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Problem Sheet 2

C3.1: Algebraic Topology

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Convention: All spaces are topological spaces. Maps of spaces are always continuous.

Question 1

Show that chain homotopy of chain maps $C_\bullet \to \widetilde{C}_\bullet$ is an equivalence relation.

Proof. Suppose that $(C_\bullet, \partial_\bullet)$ and $(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet)$ are two chain complexes in an Abelian category A.

- Reflectivity:

  Let $f_\bullet : C_\bullet \to \widetilde{C}_\bullet$ be a chain map. We take $h_\bullet : C_\bullet \to \widetilde{C}[1]_\bullet$ defined by $h_n = 0$ for all $n \in \mathbb{N}$. Then trivially we have
  $$f_n - f_n = h_{n-1} \circ \partial_n + \widetilde{\partial}_{n+1} \circ h_n$$
  Hence $f_\bullet \simeq f_\bullet$.

- Symmetry:

  Let $h_\bullet$ be a chain homotopy from $f_\bullet$ to $g_\bullet$. Then $-h_\bullet$ is a chain homotopy from $g_\bullet$ to $f_\bullet$.

- Transitivity:

  Let $h_\bullet$ be a chain homotopy from $f_\bullet$ to $f'_\bullet$, and $h'_\bullet$ be a chain homotopy from $f'_\bullet$ to $f''_\bullet$. Then $h_\bullet + h'_\bullet$ is a chain homotopy from $f_\bullet$ to $f''_\bullet$. □

Question 2

Show that the relative homology $H_1(\mathbb{R}, \mathbb{Q})$ of the pair $\mathbb{Q} \subseteq \mathbb{R}$ is a free Abelian group, and find a basis.

Proof. From the short exact sequence of chain maps

$$0 \longrightarrow C_\bullet(\mathbb{Q}) \longrightarrow C_\bullet(\mathbb{R}) \longrightarrow C_\bullet(\mathbb{R}, \mathbb{Q}) \longrightarrow 0$$

We can construct the long exact sequence of reduced homology groups

$$\cdots \longrightarrow \widetilde{H}_1(\mathbb{R}) \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\delta} \widetilde{H}_0(\mathbb{Q}) \longrightarrow \widetilde{H}_0(\mathbb{R}) \longrightarrow \cdots$$

Since $\mathbb{R}$ is contractible, we have $\widetilde{H}_0(\mathbb{R}) = 0$ and $\widetilde{H}_1(\mathbb{R}) = 0$. By exactness we have $H_1(\mathbb{R}, \mathbb{Q}) \cong \widetilde{H}_0(\mathbb{Q})$. Since the path components of $\mathbb{Q}$ are the singletons, we have

$$H_0(\mathbb{Q}) = \bigoplus_{x \in F(\mathbb{Q})} \mathbb{Z}x$$

where $F : \mathsf{Ab} \to \mathsf{Set}$ is the forgetful functor.

From Sheet 1 Question 4 we have $H_0(\mathbb{Q}) \cong \widetilde{H}_0(\mathbb{Q}) \oplus \mathbb{Z}$. Hence $H_1(\mathbb{R}, \mathbb{Q})$ is isomorphic to a direct summand of the free Abelian group $H_0(\mathbb{Q})$. $H_1(\mathbb{R}, \mathbb{Q})$ is the free Abelian group $\mathbb{Z}^{\oplus \mathbb{N}}$.

We fix a base point $x_0 \in F(\mathbb{Q})$. Then $H_1(\mathbb{R}, \mathbb{Q})$ is generated by the basis $\{[x_0, x] : x \in F(\mathbb{Q})\}$. □

Question 3

In the course notes, from a short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ we built the long exact sequence (LES):

$$\cdots \longrightarrow H_\bullet(A) \xrightarrow{i_\bullet} H_\bullet(B) \xrightarrow{\pi_\bullet} H_\bullet(C) \xrightarrow{\delta} H_{\bullet-1}(A) \xrightarrow{i_\bullet[-1]} \cdots$$

In the notes we showed exactness at $H_\bullet(C)$.

> Prove exactness at $H_\bullet(A)$ and $H_\bullet(B)$ in the LES.

