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Problem Sheet 3

B4.3: Distribution Theory

27 November, 2020

Remark. I ran out of time when doing this problem sheet. As a result, Question 4 and 6 are only partially answered.

Question 1

Find the general solutions to the ODEs

$$y'' + 2y' + y = 1 \tag{i}$$
$$y'' + 2y' + y = H \tag{ii}$$
$$y'' + 2y' + y = \delta_0 \tag{iii}$$

in $\mathscr{D}'(\mathbb{R})$, where H is Heaviside's function and δ_0 is Dirac's delta-function at 0. What are the classical solutions to (i) and (ii)?

Proof. First we consider the homogeneous equation

$$\mathscr{L}y = y'' + 2y' + y = 0 \tag{H}$$

you don't need any of this.

$\mathscr{L}$ is a linear operator on $\mathscr{D}'(\mathbb{R})$. For $y_1, y_2 \in \mathscr{D}'(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathscr{L}(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 \mathscr{L} y_1 + \lambda_2 \mathscr{L} y_2$.

In the theory of ordinary differential equations of $C^2(\mathbb{R})$ functions, the next thing is to prove that $\dim\ker\mathscr{L} = 2$, a key step of which is to invoke the Picard-Lindelöf Theorem in Part A Differential Equations I. However, I don't think it can be generalised to distributional differential equations (especially we don't have a clear definition of an initial value problem).

Assuming that $\dim\ker\mathscr{L} = 2$, for any inhomogeneous problem

$$\mathscr{L}y = y'' + 2y' + y = u \tag{N}$$

where $u \in \mathscr{D}'(\mathbb{R})$, the general solution is given by

$$y = y_p + c_1 y_1 + c_2 y_2$$

where $y_p \in \mathscr{D}'(\mathbb{R})$ is any solution of (N), $y_1, y_2 \in \mathscr{D}'(\mathbb{R})$ are linearly independent solutions of (H), and $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Let us solve (H) explicitly. Note that

$$\mathscr{L}y = \left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 1\right) y = \left(\frac{d}{dx} + 1\right)^2 y$$

Let $z = \left(\frac{d}{dx} + 1\right) y$. Then

$$\left(\frac{d}{dx} + 1\right) z = 0 \implies z' + z = 0 \implies \frac{d}{dx}(e^x z) = e^x(z' + z) = 0$$

constancy thm

By the identity theorem for distributions we have $e^x z = \text{const}$. Hence z is a regular distribution given by $z = c_2 e^{-x}$ for some $c_2 \in \mathbb{R}$. Next we solve

$$\left(\frac{d}{dx} + 1\right) y = z \implies y' + y = c_2 e^{-x} \implies \frac{d}{dx}(e^x y) = e^x(y' + y) = c_2$$

By the fundamental theorem of calculus for distributions, y is a regular distribution and we have $e^x y = c_2 x + c_1$ for some $c_1 \in \mathbb{R}$. Hence the general solution to (H) is given by

$$y(x) = (c_1 + c_2 x) e^{-x}$$

Let $y_1(x) = e^{-x}$ and $y_2(x) = x e^{-x}$. The Wronskian $W(x) = y_1 y_2' - y_2 y_1' = e^{-2x}$.

Now we consider the inhomogeneous problems (N). To find a particular solution, we use the variation of parameters and consider

$$y_p = u_1 y_1(x) + u_2 y_2(x)$$

where $u_1, u_2 \in \mathscr{D}'(\mathbb{R})$ are such that

$$u_1' = -\frac{y_2(x)}{W(x)} u = -x e^x u, \qquad u_2' = \frac{y_1(x)}{W(x)} u = e^x u$$

