A

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Problem Sheet 2

C7.6: General Relativity II

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Section A: Introductory

Question 1. Einstein-tensor conservation

Let (M, g) be a Lorentzian manifold. Show that the Bianchi identity $\nabla_{[a} R_{bc]de} = 0$ implies that the Einstein tensor is divergence free, i.e.

$$\nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 0.$$

Question 2. Scalar and Maxwell matter

Let (M, g) be a Lorentzian manifold.

(a) Let $\phi \in C^\infty(M)$. We say that ϕ satisfies the wave equation iff $\Box_g \phi := \nabla^a \nabla_a \phi = 0$. Define the symmetric 2-covariant tensor field T associated to ϕ by

$$T(X, Y) := (X\phi)(Y\phi) - \frac{1}{2} g(X, Y) g^{-1}(\mathrm{d}\phi, \mathrm{d}\phi).$$

Show that T is divergence-free, i.e. $\nabla^a T_{ab} = 0$ if, and only if, ϕ satisfies the wave equation. Find the expression for T_{00} in terms of ϕ in the special case that (M, g) is the Minkowski spacetime.

(b) Let F be a two-form and define the associated symmetric 2-covariant tensor field

$$T_{ab} = \frac{1}{4\pi} \left(F_{ac} F_b^{\ c} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right).$$

Show that T satisfies $\nabla^a T_{ab} = 0$ if F satisfies the Maxwell equations $\mathrm{d}F = 0$ and $\nabla^a F_{ab} = 0$ (the other direction does in general not hold true). In the special case that (M, g) is the Minkowski spacetime find the expression for T_{00} in terms of the electric field $E_i = -F_{0i}$ and the magnetic field $B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$.

Section B: Core

Question 3. Integrable 2-dimensional distributions

Consider the vector fields $V = \partial_x + y\partial_z$ and $W = \partial_y + x\partial_z$ in $\mathbb{R}^3$ with the standard Cartesian coordinates (x, y, z). Show that span $\{V, W\}$ is integrable and construct global coordinates (u, v, w) on $\mathbb{R}^3$ such that the integral manifolds are given as level sets of w.

Proof. For $f \in \mathrm{C}^\infty(M)$, we have

$$\begin{aligned}
[V, W]f &= VWf - WVf \\
&= (\partial_x + y\partial_z)(\partial_y f + x\partial_z f) - (\partial_y + x\partial_z)(\partial_x f + y\partial_z f) \\
&= (\partial_x\partial_y f + x\partial_x\partial_z f + y\partial_y\partial_z f + xy\partial_z^2 f + \partial_z f) - (\partial_x\partial_y f + x\partial_x\partial_z f + y\partial_y\partial_z f + xy\partial_z^2 f + \partial_z f) \\
&= 0
\end{aligned}$$

Hence $[V, W] = 0$. By Frobenius' Theorem, span $\{V, W\}$ is integrable. There exists a coordinate system (u, v, w) on $U \subseteq \mathbb{R}^3$ such that $V = \partial_u$ and $W = \partial_v$. Then we have

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial z}{\partial u} = y,$$
$$\frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = 1, \quad \frac{\partial z}{\partial v} = x.$$

Hence $x = u + f(w)$ and $y = v + g(w)$ for some $f, g \in C^\infty(\mathbb{R})$. Then $z = uv + ug(w) + vf(w)$. Next we compute the Jacobian:

$$\det J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = -(fg)'(w).$$

If we want globally defined coordinates (u, v, w), then we need $\det J \neq 0$ for all $w \in \mathbb{R}$. For this we may choose $f(w) = 1$ and $g(w) = w$. We have:

$$x = u + 1, \qquad y = v + w, \qquad z = uv + uw + v.$$

Inversion:

$$u = x - 1, \qquad v = (1 - x)y + z, \qquad w = xy - z.$$

The integral manifold is given by $\{(x, y, z) \in \mathbb{R}^3 : w(x, y, z) = xy - z = 0\}$. □

Question 4. Static spacetimes

Let (M, g) be a static spacetime, i.e. there exists a timelike and hypersurface-orthogonal Killing vector field V. Show that one can locally choose coordinates $\{y^0, y^1, \dots, y^{n-1}\}$ such that

- $V = \frac{\partial}{\partial y^0}$;
- $g_{\mu\nu}$ is independent of y^0;
- $g_{0i} = 0$ for $i = 1, \dots, n$.