Proof. Let us sweep the dirt under the carpet by using the snake lemma. Consider a chain complex $(C_\bullet, \partial_\bullet)$. The differential map $\partial_n : C_n \to C_{n-1}$ induces an exact sequence

$$0 \longrightarrow \ker\partial_n \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \operatorname{coker}\partial_n \longrightarrow 0$$

Connecting the differential maps we obtain a commutative diagram:

$$\begin{tikzcd}
 & \ker\partial_{n+1} \arrow[d, tail] & \ker\partial_n \arrow[d, tail] & \ker\partial_{n-1} \arrow[d, tail] & & {} \\
{} \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, two heads] \arrow[ur, dashed] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, two heads] & C_{n-2} \arrow[r] \arrow[d, two heads] & {} \\
 & & \operatorname{coker}\partial_{n+1} \arrow[uur, dashed, "\widetilde{\partial}_n"] & \operatorname{coker}\partial_n \arrow[uurr, dashed] & \operatorname{coker}\partial_{n-1} &
\end{tikzcd}$$

Since $\operatorname{im}\partial_n \subseteq \ker\partial_{n-1}$ and $\operatorname{im}\partial_{n+1} \subseteq \ker\partial_n$, ∂_n factors through the cokernel and the kernel as shown in the diagram. It induces $\widetilde{\partial}_n : \operatorname{coker}\partial_{n+1} \to \ker\partial_{n-1}$. Note that we have

$$\ker\widetilde{\partial}_n = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = H_n, \qquad \operatorname{coker}\widetilde{\partial}_n = \frac{\ker\partial_{n-1}}{\operatorname{im}\widetilde{\partial}_n} = \frac{\ker\partial_{n-1}}{\operatorname{im}\partial_n} = H_{n-1}$$ [1]

Now we consider the following commutative diagram:

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & \ker\partial^A_{n+1} \arrow[r, tail] \arrow[d, tail] & \ker\partial^B_{n+1} \arrow[r] \arrow[d, tail] & \ker\partial^C_{n+1} \arrow[d, tail] \arrow[dddll] & \\
0 \arrow[r] & A_{n+1} \arrow[r, tail] \arrow[d, "\partial^A_{n+1}"] & B_{n+1} \arrow[r, two heads] \arrow[d, "\partial^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, tail] \arrow[d, two heads] & B_n \arrow[r, two heads] \arrow[d, two heads] & C_n \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker}\partial^A_{n+1} \arrow[r] \arrow[d, "\widetilde{\partial}^A_n"] & \operatorname{coker}\partial^B_{n+1} \arrow[r, two heads] \arrow[d, "\widetilde{\partial}^B_n"] & \operatorname{coker}\partial^C_{n+1} \arrow[r] \arrow[d, "\widetilde{\partial}^C_n"] & 0 \\
0 \arrow[r] & \ker\partial^A_{n-1} \arrow[r, tail] \arrow[d] & \ker\partial^B_{n-1} \arrow[r] \arrow[d] & \ker\partial^C_{n-1} \arrow[d] & \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

By snake lemma, the blue line in the diagram gives a long exact sequence. In particular, every row in the diagram is exact. Then we apply snake lemma again to the last two rows, which gives a long exact sequence

$$\ker\widetilde{\partial}^A_n \longrightarrow \ker\widetilde{\partial}^B_n \longrightarrow \ker\widetilde{\partial}^C_n \xrightarrow{\delta_n} \operatorname{coker}\widetilde{\partial}^A_n \longrightarrow \operatorname{coker}\widetilde{\partial}^B_n \longrightarrow \operatorname{coker}\widetilde{\partial}^C_n$$

which is the long exact sequence of homology groups:

$$H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C)$$

[1] I think $H_n = \ker\widetilde{\partial}_n$ is a better way to define homology in a general Abelian category, so that we can avoid the annoying notion of image.

By gluing together patches of long exact sequences for different n, we obtain the desired long exact sequence of homology groups. α really good! $\square$

Question 4

a) Use the excision theorem to prove that, if each $x_i \in X_i$ has a contractible neighbourhood, then

$$\tilde{H}_\bullet\left(\bigvee_i X_i\right) = \bigoplus_i \tilde{H}_\bullet(X_i)$$

b) Construct a topological space X such that for all $k \geqslant 0$, $H_k(X) \cong \mathbb{Z}^{n_k}$, where $n_k \in \mathbb{N}$ are arbitrary.

c) Construct a connected topological space X with the same homology group as the tours T^2, which is <u>not</u> homotopy equivalent to T^2.

