V_H(r) = o(r^{-1})$ as $r\to\infty$. The sketch of $V_H(r)$: ✓

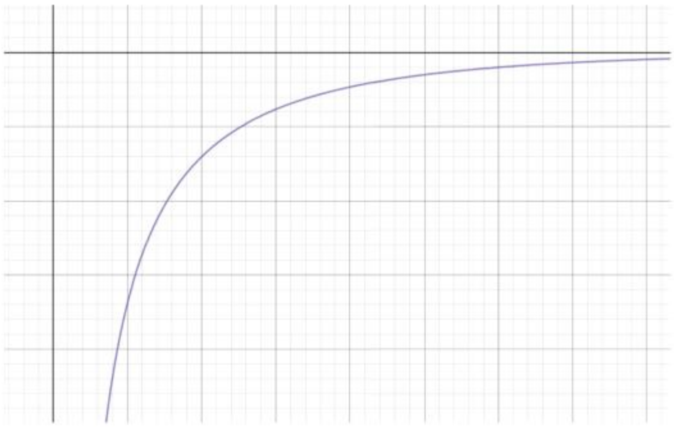

Figure 1:

2

(b) The differential cross-section given by the first order Born approximation is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}^{(1)} = \left(\frac{m}{2\pi\hbar^2}\right)^2\left|\int \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\,V_H(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}\right|^2$$

In spherical coordinates,

$$\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\,V_H(\boldsymbol{r})\,d^3\boldsymbol{r} = \iiint V_H(r)\,e^{i\|\boldsymbol{k}-\boldsymbol{k}'\| r\cos\xi}\,r^2\sin\xi\,dr\,d\xi\,d\varphi = 2\pi\int_0^\infty V_H(r)\frac{e^{i\|\boldsymbol{k}-\boldsymbol{k}'\|r}-e^{-i\|\boldsymbol{k}-\boldsymbol{k}'\|r}}{i\|\boldsymbol{k}-\boldsymbol{k}'\|r}r^2\,dr$$
$$= \frac{4\pi}{\|\boldsymbol{k}-\boldsymbol{k}'\|}\int_0^\infty V_H(r)\,r\sin(\|\boldsymbol{k}-\boldsymbol{k}'\|\,r)\,dr = \frac{2\pi}{k\sin(\theta/2)}\int_0^\infty V_H(r)\,r\sin\left(2kr\sin\frac{\theta}{2}\right)dr$$

where θ is the scattering angle. Plugging in the expression of $V_H(r)$:

$$\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\,V_H(\boldsymbol{r})\,d^3\boldsymbol{r} = \frac{2\pi}{k\sin(\theta/2)}\int_0^\infty -\frac{e^2}{4\pi\varepsilon_0}e^{-2r/a_0}\left(\frac{1}{r}+\frac{1}{a_0}\right)r\sin\left(2kr\sin\frac{\theta}{2}\right)dr$$
$$= -\frac{e^2a_0^2}{\varepsilon_0\alpha}\int_0^\infty e^{-2s}(1+s)\sin(\alpha s)\,ds \qquad \left(s := \frac{r}{a_0},\ \alpha := 2ka_0\sin\frac{\theta}{2}\right)$$
$$= -\frac{e^2a_0^2}{\varepsilon_0\alpha}\,\mathrm{Im}\left(\int_0^\infty (1+s)\,e^{-(2-i\alpha)s}\,ds\right)$$
$$= -\frac{e^2a_0^2}{\varepsilon_0\alpha}\,\mathrm{Im}\left(\frac{1}{(2-i\alpha)^2}+\frac{1}{2-i\alpha}\right) = -\frac{e^2a_0^2}{\varepsilon_0\alpha}\frac{\alpha(\alpha^2+8)}{(\alpha^2+4)^2}$$
$$= -\frac{e^2a_0^2}{\varepsilon_0}\frac{(2ka_0\sin(\theta/2))^2+8}{\left((2ka_0\sin(\theta/2))^2+4\right)^2} = -\frac{e^2a_0^2}{2\varepsilon_0}\frac{k^2a_0^2(1-\cos\theta)+4}{\left(k^2a_0^2(1-\cos\theta)+2\right)^2}$$

