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**Problem Sheet 3**

Birkhoff Theorem and Neutron Stars

# B5: General Relativity

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Affine connection for diagonal $g_{\mu\nu}$ :

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right) \quad \text{NO SUM OVER } \lambda$$

Ricci Tensor:

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}} \quad \text{FULL SUMMATION}$$

**Question 1. Birkhoff's Theorem**

**In this problem, we will set $c = 1$.**

a) Birkhoff's theorem states that outside of a spherical distribution of matter, the metric tensor must be independent of time and equal to the Schwarzschild metric - even if the matter distrbution is changing (keeping spherical symmetry) with time. A corollary is that within the hollow of an external spherical distribution of matter, the metric tensor is Minkowski spacetime. These are the precise relativistic analogues of the Newtonian results of a point mass $1/r$ potential outside any spherical distribution of matter, and the vanishing of the gravitational field inside a cavity with a spherical external distribution of matter. Birkhoff's theorem is critical to formulating cosmology. To prove the theorem is straightforward but a bit painful, because we need to calculate the Ricci tensor $R_{\mu\kappa}$, and that is always a nuisance. Both because Birkhoff's theorem is important, as well as to get practice working with the Ricci tensor, we will explicitly evaluate the key $R_{tr}$ component here, a critical step in the proof [1]. The $R_{tr}$ component vanishes identically for the static Schwarzschild metric. Consider the line element for a general time-dependent spherical system,

$$-d\tau^2 = -B(r,t)dt^2 + A(r,t)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \equiv g_{\mu\nu}dx^{\mu}dx^{\nu}$$

The nonvanishing affine connection components $\Gamma^{a}_{bc}$ from this metric tensor are the same nonvanishing set we found for the Schwarzschild metric, plus three others that used to be zero. In particular, show that

$$\Gamma^{r}_{rt} = \frac{\dot{A}}{2A}, \quad \Gamma^{t}_{rr} = \frac{\dot{A}}{2B} \quad \Gamma^{t}_{tt} = \frac{\dot{B}}{2B}$$

where we will use notation $\dot{A}$ for a time derivative and $A'$ for an $r$ derivative. A warm welcome to our new three new affine connection members.

b) We will now show that $R_{tr} = -\dot{A}/rA$, which sure looks simple but in fact involves a large cancellation. The point now is that since all the $R_{\mu\kappa}$ terms must vanish in a vacuum, $\dot{A} = 0$, and $A$ cannot depend on time. The other components of the Ricci tensor then all revert back to their Schwarzschild forms. (We won't show this explicitly, only because it is a long and dull exercise, but it is not particularly difficult). A reduction to the Schwarzschild problem means that any possible time dependence in $B$ can appear only as an overall multiplicative factor $f(t)$, which can then be completely eliminated by a simple time coordinate transformation $dt' = f dt$. The static metric is then identical to Schwarzschild. This is Birkhoff's theorem.

Using the Ricci tensor above, show that the first two groupings

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}}$$

cancel one another out precisely. This is progress. ($g$ is the determinant of $g_{\mu\nu}$.)

c) Show next that for $R_{\mu\kappa} = R_{tr}$

$$\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}} = -\frac{\dot{A}}{rA}$$

(Use §6.1 from the notes for any $\Gamma$ 's you need.) You will find that everything cancels once again, except for one final term in the $\ln|g|$ derivative, shown on the right. With $\dot{A} = 0$, Birkhoff's theorem follows relatively easily, as the remaining $R_{\mu\kappa} = 0$ equations reduce to the Schwarzschild problem.

*Proof.* a) *In general the metric should also have a cross term* $C(r,t)\mathrm{d}t\,\mathrm{d}r$. *Perhaps the question neglects this for simplicity...*

If you start with such a term, you can remove it with a coordinate transformation

The metric tensor is given by

$$g = -B(r,t)\,\mathrm{d}t^2 + A(r,t)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

