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Problem Sheet 3

Forces and Fields

B2: Symmetry & Relativity

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Remark. Personal Conventions.

The Minkowski metric is $g = \mathrm{diag}(-1, +1, +1, +1)$. Uppercase letters, like P and U, denote 4-vectors. Bold lowercase letters, like $\mathbf{p}$ and $\mathbf{u}$, denote 3-vectors. Greek letters $\mu, \nu, \ldots$ take values from 0 to 4. Latin letters $a, b, \ldots$ take values from 1 to 4. The bilinear form or pseudo-inner product $X^\mu Y_\mu = g_{\mu\nu} X^\mu Y^\nu$ is denoted by $g(X, Y)$.

Question 1

Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to t'.

[Hint: argue that the relative velocity $\mathbf{v}$ of the reference frame is constant, and use or derive an expression for $\mathrm{d}t/\mathrm{d}t'$]

Proof. Let m be a particle moving in the inertial frame S with 3-velocity $\mathbf{u}$. Let S' be another inertial frame which has velocity $\mathbf{v}$ relative to S.

Consider the Lorentz transformation of 4-momenta:

$$P'^\mu = L^\mu_\nu P^\nu$$

Taking derivative with respect to t':

$$\frac{\mathrm{d}P'^\mu}{\mathrm{d}t'} = \frac{\mathrm{d}}{\mathrm{d}t'}\left(L^\mu_\nu P^\nu\right) = \frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\mathrm{d}}{\mathrm{d}t}\left(L^\mu_\nu P^\nu\right) = \frac{\mathrm{d}t}{\mathrm{d}t'} L^\mu_\nu \frac{\mathrm{d}P^\nu}{\mathrm{d}t}$$

in which we used $\mathrm{d}L^\mu_\nu/\mathrm{d}t = 0$, because $\mathbf{d}\mathbf{v}/\mathrm{d}t = 0$ (S' is an inertial frame).

In terms of components, we have

$$\frac{\mathrm{d}P'}{\mathrm{d}t} = \left(\frac{1}{c}\frac{\mathrm{d}E'}{\mathrm{d}t'}, \frac{\mathrm{d}\mathbf{p}'}{\mathrm{d}t'}\right) = \left(\frac{1}{c}\frac{\mathrm{d}E'}{\mathrm{d}t'}, \mathbf{f}'\right), \text{?} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = \left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \mathbf{f}\right)$$

where $\mathbf{f}'$ and $\mathbf{f}$ are the 3-force acting on m in the frame S' and S respectively.

Let $\boldsymbol{\beta} := \mathbf{v}/c$. The Lorentz transformation in components:

$$\frac{1}{c}\frac{\mathrm{d}E'}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}t'}\gamma_v\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t} - \boldsymbol{\beta}\cdot\mathbf{f}\right);$$
$$\mathbf{f}' = \frac{\mathrm{d}t}{\mathrm{d}t'}\gamma_v\left(\mathbf{f} - \frac{\boldsymbol{\beta}}{c}\frac{\mathrm{d}E}{\mathrm{d}t}\right)$$

Let τ be the proper time of the particle. Then

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t'} = \frac{\gamma_u}{\gamma_{u'}}$$

From Question 2 of Sheet 2 we have

$$\gamma_{u'} = \gamma_u\gamma_v\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)$$

Hence

$$\mathbf{f}' = \frac{1}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}\left(\mathbf{f} - \frac{\boldsymbol{\beta}}{c}\frac{\mathrm{d}E}{\mathrm{d}t}\right)$$

You must have missed some factor of γ along the way. The transf. of $f_\perp$ and $f_\parallel$ also differ by a γ. See tutorial / ask there

Now we suppose that the rest mass is fixed: $\mathrm{d}m/\mathrm{d}t = 0$. We have the invariant

$$g(U, F) = \gamma(c, \mathbf{u})\cdot\gamma\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \mathbf{f}\right) = \gamma^2\left(\mathbf{u}\cdot\mathbf{f} - \frac{\mathrm{d}E}{\mathrm{d}t}\right)$$

In the (instantaneous) rest frame of m we have $g(U, F) = -c^2\dfrac{\mathrm{d}m}{\mathrm{d}t} = 0$. Hence

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{u}\cdot\mathbf{f}$$

We obtain the transformation of 3-forces:

$$\mathbf{f}' = \frac{\mathbf{f} - \mathbf{v}(\mathbf{u}\cdot\mathbf{f})/c^2}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}$$

$\square$

Question 2

Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows:

(i) Write down the solution for $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$

(ii) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$

(iii) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.