Proof. a) We use induction on n.[2] Suppose that the result holds for $n-1$. Let $Y_{n-1} = \bigvee_{i=1}^{n-1} X_i$, and $X = X_n \vee Y_{n-1} = \bigvee_{i=1}^{n} X_i$. Let $x_n \in X_n$ has contractible neighbourhood $V_n \subseteq X_n$. We have the short exact sequence of chain complexes:

$$0 \longrightarrow C_\bullet(X_n) \xrightarrow{\iota_\bullet} C_\bullet(X_n \vee Y_{n-1}) \xrightarrow{\pi_\bullet} C_\bullet(X_n \vee Y_{n-1}, X_n) \longrightarrow 0$$

which induces the long exact sequence of reduced homology groups

$$\longrightarrow \tilde{H}_q(X_n) \xrightarrow{\iota_q} \tilde{H}_q(X_n \vee Y_{n-1}) \xrightarrow{\pi_q} H_q(X_n \vee Y_{n-1}, X_n) \xrightarrow{\delta_q} \tilde{H}_{q-1}(X_n) \longrightarrow$$

The inclusion map $\iota : X_n \to X_n \vee Y_{n-1}$ has a retraction $r : X_n \vee Y_{n-1} \to X_n$. Formally we can write $X_n \vee Y_{n-1}$ as $(X_n \times \{y_0\}) \cup (\{x_n\} \times Y_{n-1})$, where $y_0 \in Y_{n-1}$. Then we have

$$\iota(x) = (x, y), \qquad r(x, y) = \begin{cases} x, & y = y_0 \\ x_n, & y \neq y_0 \end{cases}$$

and $r \circ \iota = \mathrm{id}_{X_n}$. By functoriality of $H_\bullet$, $\iota_q : \tilde{H}_q(X_n) \to \tilde{H}_q(X_n \vee Y_{n-1})$ is injective. Hence $\delta_q = 0$. The long exact sequence divides into short exact sequences

$$0 \longrightarrow \tilde{H}_q(X_n) \xrightarrow{\iota_q} \tilde{H}_q(X_n \vee Y_{n-1}) \xrightarrow{\pi_q} H_q(X_n \vee Y_{n-1}, X_n) \xrightarrow{\delta_q} 0$$

The sequence splits because ι_q has a retraction. We deduce that

$$\tilde{H}_q(X) \cong \tilde{H}_1(X_n) \oplus H_q(X_n \vee Y_{n-1}, X_n)$$

We have

$$\begin{aligned} H_q(X_n \vee Y_{n-1}, X_n) &\cong H_q(X_n \vee Y_{n-1} \setminus (X_n \setminus V_n), X_n \setminus (X_n \setminus V_n)) && \text{(excision theorem)} \\ &= H_q(V_n \vee Y_{n-1}, V_n) \\ &= H_q(Y_{n-1}, \{*\}) && (V_n \text{ is contractible}) \\ &= \tilde{H}_q(Y_{n-1}) \end{aligned}$$

correct!

Finally, by induction hypothesis, we have

$$\tilde{H}_q(X) \cong \tilde{H}_q(X_n) \oplus \tilde{H}_q(Y_{n-1}) \cong \bigoplus_{i=1}^{n} \tilde{H}_q(X_i)$$

[2] We assume that the index set is finite. The result is much easier to prove using the Mayer-Vietoris sequence.

Alex didn't write that each (x_i, X_i) is a good pair but he meant to!

why not use MV then?

which completes the induction.

b) We may have to use the infinite version of the result in (a), though I do not know how to prove it...

For each $k \geqslant 1$, we take n_k copies of k-spheres $S_1^k, \ldots, S_{n_k}^k$. We note that

$$\widetilde{H}_n(S^k) = \begin{cases} \mathbb{Z}, & n = k \\ 0, & n \neq k \end{cases}$$

Take $Y = \bigvee_{k=0}^{\infty} \bigvee_{i=1}^{n_k} S_i^k$. Then we have $\widetilde{H}_k(Y) \cong \mathbb{Z}^{n_k}$ for $k \geqslant 1$. Next we take a finite set $\{1, 2, \ldots, n_0 - 1\}$ with discrete topology. We set

$$X = Y \sqcup \{1, 2, \ldots, n_0 - 1\} = \bigvee_{k=0}^{\infty} \bigvee_{i=1}^{n_k} S_i^k \sqcup \{1, 2, \ldots, n_0 - 1\}$$