The Lagrangian is given by

$$\mathscr{L} = -B(r,t)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + A(r,t)\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)^2$$

where $\lambda$ is an affine parameter. By Euler-Lagrange equation, we have

$$\begin{aligned}
-2\frac{\mathrm{d}}{\mathrm{d}\lambda}(B\dot{t}) + \frac{\partial B}{\partial t}\dot{t}^2 - \frac{\partial A}{\partial t}\dot{r}^2 = 0 &\implies \ddot{t} + \frac{\partial_t B}{2B}\dot{t}^2 + \frac{\partial_r B}{B}\dot{r}\dot{t} + \frac{\partial_t A}{2B}\dot{r}^2 = 0\\
&\implies \Gamma^t_{tt} = \frac{\partial_t B}{2B},\quad \Gamma^t_{rt} = \frac{\partial_r B}{2B},\quad \Gamma^t_{rr} = \frac{\partial_t A}{2B}\\
2\frac{\mathrm{d}}{\mathrm{d}\lambda}(A\dot{r}) + \frac{\partial B}{\partial r}\dot{t}^2 - \frac{\partial A}{\partial r}\dot{r}^2 = 0 &\implies \ddot{r} + \frac{\partial_r B}{2A}\dot{t}^2 + \frac{\partial_t A}{A}\dot{r}\dot{t} + \frac{\partial_r A}{2A}\dot{r}^2 = 0\\
&\implies \Gamma^r_{tt} = \frac{\partial_r B}{2A},\quad \Gamma^r_{rt} = \frac{\partial_t A}{2A},\quad \Gamma^r_{rr} = \frac{\partial_r A}{2A}\\
2\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\dot{\theta}) - 2r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0 &\implies \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0\\
&\implies \Gamma^\theta_{r\theta} = \frac{1}{r},\quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\\
2\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\sin^2\theta\dot{\varphi}) = 0 &\implies \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0\\
&\implies \Gamma^\varphi_{r\varphi} = \frac{1}{r},\quad \Gamma^\varphi_{\theta\varphi} = \cot\theta
\end{aligned}$$

where the dot denotes $\mathrm{d}/\mathrm{d}\lambda$. Now all non-vanishing Christoffel symbols are given by

$$\Gamma^t_{tt} = \frac{\partial_t B}{2B},\quad \Gamma^t_{rt} = \frac{\partial_r B}{2B},\quad \Gamma^t_{rr} = \frac{\partial_t A}{2B},\quad \Gamma^r_{tt} = \frac{\partial_r B}{2A},\quad \Gamma^r_{rt} = \frac{\partial_t A}{2A},\quad \Gamma^r_{rr} = \frac{\partial_r A}{2A}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r},\quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,\quad \Gamma^\varphi_{r\varphi} = \frac{1}{r},\quad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

b) The expressions cancel when $(\mu,\kappa) = (t,r)$:

$$\frac{1}{2}\frac{\partial^2 \ln|\det g|}{\partial t\partial r} = \frac{1}{2}\frac{\partial^2}{\partial t\partial r}\ln(|AB|r^4\sin^2\theta) = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial_t A}{A} + \frac{\partial_r A}{A}\right) = \frac{1}{2}\left(\frac{A\partial_t\partial_r A - \partial_t A\partial_r A}{A^2} + \frac{B\partial_t\partial_r B - \partial_t B\partial_r B}{B^2}\right)$$

$$\frac{\partial\Gamma^\lambda_{tr}}{\partial x^\lambda} = \frac{\partial\Gamma^t_{tr}}{\partial t} + \frac{\partial\Gamma^r_{tr}}{\partial r} = \frac{\partial}{\partial t}\left(\frac{\partial_r B}{2B}\right) + \frac{\partial}{\partial r}\left(\frac{\partial_t A}{2A}\right) = \frac{1}{2}\left(\frac{A\partial_t\partial_r A - \partial_t A\partial_r A}{A^2} + \frac{B\partial_t\partial_r B - \partial_t B\partial_r B}{B^2}\right)$$

Hence $\displaystyle\frac{1}{2}\frac{\partial^2 \ln|\det g|}{\partial t\partial r} = \frac{\partial\Gamma^\lambda_{tr}}{\partial x^\lambda}$ as claimed.

c) From the formula

$$R_{tr} = \frac{1}{2}\frac{\partial^2\ln|g|}{\partial r\partial t} - \frac{\partial\Gamma^\lambda_{tr}}{\partial x^\lambda} + \Gamma^\eta_{t\lambda}\Gamma^\lambda_{r\eta} - \frac{\Gamma^\eta_{tr}}{2}\frac{\partial\ln|g|}{\partial x^\eta}$$