[*Hint: Combine elements of Proposition* 1.17 *and Corollary* 1.35 *from the lectures with V being a Killing vector field.*]

Proof. The first and third properties are are proven in Corollary 1.35. We only need to prove the second property. We have

Yes but should also check the Jacobian $D\psi$ is non-singular here.

$$\partial_0 g_{\mu\nu} = \mathcal{L}_V(g_{\mu\nu}) = \mathcal{L}_V(g(\partial_\mu, \partial_\nu)) = (\mathcal{L}_V g)(\partial_\mu, \partial_\nu) + g(\mathcal{L}_V \partial_\mu, \partial_\nu) + g(\partial_\mu, \mathcal{L}_V \partial_\nu).$$

Since $\{\partial_\mu\}$ is a coordinate frame of vector fields, we have $\mathcal{L}_V \partial_\mu = [\partial_0, \partial_\mu] = 0$. Since V is a Killing vector field, $\mathcal{L}_V g = 0$. Therefore we have $\partial_0 g_{\mu\nu} = 0$. Hence $g_{\mu\nu}$ is independent of y^0. □

Question 5. Noether charges

Let (M, g) be a Lorentzian manifold and let T be a symmetric 2-covariant tensor field satisfying the conservation equation $\nabla^a T_{ab} = 0$. Let K be a Killing vector field on (M, g).

(a) Show that the one-form $J(\cdot) = T(\cdot, K)$ is divergence free, i.e. $\nabla^a J_a = 0$.

(b) Let now (M, g) be $(3+1)$ dimensional Minkowski spacetime with the canonical coordinates $\{t, x, y, z\}$ and assume that T vanishes for $r = \sqrt{x^2 + y^2 + z^2}$ large enough. Show that for each Killing vector field K the corresponding charge

$$Q[K] := \int_{t=t_0} J_0\,(t_0, x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

is conserved, i.e. independent of time.

Proof. (a) We choose a local chart $(U; x^0, ..., x^n)$ on M. Since K is a Killing vector field, we have

$$g(\nabla_X K, Y) + g(X, \nabla_Y K).$$

for any $X, Y \in \Gamma(\mathrm{T}M)$. Now take $X = \partial_\mu$ and $Y = \partial_\nu$. In local coordinates we have

$$\nabla^\mu K^\nu + \nabla^\nu K^\mu = 0.$$

In local coordinates we have $J_\mu = T_{\mu\nu}K^\nu$. Therefore the divergence is given by

$$\nabla^\mu J_\mu = K^\nu \nabla^\mu T_{\mu\nu} + T_{\mu\nu}\nabla^\mu K^\nu = T_{\mu\nu}\nabla^\mu K^\nu = \frac{1}{2}T_{\mu\nu}(\nabla^\mu K^\nu + \nabla^\nu K^\mu) = 0.$$

The second equality follows from that T is divergence-less. The third equality follows from that T is symmetric.

(b) Let $N := \{(t, x, y, z) \in M : t \in [t_0, t_1],\ x^2 + y^2 + z^2 \leqslant r^2\}$ be a submanifold of M with boundary $\partial N = N_1 \cup N_2$, where $N_1 := \{t_0, t_1\} \times \{x^2 + y^2 + z^2 \leqslant r^2\}$ and $N_2 := [t_0, t_1] \times \{x^2 + y^2 + z^2 = r^2\}$, where we choose r sufficiently large such that $T = 0$ on N_2. Then $J = 0$ on N_2 too.

From Question 4.(c) of Sheet 2 of *C3.11 Riemannian Geometry*[1], we have $\mathcal{L}_{J^\sharp}\Omega = (\nabla_\mu J^\mu)\Omega = 0$, where $\Omega = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the standard volume form on N. Then by Cartan's formula and Stokes' theorem, we have

Ok, but should not just refer other course's problem sets on the exam.

$$0 = \int_N \mathcal{L}_{J^\sharp}\Omega = \int_N (\mathrm{d} \circ \iota_{J^\sharp}\Omega + \iota_{J^\sharp} \circ \mathrm{d}\Omega) = \int_N \mathrm{d} \circ \iota_{J^\sharp}\Omega = \oint_{\partial N} \iota_{J^\sharp}\Omega.$$

We need to compute the interior product $\iota_{J^\sharp}\Omega$ explicitly:

$$\begin{aligned}
\iota_{J^\sharp}\Omega &= \iota_{J^\sharp}(\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3) \\
&= (-1)^\mu \iota_{J^\sharp}(\mathrm{d}x^\mu)(\mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^3) \\
&= (-1)^\mu J^\mu \mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^3 \\
&= J^0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \varepsilon_{ijk} J^i \mathrm{d}t \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k
\end{aligned}$$