Hence the differential cross-section is given by

$$\frac{d\sigma}{d\Omega}^{(1)}(\theta) = \left(\frac{m}{2\pi\hbar^2}\frac{e^2a_0^2}{2\varepsilon_0}\frac{k^2a_0^2(1-\cos\theta)+4}{\left(k^2a_0^2(1-\cos\theta)+2\right)^2}\right)^2 = a_0^2\frac{\left(k^2a_0^2(1-\cos\theta)+4\right)^2}{\left(k^2a_0^2(1-\cos\theta)+2\right)^4}$$

The sketch of $\frac{d\sigma}{d\Omega}^{(1)}(\theta)$ with different value of ka_0:

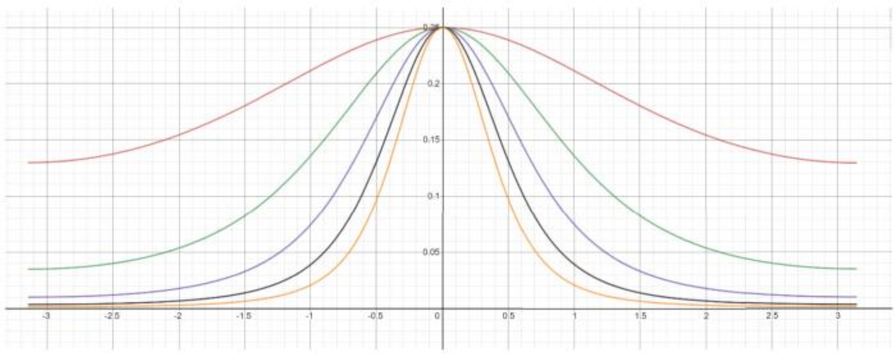

Figure 2: The graph gets lower as ka_0 gets larger.

For a anti-hydrogen atom, the potential energy $\tilde{V}_H(r) = -V_H(r)$. The differential cross-section is still the same since there is a square in the formula. 2

(c) For $ka_0 \ll 1$, we have $\left(k^2a_0^2(1-\cos\theta)+2\right)^{-4} \sim 2^{-4}\left(1-2k^2a_0^2(1-\cos\theta)\right)$. Then

$$\frac{d\sigma}{d\Omega}^{(1)}(\theta) \sim \frac{1}{16}a_0^2\left(k^2a_0^2(1-\cos\theta)+4\right)^2\left(1-2k^2a_0^2(1-\cos\theta)\right) \sim a_0^2\left(1-\frac{3}{2}k^2a_0^2(1-\cos\theta)\right)+O\left(k^4a_0^4\right)$$

In particular, $\lim_{k\to 0}\frac{d\sigma}{d\Omega}^{(1)}(\theta) = a_0^2$. $\to \sigma \approx 4\pi a_0^2$ ("sphere")

2

7/8

(d) The condition that the first order perturbation is applicable is

$$\left|\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} V_H(\boldsymbol{r})\, d^3\boldsymbol{r}\right| \ll \frac{\hbar k}{m a_0} \implies \frac{k\left(k^2 a_0^2(1-\cos\theta)+2\right)^2}{k^2 a_0^2(1-\cos\theta)+4} \gg 2\pi a_0^2$$ □

↳ can calc. an average $|\bar{V}|$

1

Question 3

Density of states for a two-body final state:

(a) Show that for two distinguishable particles A and B the momentum associated with the centre-of-mass motion is $\boldsymbol{P} = \boldsymbol{p}_A + \boldsymbol{p}_B$, and for the relative motion is $\boldsymbol{p} = (m_A\boldsymbol{p}_B - m_B\boldsymbol{p}_A)/(m_A + m_B)$. Hence show $d^3P d^3p = d^3p_A d^3p_B$ and that the number of $(\boldsymbol{P}, \boldsymbol{p})$ values with P in the range $(P, P + dP)$ and p in the range $(p, p + dp)$ is

$$\left(V/(2\pi\hbar)^3\right)^2 P^2 dP p^2 dp \int d\Omega_P \int d\Omega_p$$

(b) Use this result to show that the density of states for a two-body final state in a scattering event with the constraint $P = P_0 = \text{const}$ is given by

$$\rho(E) = \frac{V p_A^{*2}}{(2\pi\hbar)^3}\frac{dp_A^*}{dE}\int d\Omega = \frac{V p_B^{*2}}{(2\pi\hbar)^3}\frac{dp_B^*}{dE}\int d\Omega$$

where p_A^* and p_B^* are the momenta of particles A and B, respectively, in the centre-of-mass frame, and Ω the direction of the momentum associated with the relative motion.