Then

$$H_0(X) = H_0(Y) \oplus H_0(\{1, 2, \ldots, n_0 - 1\}) = \mathbb{Z}^{n_0}, \qquad H_k(X) = H_k(Y),\ (k \geqslant 1)$$

We conclude that $H_K(X) = \mathbb{Z}^{n_k}$ for all $k \geqslant 0$.

c) The homology groups of torus are given by

$$H_0(T^2) = \mathbb{Z}, \qquad H_1(T^2) = \mathbb{Z} \times \mathbb{Z}, \qquad H_2(T^2) = \mathbb{Z}, \qquad H_3(T^2) = \cdots = 0$$

By the construction in part (b), the space

$$X = S^1 \vee S^1 \vee S^2$$

has the same homology groups with those of T^2. But by Seifert-van Kampen Theorem,

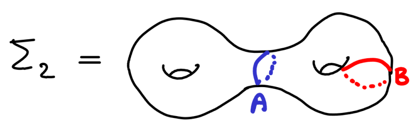
$$\pi_1(X) = \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^2) \cong \mathbb{Z} * \mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T^2)$$

Hence T^2 and X are not homotopic equivalent. $\square$

Question 5

a) Compute $H_\bullet(S^n \setminus (k+1)\text{ points})$ and $H_\bullet(\mathbb{R}^2 \setminus k\text{ points})$.

b) Compute $H_\bullet(\Sigma_2, A)$ and $H_\bullet(\Sigma_2, B)$, where



c) Using Mayor-Vietoris, calculate $H_\bullet(S^n)$ and $H_\bullet(K)$ where K is the Klein bottle.

Proof. a) From Question 1.(g) in Sheet 1, we know that

$$S^n \setminus (k+1)\text{ points} \simeq \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{k\text{ copies}}, \qquad \mathbb{R}^2 \setminus k\text{ points} \cong \mathbb{D}^2 \setminus k\text{ points} \simeq \underbrace{S^1 \vee \cdots \vee S^1}_{k\text{ copies}}$$

We know that

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = n, 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence by Question 4.(a),

$$H_q(S^n \setminus (k+1) \text{ points}) = \begin{cases} \mathbb{Z}^k & q = n-1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H_q(\mathbb{R}^2 \setminus k \text{ points}) = \begin{cases} \mathbb{Z}^k & q = 1 \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

b) We note from the picture that (Σ_2, A) and (Σ_2, B) are good pairs. Hence we have $H_n(\Sigma_2, A) \cong \widetilde{H}_n(\Sigma_2/A)$ and $H_n(\Sigma_2, B) \cong \widetilde{H}_n(\Sigma_2/B)$.

From the picture we have $\Sigma_2/A \simeq T^2 \vee T^2$ and $\Sigma_2/B \simeq T^2 \vee S^1$. Hence

$$H_n(\Sigma_2, A) = \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(T^2) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad H_n(\Sigma_2, B) = \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^3 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

c) We prove by induction on n that

$$\widetilde{H}_q(S^n) = \begin{cases} \mathbb{Z}, & q = n \\ 0, & \text{otherwise} \end{cases}$$

For $n = 0$, the result is clear. Suppose that the result holds for $n-1$. Let A be an open neighbourhood of the northern hemipshere of S^n abd B be an open neighbourhood of the southern hemipshere of S^n. Then $A \simeq B \simeq \mathbb{D}^n$, $S^n = A \cup B$ and $A \cap B \simeq S^{n-1}$. The theorem of Mayor-Vietoris sequence produces a long exact sequence

need to be more specific about A, B but I know what you meant

$$\longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(S^n) \xrightarrow{\delta_n} H_{n-1}(A \cap B) \longrightarrow H_{n-1}(A) \oplus H_{n-1}(B) \longrightarrow$$

Since A and B are contractible, $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B) = 0$. We have $\widetilde{H}_n(S^n) \cong \widetilde{H}_{n-1}(S^{n-1})$. By induction hypothesis, $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Hence

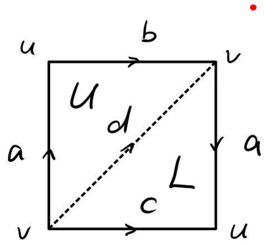
$$H_q(S^n) = \begin{cases} \mathbb{Z}, & q = 0, n \\ 0, & \text{otherwise} \end{cases}$$

This completes the induction.