The facts that $J = 0$ on N_2 and $t = \text{const}$ on N_1 show that

$$0 = \oint_{\partial N} \iota_{J^\sharp}\Omega = \int_{N_1} J^0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_{t=t_0} J_0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \int_{t=t_1} J_0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

where we have used the fact that $J_0 = -J^0$ as a special property of the Minkowski metric η. Hence

$$\int_{t=t_0} J_0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_{t=t_1} J_0 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

Since t_0 and t_1 are arbitrary, we deduce that $Q[K]$ is conserved. □

Question 6. Macroscopic motion in vacuum

Let (M, g) be the $(3+1)$-dimensional Minkowski spacetime with Cartesian coordinates $\{x^0 = t, \boldsymbol{x}\}, \boldsymbol{x} = (x^1, x^2, x^3)$. Let T_{ab} be a symmetric 2-covariant tensor field which satisfies the conservation equation $\partial_a T^{ab} = 0$

[1] The original requirement that manifold N is compact is not necessary as $J^\sharp$ is compactly supported in N.

and such that for every $t \in \mathbb{R}$ we have that $T_{ab}(t, \cdot)$ is compactly supported in space (i.e. in $\mathbb{R}^3$). Define

$$P^i(t) := \int_{\mathbb{R}^3} T^{0i}(t, \boldsymbol{x})\,\mathrm{d}^3x, \quad D^i(t) := \int_{\mathbb{R}^3} T^{00}(t, \boldsymbol{x})x^i\,\mathrm{d}^3x, \quad Q^{ij}(t) := \int_{\mathbb{R}^3} T^{00}(t, \boldsymbol{x})x^i x^j\,\mathrm{d}^3x,$$

where $i, j = 1, 2, 3$. Show that the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}D^i = P^i, \quad \frac{\mathrm{d}}{\mathrm{d}t}P^i = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}Q^{ij} = 2\int_{\mathbb{R}^3} T^{ij}(t, \boldsymbol{x})\mathrm{d}^3x.$$

Proof. We begin with a useful lemma (in fact this is divergence theorem in coordinate disguise): Suppose that $(F^1, F^2, F^3) \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is compactly supported. Then

$$\mathrm{d}(\varepsilon_{ijk}F^i\mathrm{d}x^j \wedge \mathrm{d}x^k) = \varepsilon_{ijk}\mathrm{d}F^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = \frac{\partial F^i}{\partial x^i}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \frac{\partial F^i}{\partial x^i}\mathrm{d}^3\boldsymbol{x},$$

where $\mathrm{d}^3\boldsymbol{x} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the volume form of $\mathbb{R}^3$. By Stokes' Theorem, for sufficiently large $R > 0$,

$$\int_{\mathbb{R}^3} \frac{\partial F^i}{\partial x^i}\mathrm{d}^3\boldsymbol{x} = \int_{B(0,R)} \frac{\partial F^i}{\partial x^i}\mathrm{d}^3\boldsymbol{x} = \oint_{\partial B(0,R)} \varepsilon_{ijk}F^i\mathrm{d}x^j \wedge \mathrm{d}x^k = 0.$$

Now return to the problem, where we know that T_{ab} is compactly supported in $\mathbb{R}^3$ for each fixed t.