(c) Hence show that for the decay process $M \to A + B$ the density of states is given by

$$\rho(E) = \frac{V p}{2\pi^2\hbar^3}\frac{E_A E_B}{c^2 E_M}$$

where $E_M = m_M c^2, E_A$ and E_B are the energies of the three particles involved and $p = |\boldsymbol{p}_A^*| = |\boldsymbol{p}_B^*|$ is the value of the momentum of any of the two daughter particles in the centre-of-mass system, and assuming that the decay is isotropic.

(d) What would change in this result if the daughter particles would be indistinguishable?

Solution. (a) Suppose that we are treating the non-relativistic case. The centre-of-mass velocity is

$$\boldsymbol{V} = \frac{m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B}{m_A + m_B}$$

Then the momentum of the centre of mass is

$$\boldsymbol{P} = (m_A + m_B)\boldsymbol{V} = m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B = \boldsymbol{p}_A + \boldsymbol{p}_B$$

In the centre-of-mass frame, the system has zero momentum. The relative momentum

$$\boldsymbol{p} = m_A(\boldsymbol{V} - \boldsymbol{v}_A) = m_A\frac{m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B}{m_A + m_B} - m_A\boldsymbol{v}_A = \frac{m_A m_B}{m_A + m_B}(\boldsymbol{v}_B - \boldsymbol{v}_A) = \frac{m_A\boldsymbol{p}_B - m_B\boldsymbol{p}_A}{m_A + m_B}$$

(These formulae also hold for the relativistic case, in which the centre-of-mass frame is defined to be the frame where the system has zero momentum.)

The Jacobian of the transformation $(\boldsymbol{p}_A, \boldsymbol{p}_B) \mapsto (\boldsymbol{P}, \boldsymbol{p})$ is

$$\det\frac{\partial(\boldsymbol{P}, \boldsymbol{p})}{\partial(\boldsymbol{p}_A, \boldsymbol{p}_B)} = \det\begin{pmatrix} I_3 & I_3 \\ \frac{-m_B}{m_A+m_B}I_3 & \frac{m_A}{m_A+m_B}I_3 \end{pmatrix} = 1$$

2

Hence $d^3\boldsymbol{P}d^3\boldsymbol{p} = d^3\boldsymbol{p}_A d^3\boldsymbol{p}_B$.

From statistical mechanics we know that the density of state in the $\boldsymbol{k}$-space is $\frac{Vk^2}{2\pi^2} = \frac{Vk^2}{(2\pi)^3}\int d\Omega$. The density of state in the $(\boldsymbol{P},\boldsymbol{p})$-space is

$$\left(\frac{V}{(2\pi\hbar)^3}\right)^2 P^2p^2\int \mathrm{d}\Omega_P\int \mathrm{d}\Omega_p$$

(b) For $P=P_0$ held constant, the number of states is given by

$$N=\iint_{P=P_0}\mathrm{d}P\mathrm{d}p\left(\frac{V}{(2\pi\hbar)^3}\right)^2P^2p^2\int\mathrm{d}\Omega_P\int\mathrm{d}\Omega_p=\int\mathrm{d}p\,\frac{V}{(2\pi\hbar)^3}p^2\int\mathrm{d}\Omega_p=\int\mathrm{d}E\,\frac{V}{(2\pi\hbar)^3}p^2\frac{\mathrm{d}p}{\mathrm{d}E}\int\mathrm{d}\Omega_p$$

Since $p=p_A^*=p_B^*$, the density of states

$$\rho(E)=\frac{Vp_A^{*2}}{(2\pi\hbar)^3}\frac{\mathrm{d}p_A^*}{\mathrm{d}E}\int\mathrm{d}\Omega=\frac{Vp_B^{*2}}{(2\pi\hbar)^3}\frac{\mathrm{d}p_B^*}{\mathrm{d}E}\int\mathrm{d}\Omega$$