(iv) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

$$x = \frac{c}{f}\left(m^2c^2 + p_0^2 + f^2t^2\right)^{1/2} + \text{const}$$
$$y = \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const}$$

where you may quote that $\int \left(a^2 + t^2\right)^{-1/2} dt = \log\left(t + \sqrt{a^2 + t^2}\right)$

(v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation. [N.B. the calculation as a function of proper time is best done another way, see later problems.]

Proof. (i) From $\mathbf{f} := \mathrm{d}\mathbf{p}/\mathrm{d}t$, by integration we have

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{f}t$$

for constant 3-force $\mathbf{f}$.

(ii) By invariance of 4-momentum, we have

$$\gamma^2 m^2 c^4 = E^2 = p^2c^2 + m^2c^4 \implies \gamma^2 - 1 = \frac{p^2}{m^2c^2} = \frac{\|\mathbf{p}_0 + \mathbf{f}t\|^2}{m^2c^2} = \alpha^2$$

Hence $\gamma^2 = 1 + \alpha^2$.

(iii) Since $\gamma^2 = \left(1 - v^2/c^2\right)^{-1}$, we have

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} = c\sqrt{1 - \frac{1}{1+\alpha^2}} = \frac{c\alpha}{\sqrt{1+\alpha^2}} = \frac{c\|\mathbf{p}_0 + \mathbf{f}t\|}{\sqrt{m^2c^2 + \|\mathbf{p}_0 + \mathbf{f}t\|^2}}$$

Hence

$$\mathbf{v} = \frac{c\left(\mathbf{p}_0 + \mathbf{f}t\right)}{\sqrt{m^2c^2 + \|\mathbf{p}_0 + \mathbf{f}t\|^2}}$$

From (iii) and the chosen directions we have

$$v_x = \frac{cft}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}; \qquad v_y = \frac{cp_0}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}$$

The remaining issue are some simple integrations:

$$x = \int v_x \,\mathrm{d}t = \int \frac{cft}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}\mathrm{d}t = \frac{c}{2f}\int \frac{1}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}\mathrm{d}(f^2t^2) = \frac{c}{f}\sqrt{m^2c^2 + p_0^2 + f^2t^2} + \text{const};$$
$$y = \int v_y \,\mathrm{d}t = \int \frac{cp_0}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}\mathrm{d}t = \frac{cp_0}{f}\int \frac{1}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}}\mathrm{d}(ft) = \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const}$$

(iv) I assume that the "general case" is when the angle θ between the initial 3-momentum $\mathbf{p}_0$ and the constant 3-force $\mathbf{f}$ is arbitrary.

↳ it is: what would you do then?

□

Question 3

For motion under a pure (rest mass preserving) inverse square law force $\mathbf{f} = -\alpha\mathbf{r}/r^3$, where α is a constant, derive the energy equation $\gamma mc^2 - \alpha/r = \text{constant}$.

Proof. This problem is essentially classical. From Question 1 we have $\dfrac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{u}\cdot\mathbf{f}$. The central force has potential V given by

$$\mathbf{f} = -\frac{\alpha\mathbf{r}}{r^3} = -\nabla\left(-\frac{\alpha}{r}\right) = -\nabla V$$

Note that

$$u\cdot f = -\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\nabla V = -\frac{\mathrm{d}x^a}{\mathrm{d}t}\frac{\partial V}{\partial x^a} = -\frac{\mathrm{d}V}{\mathrm{d}t}$$

Hence

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\alpha}{r}\right)$$

where $E = \gamma mc^2$ is the only relativistic thing in this problem. Integrating the expression we find that

$$\gamma mc^2 - \alpha/r = \text{const}$$

□

Question 4

Prove that the time rate of change of the angular momentum $\mathbf{L} = \mathbf{r}\wedge\mathbf{p}$ of a particle about an origin O is equal to the couple $\mathbf{r}\wedge\mathbf{f}$ of the applied force about O.