$$\begin{aligned}
\frac{\mathrm{d}D^i}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \frac{\partial T^{00}}{\partial t}x^i\,\mathrm{d}^3\boldsymbol{x} = -\int_{\mathbb{R}^3} \frac{\partial T^{j0}}{\partial x^j}x^i\,\mathrm{d}^3\boldsymbol{x} \\
&= -\int_{\mathbb{R}^3} \left(\frac{\partial}{\partial x^j}(T^{j0}x^i) - T^{j0}\frac{\partial x^i}{\partial x^j}\right)\mathrm{d}^3\boldsymbol{x} = \int_{\mathbb{R}^3} T^{j0}\frac{\partial x^i}{\partial x^j}\mathrm{d}^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} T^{0i}\,\mathrm{d}^3\boldsymbol{x} = P^i \\
\frac{\mathrm{d}P^i}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \frac{\partial T^{0i}}{\partial t}\mathrm{d}^3\boldsymbol{x} = -\int_{\mathbb{R}^3} \frac{\partial T^{ji}}{\partial x^j}\mathrm{d}^3\boldsymbol{x} = 0 \\
\frac{\mathrm{d}^2Q^{ij}}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{\partial T^{00}}{\partial t}x^ix^j\,\mathrm{d}^3\boldsymbol{x} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{\partial T^{k0}}{\partial x^k}x^ix^j\,\mathrm{d}^3\boldsymbol{x} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} T^{k0}\frac{\partial(x^ix^j)}{\partial x^k}\mathrm{d}^3\boldsymbol{x} = \int_{\mathbb{R}^3} \frac{\partial T^{k0}}{\partial t}\frac{\partial(x^ix^j)}{\partial x^k}\mathrm{d}^3\boldsymbol{x} \\
&= -\int_{\mathbb{R}^3} \frac{\partial T^{k\ell}}{\partial x^\ell}\frac{\partial(x^ix^j)}{\partial x^k}\mathrm{d}^3\boldsymbol{x} = \int_{\mathbb{R}^3} T^{k\ell}\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell}\mathrm{d}^3\boldsymbol{x}.
\end{aligned}$$

We need to prove the following lemma:

$$\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell} = \delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k.$$

If $i = j$, then $\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell} \neq 0$ only if $i = j = k = \ell$. In such case $\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell} = \frac{\partial^2(x^i)^2}{\partial(x^i)^2} = 2$. If $i \neq j$, then $\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell} \neq 0$ only if $\{i, j\} = \{k, \ell\}$. We will obtain $\frac{\partial^2(x^ix^j)}{\partial x^k\partial x^\ell} = 1$ for $(k, \ell) = (i, j)$ and $(k, \ell) = (j, i)$.

$$\frac{\mathrm{d}^2Q^{ij}}{\mathrm{d}t^2} = \int_{\mathbb{R}^3} T^{k\ell}(\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\,\mathrm{d}^3\boldsymbol{x} = 2\int_{\mathbb{R}^3} T^{ij}\,\mathrm{d}^3\boldsymbol{x}, \qquad \square$$

Question 7. Stationary stress-energy tensor

Let (M,g) be the $(3+1)$-dimensional Minkowski spacetime with Cartesian coordinates $\{x^0=t,\boldsymbol{x}\}$, $\boldsymbol{x}=(x^1,x^2,x^3)$. Let T_{ab} be a symmetric 2-covariant tensor field which satisfies the conservation equation $\partial_a T^{ab}=0$ and such that for every $t\in\mathbb{R}$ we have that $T_{ab}(t,\cdot)$ is compactly supported in space (i.e. in $\mathbb{R}^3$). Moreover assume that the stress energy distribution is stationary, i.e. that $\partial_t T_{ab}=0$. Verify the following identities which are used in the lectures to derive the far field of an isolated gravitational body in the linear approximation.

(a) $\partial_l\partial_m\left(T^{lm}x^ix^j\right)=2T^{ij}$ and show that this implies $\int_{\mathbb{R}^3}T^{ij}(t,\boldsymbol{x})\mathrm{d}^3x=0$.

(b) Show that $\int_{\mathbb{R}^3}T^{0j}(t,\boldsymbol{x})d^3x=0$.

(c) $\partial_i\left(T^i{}_jx^j\right)=T^i{}_i$ and hence show that $\int_{\mathbb{R}^3}T^i{}_i(t,\boldsymbol{x})\mathrm{d}^3x=0$.

(d) $\partial_i\left(T^{0i}x^jx^k\right)=T^{0j}x^k+T^{0k}x^j$ and hence show that $\int_{\mathbb{R}^3}\left(T^{0j}x^k+T^{0k}x^j\right)(t,\boldsymbol{x})\mathrm{d}^3x=0$.

(e) $\partial_k\left(T^k{}_ix^ix^j-\frac{1}{2}T^{jk}x_ix^i\right)=T^i{}_ix^j$ and hence show that $\int_{\mathbb{R}^3}T^i{}_i(t,\boldsymbol{x})x^j\mathrm{d}^3x=0$.

(f) $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|_{\mathbb{R}^3}}=\frac{1}{r}+\frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{r^3}+\mathcal{O}\left(\frac{1}{r^3}\right)$ as a function of $\boldsymbol{x}$, where $r^2=\boldsymbol{x}\cdot\boldsymbol{x}$ and $\boldsymbol{x}'$ is bounded.

Note that the space indices $i,j,k,l,m,\ldots$ run from 1 to 3.