2

(c) In the relativistic case, the energy and momentum is related by

$$E_i^2=m_i^2c^4+p_i^2c^2\implies E_i\mathrm{d}E_i=2c^2p_i\mathrm{d}p_i$$

In the decay process, $E_M=E_A+E_B$ and $p_A=p_B=p=p_A^*=p_B^*$. We have

$$\mathrm{d}E_M=\mathrm{d}E_A+\mathrm{d}E_B=\frac{2c^2p}{E_A}\mathrm{d}p_A+\frac{2c^2p}{E_B}\mathrm{d}p_B=\frac{2c^2pE_M}{E_AE_B}\mathrm{d}p\implies\frac{\mathrm{d}p}{\mathrm{d}E}=\frac{E_AE_B}{2c^2pE_M}$$

Substituting into the result of (b), we obtain

$$\rho(E)=\frac{Vp^2}{(2\pi\hbar)^3}\frac{E_AE_B}{2c^2pE_M}\int\mathrm{d}\Omega=\frac{Vp}{2\pi^2\hbar^3}\frac{E_AE_B}{c^2E_M}$$

2

6/7

(d) If A and B are indistinguishable, then (E_A,E_B) and (E_B,E_A) are the same state. The density of state should be ~~doubled~~:

$$\rho(E)=\frac{Vp}{\pi^2\hbar^3}\frac{E_AE_B}{c^2E_M}$$

$\times\frac{1}{2}$ □

Question 4

The Klein-Gordon equation is a quantum mechanical wave equation compatible with special relativity. This equation can be written as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{m^2c^2}{\hbar^2}\right)\Phi(\boldsymbol{x},t)=0$$

(a) Let $\Phi(\boldsymbol{x},t)=\exp(-iEt/\hbar+i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar)$ define a quantum mechanical plane wave solution describing a relativistic particle of mass m. Show that this solution satisfies the Klein-Gordon equation.

(b) Define the 4-vector $k^\mu=(E/c\hbar,\boldsymbol{p}/\hbar)$. Show that the 4-scalar product $k\cdot x=k^\mu x_\mu$, where $x_\mu=(ct,-\boldsymbol{x})$, is dimensionless.

(c) The Green's function for the Klein-Gordon equation is defined by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{m^2c^2}{\hbar^2}\right)G\left(x'-x\right)=\delta^4\left(x'-x\right)=\delta\left(\boldsymbol{x}'-\boldsymbol{x}\right)\delta\left(t'-t\right)$$

Show that $G\left(x'-x\right)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x'-x)}G(k)$ with

$$G(k)=-\frac{1}{(k^0)^2-\boldsymbol{k}\cdot\boldsymbol{k}-m^2c^2/\hbar^2}$$

satisfies this definition. (Hint: use the following definition for the Dirac function in 4d:

$$\delta^4\left(x'-x\right)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x'-x)})$$

Proof. (a) $\Phi(\boldsymbol{x},t) = \mathrm{e}^{-\mathrm{i}Et/\hbar + \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\mathrm{e}^{-\mathrm{i}\frac{Et}{\hbar}+\mathrm{i}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}} = \left(\frac{1}{c^2}\left(-\mathrm{i}\frac{E}{\hbar}\right)^2 - \sum_{j\in\{x,y,z\}}\left(\mathrm{i}\frac{p_j}{\hbar}\right)^2 + \frac{m^2c^2}{\hbar^2}\right)\mathrm{e}^{-\mathrm{i}\frac{Et}{\hbar}+\mathrm{i}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}} = \frac{-E^2+p^2c^2+m^2c^4}{c^2\hbar^2}\mathrm{e}^{-\mathrm{i}\frac{Et}{\hbar}+\mathrm{i}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}} = 0$$ √ 2

Hence the plane wave satisfies the Klein-Gorden equation.

(b) The 4-scalar product (technically this is not a scalar product as the Minkowski metric is not positive definite)

$$k^\mu x_\mu = \frac{E}{c\hbar}\cdot ct - \boldsymbol{x}\cdot\frac{\boldsymbol{p}}{\hbar} = \frac{Et - \boldsymbol{x}\cdot\boldsymbol{p}}{\hbar}$$

which is dimensionless, if we observe that $\boldsymbol{x}\cdot\boldsymbol{p}$ has the same dimension as $\hbar$. (This is true indeed by Heisenberg's Uncertainty Principle.) √ 2

(c) Starting from

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)G\left(x'-x\right) = \delta^4\left(x'-x\right)$$