If $L^{\mu\nu}$ is a particle's 4-angular momentum, and we define the 4-couple $G^{\mu\nu} \equiv X^\mu F^\nu - X^\nu F^\mu$, prove that $(d/d\tau)L^{\mu\nu} = G^{\mu\nu}$, and that the space-space part of this equation corresponds to the previous 3-vector result.

Proof.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\wedge\mathbf{p} + \mathbf{r}\wedge\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{v}\wedge\mathbf{p} + \mathbf{r}\wedge\mathbf{f} = \mathbf{r}\wedge\mathbf{f}$$

where $\mathbf{v}\wedge\mathbf{p} = 0$ because $\mathbf{v}$ and $\mathbf{p}$ are linearly dependent.

For the 4-angular momentum,

$$\begin{aligned}\frac{\mathrm{d}L^{\mu\nu}}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(X^\mu P^\nu - X^\nu P^\mu\right) = \left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}P^\nu - \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}P^\mu\right) + \left(X^\mu\frac{\mathrm{d}P^\nu}{\mathrm{d}\tau} - X^\nu\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau}\right)\\ &= m(U^\mu U^\nu - U^\nu U^\mu) + (X^\mu F^\nu - X^\nu F^\mu) = X^\mu F^\nu - X^\nu F^\mu\\ &= G^{\mu\nu}\end{aligned}$$

The spatial parts of these equations are given by

$$\frac{\mathrm{d}L^{ab}}{\mathrm{d}\tau} = G^{ab}$$

Applying Hodge dual operator on both sides we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\varepsilon_{ijk}X^i P^j = \varepsilon_{ijk}X^i F^j \implies \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_{ijk}X^i p^j = \varepsilon_{ijk}X^i f^j \implies \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}\wedge\mathbf{f}$$

which is consistent with the result above.

□

Question 5

Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$ and ϕ such that

$$\mathbf{B} = \nabla\wedge\mathbf{A}$$

$$\mathbf{E} = -\left(\frac{\partial\mathbf{A}}{\partial t}\right) - \nabla\phi$$

(i) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.

(ii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators, and Lorentz scalars (a *manifestly covariant* form).

Proof. In differential geometry, the exterior differential d is an operator defined on the exterior algebra. By Poincaré's Lemma, $\mathrm{d}: \bigwedge^k(M) \to \bigwedge^{k+1}(M)$ satisfies $\mathrm{d}^2 = 0$. In vector calculus this corresponds to $\operatorname{div}\operatorname{curl} = 0$ and $\operatorname{curl}\operatorname{grad} = 0$.

this is wrong: $d^2=0$ is simply because you take antisymm. combinations of derivatives, which commute. Poincaré lemma is "in open ball, $\subset \mathbb{R}^n$, if $d\alpha^{(p)}=0 \Rightarrow \alpha^{(p)} = d\beta^{(p-1)}$". The converse, which is $d^2=0$, is trivial.

From the expression of electromagnetic potentials we have

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\nabla\wedge\mathbf{A}) = 0$$

and

$$\nabla\wedge\mathbf{E} = \nabla\wedge\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi\right) = -\frac{\partial}{\partial t}(\nabla\wedge\mathbf{A}) - \nabla\wedge\nabla\phi = -\frac{\partial\mathbf{B}}{\partial t}$$

These are the second and the third Maxwell equations respectively.

(i) The first equation:

$$\begin{aligned}\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon} &\implies -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) - \nabla^2\phi = \frac{\rho}{\varepsilon_0}\\ &\implies \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi - \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \frac{\rho}{\varepsilon_0}\end{aligned}$$

The fourth equation:

$$\begin{aligned}\nabla\wedge B = \mu_0\mathbf{j} + \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} &\implies \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{j} - \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right)\\ &\implies \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0\mathbf{j}\end{aligned}$$

(ii) We choose the Lorenz gauge

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

Then the two equations above become

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = \frac{\rho}{\varepsilon_0}, \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = \mu_0\mathbf{j}$$