We continue the computation:

$$\begin{aligned}
\Gamma^\eta_{t\lambda}\Gamma^\lambda_{r\eta} &= \Gamma^t_{tt}\Gamma^t_{rt} + \Gamma^r_{tt}\Gamma^t_{rr} + \Gamma^t_{tr}\Gamma^r_{rt} + \Gamma^r_{tr}\Gamma^r_{rr}\\
&= \frac{\partial_t B}{2B}\frac{\partial_r B}{2B} + \frac{\partial_r B}{2A}\frac{\partial_t A}{2A} + \frac{\partial_r B}{2B}\frac{\partial_t A}{2A} + \frac{\partial_t A}{2A}\frac{\partial_r A}{2A}\\
&= \frac{1}{4}\left(\frac{\partial_t A\partial_r A}{A^2} + \frac{\partial_t B\partial_r B}{B^2} + \frac{2\partial_r B\partial_t A}{AB}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\Gamma^\eta_{tr}}{2}\frac{\partial}{\partial x^\mu}\ln|\det g| &= \frac{\Gamma^t_{tr}}{2}\frac{\partial}{\partial t}\ln(|AB|r^4\sin^\theta) + \frac{\Gamma^r_{tr}}{2}\frac{\partial}{\partial r}\ln(|AB|r^4\sin^\theta)\\
&= \frac{1}{2}\frac{\partial_r B}{2B}\left(\frac{\partial_t A}{A} + \frac{\partial_t B}{B}\right) + \frac{1}{2}\frac{\partial_t A}{2A}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B} + \frac{4}{r}\right)
\end{aligned}$$

$$= \frac{1}{4}\left(\frac{\partial_t A \partial_r A}{A^2} + \frac{\partial_t B \partial_r B}{B^2} + \frac{2\partial_r B \partial_t A}{AB}\right) + \frac{\partial_t A}{rA}$$

Then we deduce that $R_{tr} = -\frac{\partial_t A}{rA}$. As Einstein's equation requires that $R_{tr} = 0$, we deduce that $A$ is time-independent.

□

**Question 2**

In this problem we will show that the TOV equation must be obeyed if we demand that the total energy (including the gravitational contribution) is minimised when we vary $\rho(r)$ throughout a star of uniform entropy per particle, subject to the constraint that the total number of particles the same.

a) Use the method of Lagrange multipliers to show that the above statement translates into the condition that variations of the quantity

$$M - \lambda N \equiv \int_0^\infty 4\pi r^2 \rho \, dr - \lambda \int_0^\infty 4\pi n r^2 A^{1/2} dr$$

must be zero. Of course $\rho$ and $n$ vanish outside the star, so the integration limits are formal. Notation: $\rho$ is the energy density (divided by $c^2$), $n$ the number density of baryons, and $A$ is $g_{rr}$ from the interior stellar metric of §8.2 in the notes:

$$A = \left(1 - 2G\mathcal{M}(r)/rc^2\right)^{-1}, \quad \mathcal{M}(r) = \int_0^r 4\pi\rho r'^2 dr'$$

Finally, $\lambda$ is the (constant) Lagrange multiplier.

b) Prove that the first order variation $\delta$ of this equation gives:

$$\delta M - \lambda \delta N = \int_0^\infty 4\pi r^2 \left[\delta\rho - \lambda A^{1/2}\delta n - \lambda \frac{GA^{3/2}}{rc^2} n \delta\mathcal{M}\right] dr$$

c) For the constant entropy (adiabatic) perturbations, the first law of thermodynamics is $dE = -PdV$ where $E$ is the energy within some volume $V$, $P$ is the pressure and $dV$ is a small volume change. If particle number is conserved so that $nV$ also remains constant, show that $\delta n$ and $\delta\rho$ are related by:

$$\delta n = \frac{n\delta\rho}{\rho + P/c^2}$$

d) Put all these results together and show that

$$\delta M - \lambda\delta N = \int_0^\infty 4\pi r^2 \delta\rho \left[1 - \frac{\lambda n A^{1/2}}{\rho + P/c^2} - \frac{\lambda G}{c^2}\int_r^\infty \left(4\pi r' n A^{3/2} dr'\right)\right] dr = 0$$

[*Hint: In the expression for $\delta\mathcal{M}$, you will need to invert the order of integration for $r$ and $r'$. Also, they are just dummy variables, so you are allowed to interchange their names at the end!*]

e) If the result of 2d) is to hold for any $\delta\rho$, show that $1/\lambda = F(r)$, where $F$ is a function of $r$, which you should determine. But remember that $\lambda$ must be constant, so that in fact $dF/dr = 0$! Show that this leads to the TOV equation,

$$\frac{dP}{dr} = -\frac{GA}{r^2}\left(\mathcal{M} + 4\pi r^3 P/c^2\right)\left(\rho + P/c^2\right)$$