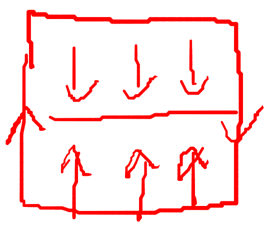
The Klein bottle is obtained by gluing the boundary circle of two Möbius bands A' and B'. Let A, B be open neighbourhoods of A and B in K respectively. Then $A \simeq A' \simeq B' \simeq B$, $K = A \cup B$, and $A \cap B \simeq S^1$.

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_2(S^1) \arrow[r] & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \arrow[r] & \widetilde{H}_2(K) \arrow[dll] & \\ & \widetilde{H}_1(S^1) \arrow[r] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r] & \widetilde{H}_1(K) \arrow[r] & 0 \end{tikzcd}$$

We need to compute the homology groups of the Möbius band A. We impose a Δ-complex structure on A as follows.



$M \simeq S^1$



The boundary maps are given by

$$\partial_1 : a \mapsto u - v,\ b \mapsto v - u,\ c \mapsto 0,\ d \mapsto u - v$$
$$\partial_2 : U \mapsto a + b - c,\ L \mapsto a + c - d$$

Hence

$$H_0(A) \cong \mathbb{Z}, \qquad H_1(A) = \frac{\langle a+b, a-d, c\rangle}{\langle a+b-c, a+c-d\rangle} \cong \mathbb{Z}, \qquad H_2(A) = 0$$

Put these into the long exact sequence we obtain

$$0 \longrightarrow \widetilde{H}_2(K) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{(i_*,-j_*)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\pi_*} \widetilde{H}_1(K) \longrightarrow 0$$

Note that the inclusion $(i_*, -j_*) : H_1(S^1) \to H_1(A) \oplus H_1(B)$ is induced by the inclusions $i : S^1 \to A$ and $j : S^1 \to B$. So $(i_*, -j_*) : 1 \mapsto (2, -2)$. By first isomorphism theorem,

$$\widetilde{H}_1(K) = \operatorname{im}\pi_* \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\ker\pi_*} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\operatorname{im}(i_*,-j_*)} \cong \frac{\mathbb{Z}a\oplus\mathbb{Z}b}{\langle 2a-2b\rangle} \cong \mathbb{Z}\oplus\mathbb{Z}/2$$

Since $(i_*, -j_*)$ is injective, we have $\delta = 0$ and hence $\widetilde{H}_2(K) = 0$.

In conclusion, the homology groups of K are given by [3]

$$H_n(K) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

□

Question 6

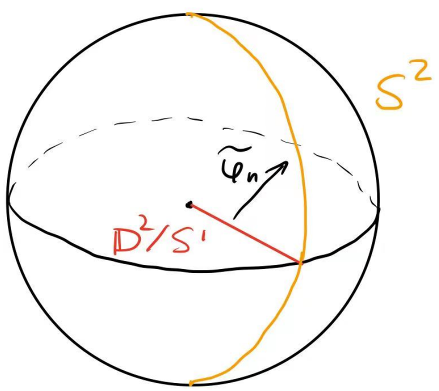
Build an explicit homeomorphism $\mathbb{D}^n / S^{n-1} \cong S^n$ in a way that preserves the orientation.

Hint: Parametrise points of $\mathbb{D}^n$ by $(tx_1, ..., tx_n)$ where $(x_1, ..., x_n) \in S^{n-1}$ and $t \in [0, 1]$.

Proof. We parametrise points of $\mathbb{D}^n$ by $(tx_1, ..., tx_n)$ where $(x_1, ..., x_n) \in S^{n-1}$ and $t \in [0, 1]$. We define $\varphi : D^n \to S^n$ by

$$\varphi(tx_1, ..., tx_n) = \left(\sqrt{1-(2t-1)^2}x_1, ..., \sqrt{1-(2t-1)^2}x_n, 2t-1\right)$$

The map clearly descends to $\widetilde{\varphi} : D^n / S^{n-1} \to S^n$, which is a homeomorphism. Geometrically $\widetilde{\varphi}$ maps each ray in $\mathbb{D}^n$ to the corresponding longitude on S^n. For $n = 2$ we have the following picture:



□

you could use $M \cup S^1$:)

[3] This method is much more complicated than direct computation of simplicial homology groups of K!!!