Now we introduce the 4-current $J = (\rho c, \mathbf{j})$ and the 4-potential $A = (\phi/c, \mathbf{A})$. The continuity equation of charge becomes

$$0 = \nabla\cdot j - \frac{\partial\rho}{\partial t} = \partial_\mu J^\mu$$

The Lorenz gauge becomes

$$0 = \nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = \partial_\mu A^\mu$$

The Maxwell equations become

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)A^\mu = \mu_0 J^\mu$$

which is equivalent to

$$\partial^\mu\partial_\mu A^\nu = -\mu_0 J^\nu$$

The corresponds form in terms of components of covectors:

$$\partial^\mu\partial_\mu A_\nu = -\mu_0 J_\nu$$

We can define the electromagnetic field tensor components:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

Then applying the Lorenz gauge we have

$$\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu = \partial^\mu\partial_\mu A_\nu$$

The Maxwell equations become

$$\partial^\mu F_{\mu\nu} = -\mu_0 J_\nu$$

We can take this one step further to make it **coordinate-independent**. Then it is surely covariant under Lorentz transformations!

The electromagnetic potential defines a 1-form $\mathsf{A} = A_\mu \mathrm{d}x^\mu$ on the Minkowski space, whose exterior differential is given by

see above

$$\mathrm{d}\mathsf{A} = \partial_\mu A_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu =: \mathsf{F}$$

By Poincaré's Lemma, we have

$$\mathrm{d}\mathsf{F} = \mathrm{d}^2\mathsf{A} = 0$$

Now we introduce the Hodge dual operator defined on the exterior algebra such that $\star : \bigwedge^k(M) \to \bigwedge^{4-k}(M)$ satisfies $\lambda \wedge \omega = \langle \star\lambda, \omega\rangle\,\sigma$, where $M = (\mathbb{R}^4, g)$ is the Minkowski spacetime, $\lambda \in \bigwedge^k(M)$, $\omega \in \bigwedge^{4-k}(M)$, and $\sigma = c\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. In particular we have

$$\star\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) = \frac{1}{(4-k)!}\varepsilon^{i_1\cdots i_k i_{k+1}\cdots i_4}\mathrm{d}x_{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x_{i_4} \qquad \text{(summation over } \{0,1,2,3\})$$

where $\varepsilon^{i_1\cdots i_4}$ is the Levi-civita symbol in 4 dimensions.

Now we compute $\mathrm{d}\star\mathsf{F}$:

$$\begin{aligned}
\mathrm{d}\star\mathsf{F} &= \frac{1}{2}\mathrm{d}\star\left(F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right)\\
&= \frac{1}{4}\mathrm{d}\left(F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}x_\rho \wedge \mathrm{d}x_\sigma\right)\\
&= \frac{1}{4}\partial_\lambda F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}x^\lambda \wedge \mathrm{d}x_\rho \wedge \mathrm{d}x_\sigma\\
&= \frac{1}{4}\partial_\lambda F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}g^{\lambda\tau}\mathrm{d}x_\tau \wedge \mathrm{d}x_\rho \wedge \mathrm{d}x_\sigma\\
&= \frac{1}{4}\partial_\lambda F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\tau\rho\sigma\eta}g^{\lambda\tau}\star(\mathrm{d}x^\eta)\\
&= \frac{1}{4}\partial_\lambda F_{\mu\nu}\cdot 2\left(\delta^\mu_\tau\delta^\nu_\eta - \delta^\nu_\tau\delta^\mu_\eta\right)g^{\lambda\tau}\star(\mathrm{d}x^\eta)\\
&= \frac{1}{2}\left(\partial^\mu F_{\mu\nu}\star(\mathrm{d}x^\nu) - \partial^\nu F_{\mu\nu}\star(\mathrm{d}x^\mu)\right)\\
&= \partial^\mu F_{\mu\nu}\star(\mathrm{d}x^\nu)\\
&= -\mu_0 J_\nu\star(\mathrm{d}x^\nu)\\
&= -\mu_0\star\mathsf{J}
\end{aligned}$$

where $\mathsf{J} := J_\nu \mathrm{d}x^\nu$.