We have the Fourier inversion formulae:

$$G(x'-x) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\widehat{G}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\,\mathrm{d}^4\boldsymbol{k}, \qquad \delta^4(x'-x) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\,\mathrm{d}^4\boldsymbol{k}$$

Then

$$\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\widehat{G}(\boldsymbol{k})\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\,\mathrm{d}^4\boldsymbol{k} = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\,\mathrm{d}^4\boldsymbol{k}$$

By injectivity of Fourier transform we can equate the integrand:

$$\begin{aligned}
&\widehat{G}(\boldsymbol{k})\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)} = \mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\\
\Longrightarrow\ &\widehat{G}(\boldsymbol{k})\frac{E^2-p^2c^2+m^2c^4}{c^2\hbar^2}\mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)} = \mathrm{e}^{-\mathrm{i}k^\mu(x'_\mu - x_\mu)}\\
\Longrightarrow\ &\widehat{G}(\boldsymbol{k}) = \frac{c^2\hbar^2}{E^2-p^2c^2+m^2c^4} = \frac{1}{k^\mu k_\mu + m^2c^2/\hbar^2}
\end{aligned}$$ √ □ 3

7/7

Question 5

Write expressions for the following quantities in natural units ($\hbar = c = 1$, and using powers of eV) , and evaluate them in i) natural units, and ii in practical units ($[E] = eV$, $[m] = eV/c^2$, $[t] = s$, $[l] = m$)

(a) The reduced Compton wavelength of the proton.

(b) The lifetime of a particle which decays after it propagated by 1 mm, if it moved at 90% of the speed of light.

(c) The mass of a particle which gets produced in the head-on collision of an electron and a positron, both with an energy of 500 GeV.

(d) The radius of the trajectory of a particle with charge e with a momentum of 1 GeV/c in a magnetic field of 1 T.

Solution. In natural units, we have:

$$\begin{aligned}
1\ eV &= 1.785\cdot 10^{-36}\ \mathrm{kg}\\
1\ (eV)^{-1} &= 6.586\cdot 10^{-16}\ \mathrm{s}\\
1\ (eV)^{-1} &= 1.973\cdot 10^{-7}\ \mathrm{m}
\end{aligned}$$

(a) The reduce Compton wavelength is given by $\lambda = \dfrac{\hbar}{mc}$. In natural units, the reduce Compton wavelength of proton is

$$\lambda = \frac{1}{m_p} = \frac{1}{938\ \mathrm{MeV}} = 1.07\cdot 10^{-9}\ (eV)^{-1}$$ √

2

In SI units:

$$\lambda = 1.07 \cdot 10^{-9} \cdot 1.973 \cdot 10^{-7}\ \text{m} = 2.1 \cdot 10^{-16}\ \text{m}$$

(b) In the lab frame, the lifetime of the particle is $t = \frac{\ell}{c} = \frac{1\ \text{mm}}{90\% \cdot 3 \cdot 10^{8}\ \text{m/c}} = 3.7 \cdot 10^{-12}$ s. In the frame of the particle, the lifetime is $\tau = \frac{t}{\sqrt{1 - v^2/c^2}} = 8.5 \cdot 10^{-12}$ s.

β

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In the natural units, these numbers are

$$t = \frac{3.7 \cdot 10^{-12}}{6.586 \cdot 10^{-16}}\ (\text{eV})^{-1} = 5.6 \cdot 10^{3}\ (\text{eV})^{-1} \qquad \tau = \frac{8.5 \cdot 10^{-12}}{6.586 \cdot 10^{-16}}\ (\text{eV})^{-1} = 1.3 \cdot 10^{4}\ (\text{eV})^{-1}$$

2

(c) The produced particle has mass $m = 1000\ \text{GeV}/c^2$. In natural units, $m = 1000$ GeV.

(d) The radius is given by

7/8

$$\frac{mv^2}{r} = evB \implies r = \frac{mv}{eB} = \frac{p}{eB}$$

In practical units, $r = \frac{1\ \text{GeV}/c}{1\ \text{T} \cdot e} = \frac{10^{9}}{3 \cdot 10^{8}}\ \text{m} = 3.33$ m. In natural units, $r = \frac{3.33}{1.973 \cdot 10^{-7}}\ (\text{eV})^{-1} = 1.689 \cdot 10^{-7}\ (\text{eV})^{-1}$. □ 2