[*Hint: in a constant entropy star, the equation of part 2c) also holds with $\delta n$ and $\delta\rho$ replaced by $dn/dr$ and $d\rho/dr$. Why is that?*]

We have thus shown that when the TOV is satisfied, the total energy is minimised for a star of constant entropy. The TOV equation itself is valid independently of any thermodynamics: it is a direct consequence of the field equations of gravity. But the constant entropy case allows a variational formulation for it. Notice as well that we reach the Newtonian limit in our final equation by setting $A = 1$ and letting $c \to \infty$. Could we set $A = 1$ at the start of our derivation and reach the Newtonian limit?

*Proof.* a) The metric is given by

$$g = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$

Note that the spatial volume element is given by

$$\mathrm{d}V = \sqrt{|g_{ii}|}\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi = A^{1/2}r^2\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi$$

The total energy is $\mathscr{M}$ at infinity, which is given by

$$M = \mathscr{M}(+\infty) = 4\pi\int_0^\infty \rho r^2\,\mathrm{d}r$$

The total number of particles is given by

$$N = \int_\Sigma n\,\mathrm{d}V = \int_{\mathbb{R}^3} n\sqrt{|g_{ii}|}\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi = 4\pi\int_0^\infty nA^{1/2}r^2\,\mathrm{d}r$$

(The difference in the factor $A^{1/2}$ may be due to that $n$ is defined in the rest inertial frame...)

Now we need to minimise $M$ while keeping $N$ constant. By the method of Lagrange multipliers (in the form of variational calculus), we should minimise the functional

$$I[\rho] := M - \lambda N = \int_0^\infty 4\pi r^2\rho\,dr - \lambda\int_0^\infty 4\pi n r^2 A^{1/2}\,dr$$

So the variation of $I$ is given by

$$\frac{\delta I[\rho]}{\delta\rho} := \frac{\mathrm{d}}{\mathrm{d}s}I[\rho(r)+s\eta(r)]\bigg|_{s=0} = 0$$

for any $\eta\in C_c^\infty(0,\infty)$.

b) The variation $\delta I$ of the functional $I$ satisfies

$$I[\rho+s\eta] = I[\rho] + \delta I[\rho] + o(\varepsilon)$$

and the variation $\delta\rho$ is simply $s\eta$ for small $s$.

Let $f(r,\rho) := 4\pi r^2\left(\rho - \lambda A^{1/2}(\rho,r)n(\rho,r)\right)$. Then

$$\delta I = \int_0^\infty \frac{\partial f}{\partial\rho}\delta\rho\,\mathrm{d}r = \int_0^\infty 4\pi r^2\left(\delta\rho - \lambda A^{1/2}\delta n - \lambda\frac{\partial A^{1/2}}{\partial\mathscr{M}}\delta\mathscr{M}\right)\mathrm{d}r = \int_0^\infty 4\pi r^2\left(\delta\rho - \lambda A^{1/2}\delta n - \lambda\frac{GA^{3/2}}{rc^2}\delta\mathscr{M}\right)\mathrm{d}r$$

c) The energy is given by

$$E = \int_V \rho c^2\,\mathrm{d}V$$

So the variation is given by

$$\delta E = c^2(\rho\delta V + V\delta\rho)$$

By first law of thermodynamics we have

$$c^2(\rho\delta V + V\delta\rho) = -P\delta V \implies \delta V = -\frac{V}{\rho + P/c^2}\delta\rho$$

With $nV = \text{const}$, we have $\delta n = -\frac{n}{V}\delta V$. Hence

$$\delta n = \frac{n}{\rho + P/c^2}\delta\rho$$

d) By definition

$$\mathscr{M} = 4\pi\int_0^r \rho r'^2\,\mathrm{d}r' \implies \delta\mathscr{M} = 4\pi\int_0^r \delta\rho r'^2\,\mathrm{d}r'$$

Hence by Fubini's Theorem,

$$\int_0^\infty -4\pi r^2\lambda\frac{GA^{3/2}(r)}{rc^2}\delta M\,\mathrm{d}r = \int_0^\infty -4\pi r\lambda\frac{GA^{3/2}(r)}{c^2}\int_0^r \delta\rho 4\pi r'^2\,\mathrm{d}r'\,\mathrm{d}r = \int_0^\infty -4\pi r^2\delta\rho\frac{\lambda G}{c^2}\int_r^\infty 4\pi r' nA^{3/2}(r')\,\mathrm{d}r'\,\mathrm{d}r$$