Question 7

If X retracts onto A, prove that $H_\bullet(X) \cong H_\bullet(A) \oplus H_\bullet(X, A)$.

Proof. We have essentially proven this as a byproduct in the proof of Question 4.(a).

Let $\iota : A \to X$ be the inclusion map and $r : X \to A$ be a retraction map. Let $\iota_n : H_n(A) \to H_n(X)$ and $r_n : H_n(X) \to H_n(A)$ be the induced maps for the respective homology groups. We have a long exact sequence of relative homology groups

$$\longrightarrow H_n(A) \xrightarrow{\iota_n} H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow$$

Since $r \circ \iota = \mathrm{id}$, $r_n \circ \iota_n = \mathrm{id}$. Hence ι_n is injective. Note that by exactness at $H_{n-1}(A)$, we have $\operatorname{im} \delta_n = \ker \iota_{n-1} = 0$. Hence $\delta_n = 0$. We break the long exact sequence into short exact sequences

$$0 \longrightarrow H_n(A) \xrightarrow{\iota_n} H_n(X) \longrightarrow H_n(X, A) \longrightarrow 0$$

Now ι_n has a retraction r_n. By splitting lemma the sequence splits. We have $H_n(X) \cong H_n(A) \oplus H_n(X, A)$. □

Question 8

a) Viewing points as singular 1-chains ($\Delta^1 \cong I$), prove that a <u>constant</u> path c is a boundary: $c \in \partial C_2(X)$.

b) For paths $f, g : I \to X$ with $f(1) = g(0)$, let $f * g : I \to X$ be the **concatenated path**. Prove that $f * g - f - g \in \partial C_2(X)$.

c) Let f^{-1} denote the **reversed path**. Prove that $f + f^{-1} \in \partial C_2(X)$.

d) If f, g are homotopic paths relative to ∂I, prove that $f - g \in \partial C_2(X)$.

e) Deduce that there exist group homomorphisms (**Hurewicz homomorphism**)

$\pi_1(X, x) \longrightarrow \pi_1^{ab}(X, x) \longrightarrow H_1(X)$, where π_1^{ab} is the Abelianisation.

f) Assume from now on that X is <u>path-connected</u>. Fix $x \in X$. Pick a path $\gamma_y : I \to X$ from x to y, for each $y \in X$, with $\gamma_x = c_x$. Show that there exists a homomorphism $H_1(X) \to \pi_1^{ab}(X, x)$ which on chains is the group homomorphism: $\varphi : C_1(X) \to \pi_1^{ab}(X, x)$, $\varphi(f : I \to X) = \gamma_{f(0)} * f * \gamma_{f(1)}^{-1}$.

Deduce that $\boldsymbol{H_1(X) \cong \pi_1^{ab}(X, x)}$ for any path-connected X.

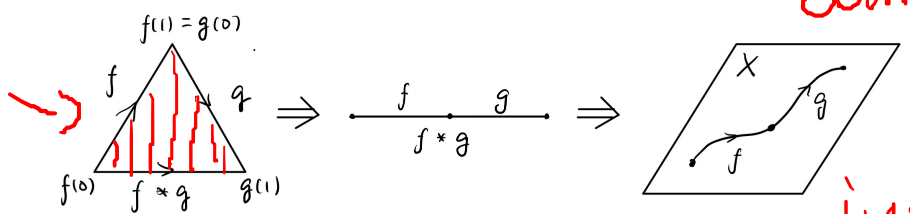
Proof. a) Consider the singular 2-simplex $\sigma : \Delta^2 \to \{c_0\} \subseteq X$. Then $\partial \sigma = c - c + c = c \in C_1(X)$, where $c : \Delta^1 \to \{c_0\} \subseteq X$ is the constant map.

b) For parts (b), (c) and (d), it is easier to argue with pictures.