We obtain the Maxwell equations which describe the current 1-form and electromagnetic field 2-form on the Minkowski spacetime:

$$\mathrm{d}\mathsf{F} = 0, \qquad \mathrm{d}\star\mathsf{F} = -\mu_0\star\mathsf{J}$$ ✓ □

Question 6

How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel$$
$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}\wedge\mathbf{B}) \qquad \mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \mathbf{v}\wedge\mathbf{E}/c^2\right)$$

[Hint: you may find the algebra easier if you treat **E** and **B** separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

Proof. Let L be a Lorentz transformation. The transformation of the electromagnetic field tensor F is given by

$$\tilde{F}^{\mu\nu} = L^{\mu}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}$$

usually people use $\tilde{F}$ to denote $\star F$

In this matrix form, this is given by $\mathbf{F} = L\mathbf{F}L^{\mathrm{T}}$ where $\mathbf{F}^{\mu}_{\nu} := F^{\mu\nu}$.

Informally, we can derive the transformation rules when the 3-velocity $\mathbf{v}$ is aligned with x-direction, and then argue that the general case follows similarly. Then we have

$$\mathbf{F} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
L\mathbf{F}L^{-1} &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta\gamma E_x/c & \gamma E_x/c & E_y/c & E_z/c \\ -\gamma E_x/c & \beta\gamma E_x/c & B_z & -B_y \\ \gamma E_y/c + \beta\gamma B_z & \beta\gamma E_y/c - \gamma B_z & 0 & B_x \\ -\gamma E_z/c - \beta\gamma B_y & \beta\gamma E_z/c + \gamma B_y & -B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \gamma^2 E_x/c - \beta^2\gamma^2 E_x/c & \gamma E_y/c - \beta\gamma B_z & \gamma E_z/c + \beta\gamma B_y \\ \beta^2\gamma^2 E_x/c - \gamma^2 E_x/c & 0 & -\beta\gamma E_y/c + \gamma B_z & -\beta\gamma E_z/c - \gamma B_y \\ -\gamma E_y/c + \beta\gamma B_z & \beta\gamma E_y/c - \gamma B_z & 0 & B_x \\ -\gamma E_z/c - \beta\gamma B_y & \beta\gamma E_z/c + \gamma B_y & -B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & E_x/c & \gamma(E_y - \beta B_z)/c & \gamma(E_z/c + \beta B_y) \\ E_x/c & 0 & \gamma(B_z - \beta E_y/c) & -\gamma(B_y + \beta E_z/c) \\ \gamma(E_y - \beta B_z)/c & -\gamma(B_z - \beta E_y/c) & 0 & B_x \\ -\gamma(E_z/c + \beta B_y) & \gamma(B_z - \beta E_y/c) & -B_x & 0 \end{pmatrix}
\end{aligned}$$

Hence

$$\tilde{E}_x = E_x \qquad \tilde{E}_y = \gamma(E_y - vB_z) \qquad \tilde{E}_z = \gamma(E_z + vB_y)$$
$$\tilde{B}_x = B_x \qquad \tilde{B}_y = \gamma(B_y + vE_z/c^2) \qquad \tilde{B}_z = \gamma(B_z - vE_y/c^2)$$

The general forms are:

$$\tilde{\mathbf{E}}_{/\!/} = \mathbf{E}_{/\!/} \qquad \tilde{\mathbf{B}}_{/\!/} = \mathbf{B}_{/\!/}$$
$$\tilde{\mathbf{E}}_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}) \qquad \tilde{\mathbf{B}}_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2\right)$$

By Gauss' Theorem, the electric field of a line charge λ is given by

why? is!

$$\mathbf{E} = \frac{\lambda}{2\pi r \varepsilon_0} \mathbf{e}_r$$

The current $I = \gamma\lambda v$ where $\mathbf{v} = v\mathbf{e}_z$ can be regarded as the velocity of the moving charges. We transform to the frame where the charges are stationary. By the transformation rules above, we have

$$\tilde{\mathbf{B}}_{/\!/} = \mathbf{B}_{/\!/} = 0, \qquad \tilde{\mathbf{B}}_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2\right) = \frac{\gamma v \lambda}{2\pi r \varepsilon_0 c^2} \mathbf{e}_z \wedge \mathbf{e}_r = \frac{I}{2\pi r \mu_0} \mathbf{e}_\theta$$