Combining all results we obtain

$$\delta M - \lambda \delta N = \int_0^\infty 4\pi r^2 \delta\rho \left(1 - \frac{\lambda n A^{1/2}}{\rho + P/c^2} - \frac{\lambda G}{c^2} \int_r^\infty 4\pi r' n A^{3/2}(r')\,\mathrm{d}r'\right) \mathrm{d}r = 0$$

e) By fundamental lemma of calculus of variations, we have

$$1 - \frac{\lambda n A^{1/2}}{\rho + P/c^2} - \frac{\lambda G}{c^2} \int_r^\infty 4\pi r' n A^{3/2}(r')\,\mathrm{d}r' = 0$$

Hence we have a constant

$$\frac{1}{\lambda} = \frac{n A^{1/2}}{\rho + P/c^2} + \frac{G}{c^2} \int_r^\infty 4\pi r' n A^{3/2}(r')\,\mathrm{d}r' =: F(r)$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n A^{1/2}}{\rho + P/c^2} + \frac{G}{c^2} \int_r^\infty 4\pi r' n A^{3/2}(r')\,\mathrm{d}r'\right) = 0$$

It is quite a painful process to compute the derivative... For the first term, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n A^{1/2}}{\rho + P/c^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n}{\rho + P/c^2}\right) A^{1/2} + \frac{n}{\rho + P/c^2}\frac{\mathrm{d}A^{1/2}}{\mathrm{d}r} \\
&= \frac{\frac{\mathrm{d}n}{\mathrm{d}r}\left(\rho + \frac{P}{c^2}\right) - n\left(\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{c^2}\frac{\mathrm{d}P}{\mathrm{d}r}\right)}{(\rho + P/c^2)^2} A^{1/2} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathcal{M}}{r}\right)
\end{aligned}$$

The result in (c) shows that

$$\frac{\mathrm{d}n}{\mathrm{d}r}\left(\rho + \frac{P}{c^2}\right) = n\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n A^{1/2}}{\rho + P/c^2}\right) &= -\frac{n A^{1/2}}{(\rho + P/c^2)^2 c^2}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathcal{M}}{r}\right) \\
&= -\frac{n A^{1/2}}{(\rho + P/c^2)^2 c^2}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\left(\frac{1}{r}\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}r} - \frac{\mathcal{M}}{r^2}\right) \\
&= -\frac{n A^{1/2}}{(\rho + P/c^2)^2 c^2}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r 4\pi r'^2 \rho\,\mathrm{d}r' - \frac{\mathcal{M}}{r^2}\right) \\
&= -\frac{n A^{1/2}}{(\rho + P/c^2)^2 c^2}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\left(4\pi r\rho - \frac{\mathcal{M}}{r^2}\right)
\end{aligned}$$

Hence we have

$$-\frac{n A^{1/2}}{(\rho + P/c^2)^2 c^2}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{n}{\rho + P/c^2}\frac{G A^{3/2}}{c^2}\left(4\pi r\rho - \frac{\mathcal{M}}{r^2}\right) - \frac{G}{c^2} 4\pi r n A^{3/2} = 0$$

Rearranging the expression, we obtain the TOV equation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -GA\left(\rho + \frac{P}{c^2}\right)\left(\frac{\mathcal{M}}{r^2} + \frac{4\pi r P}{c^2}\right)$$

In the Newtonian limit, $A = 1$ and $c \to \infty$. We obtain

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\mathcal{M}\rho}{r^2}$$

We cannot set $A = 1$ at the start, otherwise we will not obtain the dependence of $\mathcal{M}$ in our final expression. □