Let $\sigma : \Delta^2 \to X$ be a singular 2-simplex defined in the following way:

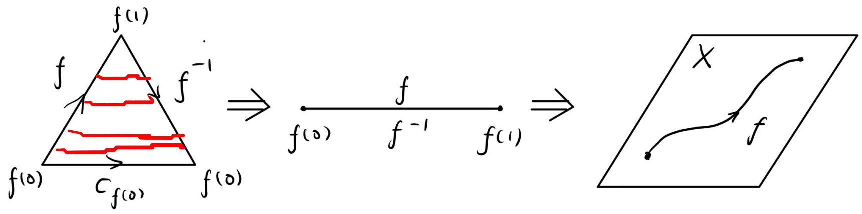
you should comment on how σ is defined inside Δ²

level sets of σ.



Then $\partial \sigma = f \star g - f - g \in \partial C_2(X)$.

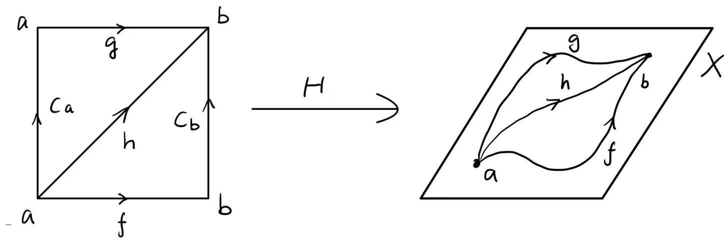
c) $\sigma : \Delta^2 \to X$ be a singular 2-simplex defined in the following way, where $c_{f(0)}$ is the singular 1-simplex representing the constant path at $f(0) \in X$.



same

Then $\partial\sigma = f + f^{-1} - c_{f(0)} \in \partial C_2(X)$. Since $c_{f(0)} \in \partial C_2(X)$, we have $f + f^{-1} \in \partial C_2(X)$. ✓

d) Let $H: I \times I \to X$ be the homotopy from f to g. We impose a Δ-complex structure on $I \times I$.



We have $f + c_b - h \in \partial C_2(X)$ and $h - g - c_a \in \partial C_2(X)$. Since $c_a, c_b \in \partial C_2(X)$, we have $f - g \in \partial C_2(X)$. ✓

e) For a based loop $f: I \to X$ with $f(0) = f(1) = x$, let $[f]_\pi$ denotes the homotopy class of f, and $[f]_H$ denotes the class $f + \partial C_2(X)$ in $H_1(X)$. Let $\psi: \pi_1(X) \to H_1(X)$ defined by $[f]_\pi \mapsto [f]_H$. This is well-defined by (d): if $f, g: I \to X$ are such that $[f]_\pi = [g]_\pi$, then $f \simeq g$ and hence $f - g \in \partial C_2(X)$.

also neat $\partial\sigma = f(1) - f(0) = 0$ so H_1/H makes sense

ψ is a group homomorphism. For $f, g: I \to X$, we have

$$[f]_H + [g]_H = f + g + \partial C_2(X) = f \star g + \partial C_2(X) = [f \star g]_H = \psi([f]_\pi \cdot [g]_\pi)$$ ✓

Finally, since $H_1(X)$ is Abelian, $[\pi_1(X,x), \pi_1(X,x)] \in \ker\psi$. ψ descends to the group homomorphism

$$\widetilde{\psi}: \pi_1^{\mathrm{ab}}(X,x) = \frac{\pi_1(X,x)}{[\pi_1(X,x), \pi_1(X,x)]} \to H_1(X)$$ ✓

both abelian

f) For a singular 1-simplex f, let $\varphi(f) = [\gamma_{f(0)} \star f \star \gamma_{f(1)}^{-1}] \in \pi_1^{\mathrm{ab}}(X,x)$ and extend this $\mathbb{Z}$-linearly. That is, $\varphi(nf + mg) = n[f] + m[g] \in \pi_1^{\mathrm{ab}}(X,x)$. Then $\varphi: C_1(X) \to \pi_1^{\mathrm{ab}}(X,x)$ is a group homomorphism. ✓

For $f \in \partial C_2(X)$, we have $f = \partial\left(\sum_{i=1}^{n} m_i \sigma_i\right)$, where $m_i \in \mathbb{Z}$ and σ_i are singular 2-simplices. We write $\partial\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$. After some labourious manipulation of Δ-complexes we may assume that each τ_{ij} is a loop based at $x \in X$.[4] ✓