□

Question 7

The electromagnetic field tensor $F^{\mu\nu}$ (sometimes called the Faraday tensor) is defined such that the 4-force on a charged particle is given by

$$f^{\mu} = qF^{\mu\nu}U_{\nu}$$

By comparing this to the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

which defines the electromagnetic and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $dP^{\mu}/d\tau$), show that the components of the field tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Proof. The 4-force is $f = \dfrac{\mathrm{d}P}{\mathrm{d}\tau} = \gamma\dfrac{\mathrm{d}P}{\mathrm{d}t} = \gamma\left(\dfrac{1}{c}\dfrac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}, \mathbf{F}\right) = \gamma\left(\dfrac{1}{c}\mathbf{v}\cdot\mathbf{F}, \mathbf{F}\right)$. The 4-velocity is $U = \dfrac{\mathrm{d}X}{\mathrm{d}\tau} = \gamma\dfrac{\mathrm{d}X}{\mathrm{d}t} = \gamma(c, \mathbf{v})$. Hence the equation $f^{\mu} = qF^{\mu\nu}U_{\nu}$ leads to

$$\begin{aligned} \frac{1}{c}\mathbf{v}\cdot\mathbf{F} &= q\left(-cF^{00} + v_a F^{0a}\right) \\ f^a &= q\left(-cF^{a0} + v_b F^{ab}\right) \end{aligned}$$

where f^a is the component of $\mathbf{F}$ rather than f. From $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$, we have

$$f^a = q\left(E^a + \varepsilon^{abc} v_b B_c\right)$$

and

$$\mathbf{v}\cdot\mathbf{F} = q\mathbf{v}\cdot\mathbf{E} = q v_a E^a$$

Comparing the two sets of equations we have

$$\begin{aligned} -cF^{00} + v_a F^{0a} &= v_a E^a \\ -cF^{a0} + v_b F^{ab} &= E^a + \varepsilon^{abc} v_b B_c \end{aligned}$$

Therefore $F^{00} = 0$, $F^{0a} = E^a$, and $F^{ab} = \varepsilon^{abc} B_c$. The matrix of F is given by

$$\mathbf{F} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

□

Question 8

Show that the field equation

$$\partial_{\lambda} F^{\lambda\nu} = -\mu_0 \rho_0 U^{\nu}$$

is equivalent to

$$\partial^{\lambda}\partial_{\lambda} A^{\nu} - \partial^{\nu}\left(\partial_{\lambda} A^{\lambda}\right) = -\mu_0 J^{\nu}$$

where $J^{\nu} \equiv \rho_0 U^{\nu}$ (here ρ_0 is the proper charge density, and J^{ν} is the 4-current density). Comment.

Proof. The equation follows from the definition of the electromagnetic field tensor

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$$

See Question 5 for a detailed discussion. □

Question 9

Show that the following two scalar quantities are Lorentz invariant:

$$D = B^2 - E^2/c^2$$
$$\alpha = \mathbf{B}\cdot\mathbf{E}/c$$

[Hint: for the second, introduce the dual field tensor $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$.]

Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are $\mathbf{E}, \mathbf{B}$). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

Proof. D is obtained by contracting the contravariant and covariant electromagnetic field tensors:

$$F^{\mu\nu} = \mathbf{F} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \qquad F_{\mu\nu} = \mathbf{F}' = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Let $\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ and $\mathscr{B} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$. Then