**Question 3. Rotating, relativistic stars.**

a) Do the following Exercise from §4.6 in the notes. Starting with the formal stress tensor conservation equation for an ideal fluid,

$$0=\frac{\partial P}{\partial x^\nu}+\frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}\left(\rho+P/c^2\right)U^\mu U_\nu\right]-\Gamma^\mu_{\nu\lambda}\left(\rho+P/c^2\right)U_\mu U^\lambda$$

use the general expression for $\Gamma^\mu_{\nu\lambda}$ to show that this may be written (more simply) as:

$$0=\frac{\partial P}{\partial x^\nu}+\frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}\left(\rho+P/c^2\right)U^\mu U_\nu\right]+\left(\rho+P/c^2\right)U_\mu\frac{\partial U^\mu}{\partial x^\nu}$$

b) Next, consider the case of a rotating star. The uniform rotation rate $\Omega=d\phi/dt$ is assumed to be constant. On the surface (and in the interior) of the star, the 4-velocity component $U^\phi=d\phi/d\tau=\Omega U^0$, where as usual $U^0=dt/d\tau$. There are no other 4 -velocity $U^\mu$ components, and no $t$ or $\phi$ dependence of the stellar structure. Show that, under these conditions, our equation becomes

$$0=\frac{\partial P}{\partial x^\nu}-\left(\rho c^2+P\right)\frac{\partial\ln U^0}{\partial x^\nu}$$

c) Work now with spatial coordinates, $i,j,k$ for $\nu$. Recall that if $A_i$ is a covariant vector, the curl operator $\nabla\times A=\partial_j A_i-\partial_i A_j$ is also a vector, as the affine connection terms from the covariant derivatives cancel. Show that if surfaces of constant of $\rho$ and constant $P$ (or any two functions at all) coincide, then $(\partial_i P)\left(\partial_j\rho\right)=\left(\partial_j P\right)\left(\partial_i\rho\right)$.

d) Using 3c), show that for a rotating star, surfaces of constant $\rho$, $P$, and $U^0$ all coincide. Viewed by an observer at infinity, do surface clocks on a rotating neutron star run faster at the equator or the poles? Does spherical symmetry matter?

*Proof.* a) The Christoffel symbol is given in terms of the metric by

$$\Gamma^\mu_{\nu\lambda}=\frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda}\right)$$

Hence

$$\begin{aligned}\Gamma^\mu_{\nu\lambda}U_\mu U^\lambda&=\frac{1}{2}g^{\mu\sigma}U_\mu U^\lambda\left(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda}\right)\\&=\frac{1}{2}U^\sigma U^\lambda\left(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda}\right)\\&=\frac{1}{2}U^\sigma U^\lambda\partial_\nu g_{\lambda\sigma}\end{aligned}$$

Note that $-c^2=U^\lambda U^\sigma g_{\lambda\sigma}$. We have

$$\begin{aligned}0=\partial_\nu(-c^2)&=\partial_\nu(U^\lambda U^\sigma g_{\lambda\sigma})=U^\lambda U^\sigma\partial_\nu g_{\lambda\sigma}+U^\lambda g_{\lambda\sigma}\partial_\nu U^\sigma+U^\sigma g_{\lambda\sigma}\partial_\nu U^\lambda\\&=U^\lambda U^\sigma\partial_\nu g_{\lambda\sigma}+2U_\mu\partial_\nu U^\mu\end{aligned}$$

Hence

$$\Gamma^\mu_{\nu\lambda}U_\mu U^\lambda=-U_\mu\partial_\nu U^\mu$$

We deduce that

$$0=\frac{\partial P}{\partial x^\nu}+\frac{1}{|\det g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|\det g|^{1/2}\left(\rho+P/c^2\right)U^\mu U_\nu\right]+\left(\rho+P/c^2\right)U_\mu\frac{\partial U^\mu}{\partial x^\nu}$$

b) The 4-velocity is given by

$$\mathsf{U}=(U^0,0,0,U^\varphi)=U^0(1,0,0,\Omega)$$

We look at the third term:

$$U_\mu\frac{\partial U^\mu}{\partial x^\nu}=U_0\frac{\partial U^0}{\partial x^\nu}+U_\varphi\frac{\partial U^\varphi}{\partial x^\nu}=\left(U_0+\Omega U_\varphi\right)\frac{\partial U^0}{\partial x^\nu}=\frac{U_0U^0+U_\varphi U^\varphi}{U^0}\frac{\partial U^0}{\partial x^\nu}=\frac{-c^2}{U^0}\frac{\partial U^0}{\partial x^\nu}=-c^2\frac{\partial\ln U^0}{\partial x^\nu}$$

For the second term, since the metric is diagonal, it vanishes automatically for $\nu=r,\theta$. For $\nu=0,\varphi$, we can use the fact that $\det g$, $\rho$ and $P$ have no dependence on $x^\nu$. We have

$$\frac{1}{|\det g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|\det g|^{1/2}\left(\rho+P/c^2\right)U^\mu U_\nu\right]=(\rho+P/c^2)\partial_\mu(U^\mu U_\nu)=\begin{cases}(\rho+P/c^2)g_{00}\partial_0(U^0U^0), & \nu=0\\ (\rho+P/c^2)g_{\varphi\varphi}\Omega^2\partial_\varphi(U^0U^0), & \nu=\varphi\end{cases}$$