Proof. Since T is stationary and divergence-less, we have $\partial_i T^{i\mu}=0$.

(a) Note that $\partial_m T^{\ell m}=0$ and $\partial_\ell T^{\ell m}=0$. Then

$$\frac{\partial^2}{\partial x^\ell\partial x^m}(T^{\ell m}x^ix^j)=T^{\ell m}\frac{\partial^2(x^ix^j)}{\partial x^\ell\partial x^m}=2T^{ij},$$

where we used the result in Question 6. Therefore for sufficiently large $R>0$,

$$\begin{aligned}\int_{\mathbb{R}^3}T^{ij}\,\mathrm{d}^3\boldsymbol{x}&=\frac{1}{2}\int_{B(0,R)}\partial_\ell\partial_m(T^{\ell m}x^ix^j)\,\mathrm{d}^3\boldsymbol{x}=\frac{1}{2}\int_{B(0,R)}\mathrm{d}(\varepsilon_{nk\ell}\partial_m(T^{mn}x^ix^j)\,\mathrm{d}x^k\wedge\mathrm{d}x^\ell)\\&=\frac{1}{2}\oint_{\partial B(0,R)}\varepsilon_{nk\ell}\partial_m(T^{mn}x^ix^j)\,\mathrm{d}x^k\wedge\mathrm{d}x^\ell=0\end{aligned}$$

(b) From Question 6 we have

$$\int_{\mathbb{R}^3}T^{0j}\,\mathrm{d}^3\boldsymbol{x}=P^j=\frac{\mathrm{d}D^j}{\mathrm{d}t}=\int_{\mathbb{R}^3}\frac{\partial T^{00}}{\partial t}x^j\,\mathrm{d}^3\boldsymbol{x}=0.$$

(c) Since $\partial_i T^i{}_j=0$, we have $\partial_i(T^i{}_jx^j)=T^i{}_j\partial_ix^j=T^i{}_j\delta^j_i=T^i{}_i$. For sufficiently large $R>0$,

$$\begin{aligned}\int_{\mathbb{R}^3}T^i{}_i\,\mathrm{d}^3\boldsymbol{x}&=\int_{B(0,R)}\partial_i(T^i{}_jx^j)\,\mathrm{d}^3\boldsymbol{x}=\int_{B(0,R)}\mathrm{d}(\varepsilon_{k\ell m}T^k{}_jx^j\mathrm{d}x^\ell\wedge\mathrm{d}x^m)\\&=\oint_{\partial B(0,R)}\varepsilon_{k\ell m}T^k{}_jx^j\mathrm{d}x^\ell\wedge\mathrm{d}x^m=0\end{aligned}$$

(d) $\partial_i(T^{0i}x^jx^k)=T^{0i}x^j\partial_ix^k+T^{0i}x^k\partial_ix^j=T^{0k}x^j+T^{0j}x^k$. For sufficiently large $R>0$,

$$\begin{aligned}\int_{\mathbb{R}^3}(T^{0k}x^j+T^{0j}x^k)\,\mathrm{d}^3\boldsymbol{x}&=\int_{B(0,R)}\partial_i(T^{0i}x^jx^k)\,\mathrm{d}^3\boldsymbol{x}=\int_{B(0,R)}\mathrm{d}(\varepsilon_{i\ell m}T^{0i}x^jx^k\mathrm{d}x^\ell\wedge\mathrm{d}x^m)\\&=\oint_{\partial B(0,R)}\varepsilon_{i\ell m}T^{0i}x^j\mathrm{d}x^\ell\wedge\mathrm{d}x^m=0\end{aligned}$$

(e) $\partial_k\left(T^k{}_ix^ix^j-\frac{1}{2}T^{jk}x_ix^i\right)=\partial_k\left(T^k{}_ix^ix^j-\frac{1}{2}T^j{}_kx^ix^i\right)=T^j{}_ix^i+T^i{}_ix^j-T^j{}_ix^i=T^i{}_ix^j$. For sufficiently large $R > 0$,

$$\begin{aligned}\int_{\mathbb{R}^3} T^i{}_i x^j \, \mathrm{d}^3\boldsymbol{x} &= \int_{B(0,R)} \partial_k \left(T^k{}_i x^i x^j - \frac{1}{2} T^{jk} x_i x^i \right) \mathrm{d}^3\boldsymbol{x} \\ &= \int_{B(0,R)} \mathrm{d} \left(\varepsilon_{k\ell m} \left(T^k{}_i x^i x^j - \frac{1}{2} T^{jk} x_i x^i \right) \mathrm{d}x^\ell \wedge \mathrm{d}x^m \right) \\ &= \oint_{\partial B(0,R)} \varepsilon_{k\ell m} \left(T^k{}_i x^i x^j - \frac{1}{2} T^{jk} x_i x^i \right) \mathrm{d}x^\ell \wedge \mathrm{d}x^m = 0\end{aligned}$$