$$\begin{aligned}
F^{\mu\nu}F_{\mu\nu} &= \operatorname{tr}(\mathbf{F}^{\mathrm{T}}\mathbf{F}') = \operatorname{tr}\left(\begin{pmatrix} 0 & -\mathbf{E}^{\mathrm{T}}/c \\ \mathbf{E}/c & -\mathscr{B} \end{pmatrix}\begin{pmatrix} 0 & -\mathbf{E}^{\mathrm{T}}/c \\ \mathbf{E}/c & \mathscr{B} \end{pmatrix}\right) \\
&= \operatorname{tr}\begin{pmatrix} -E^2/c^2 & (\mathscr{B}\mathbf{E})^{\mathrm{T}}/c \\ -\mathscr{B}\mathbf{E}/c & -\mathbf{E}\mathbf{E}^{\mathrm{T}}/c^2 - \mathscr{B}^2 \end{pmatrix} \\
&= -E^2/c^2 + \operatorname{tr}\left(-\mathbf{E}\mathbf{E}^{\mathrm{T}}/c^2\right) - \operatorname{tr}\left(\mathscr{B}^2\right) \\
&= -\frac{E^2}{c^2} - \frac{1}{c^2}\operatorname{tr}\begin{pmatrix} E_x^2 & E_xE_y & E_xE_z \\ E_xE_y & E_y^2 & E_yE_z \\ E_xE_z & E_yE_z & E_z^2 \end{pmatrix} + \operatorname{tr}\begin{pmatrix} B_y^2+B_z^2 & -B_xB_y & -B_xB_z \\ -B_xB_y & B_x^2+B_z^2 & -B_yB_z \\ -B_xB_z & -B_yB_z & B_x^2+B_y^2 \end{pmatrix} \\
&= -\frac{2E^2}{c^2} + 2B^2 = 2D
\end{aligned}$$

Hence D is a scalar invariant.

Next we consider the Hodge dual (see Question 5) of the electromagnetic field tensor:

$$(\star F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{B}^{\mathrm{T}} \\ -\mathbf{B} & -\mathscr{E}/c \end{pmatrix}$$

where $\mathbf{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$ and $\mathscr{E} = \begin{pmatrix} 0 & E_z & -E_y \\ -E_z & 0 & E_x \\ E_y & -E_x & 0 \end{pmatrix}$. Then

$$\begin{aligned}
F^{\mu\nu}(\star F)_{\mu\nu} &= \operatorname{tr}\left(\begin{pmatrix} 0 & -\mathbf{E}^{\mathrm{T}}/c \\ \mathbf{E}/c & -\mathscr{B} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{B}^{\mathrm{T}} \\ -\mathbf{B} & -\mathscr{E}/c \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} \mathbf{E}\cdot\mathbf{B}/c & -(\mathscr{E}\mathbf{E})^{\mathrm{T}}/c^2 \\ \mathscr{B}\mathbf{B} & \mathbf{E}\mathbf{B}^{\mathrm{T}}/c + \mathscr{B}\mathscr{E}/c \end{pmatrix} \\
&= \frac{\mathbf{E}\cdot\mathbf{B}}{c} + \frac{1}{c}\operatorname{tr}(\mathbf{E}\mathbf{B}^{\mathrm{T}}) + \frac{1}{c}\operatorname{tr}(\mathscr{B}\mathscr{E}) = \frac{4\mathbf{E}\cdot\mathbf{B}}{c^2} = 4\alpha
\end{aligned}$$

4α is the contraction of a rank 2 contravariant tensor with a rank 2 covariant tensor. Hence α is a scalar invariant.

Now suppose that $\alpha = 0$ and $D \neq 0$. Then **E** and **B** are orthogonal in any inertial frame. We can first apply a spatial rotation such that $\mathbf{E} = E\mathbf{e}_x$ and $\mathbf{B} = B\mathbf{e}_y$ after rotation. Now consider a Lorentz boost with $\mathbf{v} = v\mathbf{e}_z$. By the transformation rules given in Question 6, we find that

$$\widetilde{\mathbf{E}} = \gamma_v(E - Bv)\mathbf{e}_x, \qquad \widetilde{\mathbf{B}} = \gamma_v(B - Ev/c^2)\mathbf{e}_y$$

If $D > 0$, then $E/B < c$. We take $v = E/B$. Then $\widetilde{\mathbf{E}} = 0$, and the transformed field is purely magnetic. If $D < 0$, then $E/B > c$. We take $v = c^2B/E$. Then $\widetilde{\mathbf{B}} = 0$, and the transformed field is purely electric.

An example of the field with $\alpha = 0$ and $D = 0$ is the electromagnetic plane wave in vaccum:

$$\mathbf{E} = E\cos(kz - \omega t)\mathbf{e}_x, \qquad \mathbf{B} = Ec\cos(kz - \omega t)\mathbf{e}_y, \qquad \mathbf{k} = \frac{\omega}{c}\mathbf{e}_z \qquad \square$$