But

$$-c^2=U^0U_0+U^\varphi U_\varphi=\left(g_{00}+\Omega^2 g_{\varphi\varphi}\right)U^0U^0$$

So the fact that $g_{00}$ and $g_{\varphi\varphi}$ are independent of $t$ and $\varphi$ implies that $U^0$ is independent of $t$ and $\varphi$. We deduce that the second term vanishes for $\nu=0,r,\theta,\varphi$. The equation of hydrostatic equilibrium becomes

$$0=\frac{\partial P}{\partial x^\nu}-\left(\rho c^2+P\right)\frac{\partial \ln U^0}{\partial x^\nu}$$

c) For the Euclidean space $\mathbb{R}^3$, we know that if two scalar fields have the same surfaces of constant, then their gradient vectors are parallel. With some similar idea we may generalise the result to the manifold. That is, $\nabla P=\lambda\nabla\rho$ for some $\lambda\neq 0$. In terms of components this is

$$\nabla_i P\cdot\nabla_j\rho=\nabla_j P\cdot\nabla_i\rho$$

And we know that $\nabla_i=\partial_i$ when acting on scalar fields.

d) Assuming that $P$ is a function of $\rho$. Then the surfaces of constant of $P$ and $\rho$ coincide. Note that the gradient vector of $P$ and $\ln U^0$ are parallel as indicated by the hydrostatic equilibrium equation.

no need to assume $P=P(\rho)$

*It seems that we may need some theorem like: Let $N$ be a hypersurface of the Riemannian manifold $(M,g)$. There exists a function $\xi$ on $M$ such that $(\mathrm{d}\xi)^\sharp\in(\mathrm{T}_pN)^\perp$ for all $p\in N$. Then $\xi=$ const on $N$. I don't know if this is correct.* If we assume this then $U^0$ is constant on the level sets of $P$. So the surfaces of constant of $P$, $\rho$ and $U^0$ coincide.

$U^0$ measures the local time dilation effect. On the surface of the rotating star, $P=$ const. Hence $U^0=$ const. In other words, the time dilation effect is uniform on the surfaec of the neutron star. This result does not require the rotating star to be spherically symmetric. □

**Question 4. Explicit degenerate ideal gas neutron star equations.**

a) Show that the differential equations for a star neutron star with a fully degenerate ideal gas equation of state are (refer to and use §8.5 of the notes for complete background and definitions):

$$\frac{dt}{dy}=-\frac{4m}{y^2(1-2m/y)}\left(\frac{\sinh t-2\sinh(t/2)}{\cosh t-4\cosh(t/2)+3}\right)\left(1+\frac{\pi y^3}{8m}[\sinh t-8\sinh(t/2)+3t]\right)$$

$$\frac{dm}{dy}=\frac{3\pi y^2}{8}(\sinh t-t)$$

The dimensionless variables $m$ and $y$ are defined in terms of radius $r$ and $\mathcal{M}$ (mass within $r$) by:

$$\rho_{\text{char}}=8\pi m_n^4c^3/3h^3,\quad r\equiv c\left(G\rho_{\text{char}}\right)^{-1/2}y,\quad \mathcal{M}\equiv c^3G^{-3/2}\rho_{\text{char}}^{-1/2}m$$

Recall the equation of state parameterisation from the notes:

$$\rho(t)=\frac{3\rho_{\text{char}}}{32}(\sinh t-t),\quad P(t)=\frac{\rho_{\text{char}}c^2}{32}[\sinh t-8\sinh(t/2)+3t]$$

A neutron star model consists of picking some $t=t_0$ at $y=0$ to fix the central density $\rho_0=(3/32)\rho_{\text{char}}(\sinh t_0-t_0)$, setting $m=\pi(\sinh t_0-t_0)y^3/8$ at small $y$ (justify!), and integrating the equations until $t=0$ at some finite value of $y$. This defines the outer edge of the star. A value of $t_0\simeq 3$ yields the maximum mass of $0.7M_\odot$.