(f) This is familiar computation in multipole expansion:

$$\begin{aligned}\frac{1}{\|\boldsymbol{x} - \boldsymbol{x}'\|} &= ((\boldsymbol{x} - \boldsymbol{x}') \cdot (\boldsymbol{x} - \boldsymbol{x}'))^{-1/2} = \left(r^2 - 2\boldsymbol{x} \cdot \boldsymbol{x}' + \|\boldsymbol{x}'\|^2 \right)^{1/2} \\ &= \frac{1}{r} \left(1 - \frac{2\boldsymbol{x} \cdot \boldsymbol{x}'}{r^2} + \frac{\|\boldsymbol{x}'\|^2}{r^2} \right)^{-1/2} = \frac{1}{r} + \frac{\boldsymbol{x} \cdot \boldsymbol{x}'}{r^3} + \mathcal{O}(r^{-3}).\end{aligned}$$

□

Question 8. Geodesics in static spacetimes

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and let $M = \mathbb{R} \times \overline{M}$ with Lorentzian metric $g = -dx_0^2 + \overline{g}$. Show that $s \mapsto (\sigma^0(s), \overline{\sigma}(s))$ is an affinely parametrised geodesic in M if, and only if, $s \mapsto \overline{\sigma}(s)$ is an affinely parametrised geodesic in $\overline{M}$ and $\sigma^0(s) = \lambda s, \lambda \in \mathbb{R}$.

Proof. Let ∇^0 and $\overline{\nabla}$ be the Levi-Civita connections on $\mathbb{R}$ and $\overline{M}$ respectively. Let $\sigma(s) = (\sigma^0(s), \overline{\sigma}(s))$ be a path in M. The key observation is that (Question 3 of Sheet 1 of *C3.11 Riemannian Geometry*):

$$\nabla_{\dot\sigma} \dot\sigma = \nabla_{(\dot\sigma^0, \dot{\overline{\sigma}})} (\dot\sigma^0, \dot{\overline{\sigma}}) = \left(\nabla^0_{\dot\sigma^0} \dot\sigma^0, \overline{\nabla}_{\dot{\overline{\sigma}}} \dot{\overline{\sigma}} \right) = 0 \iff \nabla^0_{\dot\sigma^0} \dot\sigma^0 = 0 \wedge \overline{\nabla}_{\dot{\overline{\sigma}}} \dot{\overline{\sigma}} = 0.$$

That is, σ is an (affinely parametrised) geodesic in M if and only if σ^0 is an geodesic in $\mathbb{R}$ and $\overline{\sigma}$ is an geodesic in $\overline{M}$.

Yes, but should explicitly show that $\Gamma^0_{\mu\nu} = \Gamma^i_{0\nu} = 0$, $\Gamma^i_{jk} = \overline{\Gamma}^i_{jk}$, since they can be checked directly.

Next we expand the condition of being a geodesic on $\mathbb{R}$. We have

$$\nabla^0_{\dot\sigma^0} \dot\sigma^0 = \dot\sigma^0 \cdot \ddot\sigma^0 = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} (\dot\sigma^0)^2 = 0 \iff \dot\sigma^0 = \lambda \text{ for some } \lambda \in \mathbb{R}$$

Hence σ^0 is a geodesic if and only if $\sigma^0(s) = a + \lambda s$ for some $a, \lambda \in \mathbb{R}$. □

Section C: Optional

Question 9. Noether charges in Minkowski space

Let (M, g) be $(3+1)$-dimensional Minkowski spacetime with canonical $\{t, x, y, z\}$ coordinates and assume that T vanishes for $r = \sqrt{x^2 + y^2 + z^2}$ large enough. From Problem 5 we know that each Killing vector field K results in the corresponding conserved charge

$$Q[K] := \int_{t=t_0} J_0 \left(t_0, x, y, z \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Express the conserved charges corresponding to the 10 linearly independent Killing vector fields found in Problem 6.(b) from the first problem sheet in terms of the components of the stress-energy tensor. Give a physical interpretation of each of these charges.