Now consider $\varphi(f) = \sum_{i=1}^{n} m_i[\partial\sigma_i] \in \pi_1^{\mathrm{ab}}(X,x)$, since σ_i gives a null-homotopy of the loop $\tau_{i0} - \tau_{i1} + \tau_{i2}$, then $[f] = 0$. We deduce that $\partial C_2(X) \subseteq \ker\varphi$. Hence φ induces a group homomorphism $\widetilde{\varphi}: H_1(X) \to \pi_1^{\mathrm{ab}}(X,x)$. It is easy to check that $\widetilde{\varphi}$ is the inverse of $\widetilde{\psi}$. In conclusion, $H_1(X) \cong \pi_1^{\mathrm{ab}}(X,x)$, □

:)

Question 9

Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}_{>0}\}$. $H = \bigcup_{n \in \mathbb{Z}_{>0}} B\left((\frac{1}{n}, 0), \frac{1}{n}\right) \subseteq \mathbb{R}^2$ ("Hawaiian earring"). $W = \bigvee_{n \in \mathbb{Z}_{>0}} S^1$.

- Show that H and W are not homeomorphic.

[4] See the second-last paragraph of p. 167, *Algebraic Topology* by Allen Hatcher for detail.

- Is X/A homeomorphic to H or to W?
- Show that $H_1(X, A) \ncong \widetilde{H}_1(X/A)$. (Note $A \subseteq X$ is not a good pair.)

(You do not need ot fully compute $\widetilde{H}_1(X/A)$.)

Proof.

- Suppose that $f : H \to W$ is a homeomorphism. We must have $f(0,0) = (-1,0)$, because every point on $H \setminus \{(0,0)\}$ and on $W \setminus \{(-1,0)\}$ has a neighbourhood homeomorphic to I. For any neighbourhood $V \subseteq H$ of $(0,0)$, there exists $n \in \mathbb{N}$ such that $B((\frac{1}{n},0),\frac{1}{n}) \subseteq V$. So V is not contractible. But $f(V) \subseteq W$ is contractible for sufficiently small diam V. This is a contradiction. $H \ncong W$.
- $X/A \cong H$ is intuitively correct.
- We compute $H_1(X, A)$. The long exact sequence of relative homology groups is given by

$$\longrightarrow \widetilde{H}_1(A) \longrightarrow \widetilde{H}_1(X) \longrightarrow H_1(X,A) \longrightarrow \widetilde{H}_0(A) \longrightarrow \widetilde{H}_0(X) \longrightarrow$$

Since X is contractible, we have $\widetilde{H}_1(X) = 0$, $\widetilde{H}_0(X) = 0$. Hence $H_1(X, A) \cong \widetilde{H}_0(A) \cong \mathbb{Z}^{\oplus\mathbb{N}}$. In particular $H_1(X, A)$ is countable.

Since $X/A \cong H$, for each $n \in \mathbb{Z}_+$, there exists a retraction $r_n : X/A \to C_n := B((\frac{1}{n},0),\frac{1}{n})$. Then r_n induces the group epimorphism $r_{n*} : \pi_1(X/A) \to \mathbb{Z}$. So there exists a group homomorphism $\varphi : \pi_1(X/A) \to \prod_{n=1}^{\infty} \mathbb{Z}$. For each $\{a_n\} \in \prod_{n=1}^{\infty} \mathbb{Z}$, we can define a loop $f : I \to X/A$ such that f goes a_n times around C_n in the time interval $\left[1 - \dfrac{1}{n}, 1 - \dfrac{1}{n+1}\right]$. This implies that φ is in fact surjective.

Since $\prod_{n=1}^{\infty} \mathbb{Z}$ is Abelian, φ descends to a group epimorphism $\widetilde{\varphi} : \pi_1^{\mathrm{ab}}(X/A) \to \prod_{n=1}^{\infty} \mathbb{Z}$. By Question 8, since X/A is path-connected, we have $H_1(X/A) \cong \pi_1^{\mathrm{ab}}(X/A)$. Therefore $\prod_{n=1}^{\infty} \mathbb{Z}$ is isomorphic to a quotient of $\widetilde{H}_1(X/A)$. Since $\prod_{n=1}^{\infty} \mathbb{Z}$ is uncountable, $\widetilde{H}_1(X/A)$ is also uncountable.

We conclude that $\widetilde{H}_1(X/A) \ncong H_1(X, A)$. □

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