b) Take the limit $t\to\infty$ and recover the extreme relativistic TOV equation:

$$\frac{d\rho}{dr}=-\frac{4G\mathcal{M}(r)\rho}{r^2c^2}\left(1+\frac{4\pi\rho r^3}{3\mathcal{M}(r)}\right)\left(1-\frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1}$$

*Proof.* a) From the notes, the parameter $t$ is given by

$$t = 4\,\mathrm{arsh}\left(\frac{p_F}{m_n c}\right)$$

First we compute $\mathrm{d}P/\mathrm{d}t$:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\rho_{\text{char}} c^2}{32}\frac{\mathrm{d}}{\mathrm{d}t}(\sinh t - 8\sinh(t/2) + 3t) = \frac{\rho_{\text{char}} c^2}{32}(\cosh t - 4\cosh(t/2) + 3)$$

Then

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\sqrt{G\rho_{\text{char}}}}{c}\frac{\mathrm{d}P}{\mathrm{d}y} = \frac{\sqrt{G\rho_{\text{char}}}}{c}\frac{\mathrm{d}P}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{\mathrm{d}t}{\mathrm{d}y}\frac{G^{1/2}\rho_{\text{char}}^{3/2} c}{32}(\cosh t - 4\cosh(t/2) + 3)$$

where $\mathrm{d}P/\mathrm{d}r$ is given by the TOV equation:

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} &= -G\left(\rho + \frac{P}{c^2}\right)\left(\frac{\mathcal{M}}{r^2} + \frac{4\pi r P}{c^2}\right)\left(1 - \frac{2G\mathcal{M}}{rc^2}\right)^{-1} \\
&= -G\cdot\frac{\rho_{\text{char}}}{8}(\sinh t - 2\sinh(t/2))\cdot\frac{c\rho_{\text{char}}^{1/2}}{G^{1/2}}\left(\frac{m}{y^2} + \frac{1}{8}\pi y(\sinh t - 8\sinh(t/2) + 3t)\right)\cdot\left(1 - \frac{2m}{y}\right)^{-1} \\
&= -\frac{1}{8}G^{1/2} c\rho_{\text{char}}^{3/2}\frac{m}{y^2(1 - 2m/y)}(\sinh t - 2\sinh(t/2))\left(1 + \frac{\pi y^2}{8m}(\sinh t - 8\sinh(t/2) + 3t)\right)
\end{aligned}$$

We obtain the expression for $\mathrm{d}t/\mathrm{d}y$:

$$\frac{\mathrm{d}t}{\mathrm{d}y} = -\frac{4m}{y^2(1-2m/y)}\left(\frac{\sinh t - 2\sinh(t/2)}{\cosh t - 4\cosh(t/2) + 3}\right)\left(1 + \frac{\pi y^3}{8m}[\sinh t - 8\sinh(t/2) + 3t]\right)$$

Since

$$\mathcal{M} = \int_0^r 4\pi r' \rho\,\mathrm{d}r'$$

We have

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{y}{\rho_{\text{char}}}4\pi\cdot\frac{3\rho_{\text{char}}}{32}(\sinh t - t) = \frac{3\pi y}{8}(\sinh t - t)$$

Regarding the comments in part (a), for small $y$ we may treat $t = t_0$ as a constant, and integrate $\mathrm{d}m/\mathrm{d}y$:

$$m = \int_0^y \frac{\mathrm{d}m}{\mathrm{d}y}\,\mathrm{d}y = \int_0^y \frac{3\pi y^2}{8}(\sinh t_0 - t_0)\,\mathrm{d}y = \frac{\pi y^3}{8}(\sinh t_0 - t_0)$$

b) Note that in general we have

$$P(t) = \frac{c^2}{3}(\rho(t) - 8\rho(t/2))$$

For large $t$, $\rho(t) \propto (\sinh t - t)$ grows like $\exp t$. Hence $\rho(t) \gg \rho(t/2)$. In the ultra-relativistic approximation, we have

$$P = \frac{c^2}{3}\rho$$

Substituting into the TOV equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{3G}{c^2}\left(\rho + \frac{P}{c^2}\right)\left(\frac{\mathcal{M}}{r^2} + \frac{4\pi r P}{c^2}\right)\left(1 - \frac{2G\mathcal{M}}{rc^2}\right)^{-1} = -\frac{4G\mathcal{M}\rho}{r^2c^2}\left(1 + \frac{4\pi r^3\rho}{3\mathcal{M}}\right)\left(1 - \frac{2G\mathcal{M}}{rc^2}\right)^{-1} \qquad \square$$