To show that y_p is a solution to (N), first note that

$$u_1' y_1 + u_2' y_2 = \frac{-y_1 y_2 + y_1 y_2}{W} u = 0$$

Hence $y_p' = u_1 y_1' + u_2 y_2'$ by Leibniz's rule. And

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

We compute:

$$\begin{aligned}
\mathscr{L} y_p &= y_p'' + 2y_p' + y_p \\
&= u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + 2(u_1 y_1' + u_2 y_2') + u_1 y_1 + u_2 y_2 \\
&= u_1' y_1' + u_2' y_2' \\
&= \frac{y_1 y_2' - y_2 y_1'}{W} u = u
\end{aligned}$$

Therefore y_p is indeed a solution to (N).

$$y_p = -\mathrm{e}^{-x} \int x \mathrm{e}^x u + x \mathrm{e}^{-x} \int \mathrm{e}^x u$$

where $\int v \in \mathscr{D}'(\mathbb{R})$ is such that $(\int v)' = v$ in $\mathscr{D}'(\mathbb{R})$.

Not sure what the pt of this was as not needed?

(i) Note that the regular distribution $y = 1$ is a particular solution to equation (i). Therefore the general solution to (i) is a regular distribution given by

$$y(x) = 1 + (c_1 + c_2 x) \mathrm{e}^{-x}$$

for some $c_1, c_2 \in \mathbb{R}$. The solution is completely classical because it is C^∞.

(ii) From the previous discussions, we need to find the primitives of the distributions $x \mathrm{e}^x H$, and $\mathrm{e}^x H$.

Note that H is given by the regular distribution $H(x) = \mathbf{1}_{(0,\infty)}$. We have

$$\int x \mathrm{e}^x H = \begin{cases} (x-1)\mathrm{e}^x + 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases}, \qquad \int \mathrm{e}^x H = \begin{cases} \mathrm{e}^x - 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

dodgy notation but ok.

Here is a quick check of the above equations:

$$\begin{aligned}
\left\langle \left(\int x \mathrm{e}^x H \right)', \varphi \right\rangle &= -\left\langle \int x \mathrm{e}^x H, \varphi' \right\rangle = -\int_0^\infty ((x-1)\mathrm{e}^x + 1)\varphi'(x)\,\mathrm{d}x \\
&= \int_0^\infty x \mathrm{e}^x \varphi \,\mathrm{d}x - ((x-1)\mathrm{e}^x + 1)\varphi(x) \Big|_0^\infty \\
&= \langle x \mathrm{e}^x H, \varphi \rangle \\
\left\langle \left(\int \mathrm{e}^x H \right)', \varphi \right\rangle &= -\left\langle \int \mathrm{e}^x H, \varphi' \right\rangle = -\int_0^\infty (\mathrm{e}^x - 1)\varphi'(x)\,\mathrm{d}x \\
&= \int_0^\infty \mathrm{e}^x \varphi \,\mathrm{d}x - (\mathrm{e}^x - 1)\varphi(x) \Big|_0^\infty \\
&= \langle \mathrm{e}^x H, \varphi \rangle
\end{aligned}$$

It is easier to just prove this result directly using the constancy thm.

Then the particular solution is given by

$$y_p(x) = \begin{cases} 1 - (x+1)\mathrm{e}^{-x}, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

The general solution is given by

$$\begin{aligned}
y(x) &= \begin{cases} 1 + (c_2 - 1)x\mathrm{e}^{-x} + (c_1 - 1)\mathrm{e}^{-x}, & x \geqslant 0 \\ c_2 x \mathrm{e}^{-x} + c_1 \mathrm{e}^{-x}, & x < 0 \end{cases} \\
&= \left(1 - (x+1)\mathrm{e}^{-x}\right) H(x) + (c_1 + c_2 x)\mathrm{e}^{-x}
\end{aligned}$$

The general solution is still regular distribution induced by a continuous function. This distributional solution agree with the classical solution everywhere except at $x = 0$, where the classical derivative y'' does not exist.

(iii) We need to find the primitives of the distributions $x\mathrm{e}^x \delta_0$, and $\mathrm{e}^x \delta_0$. Note that $x\delta_0 = 0$ by Question 5 in Sheet 2. So bad notation ~ better to prove directly using const. thm & the fact that $H' = \delta_0$

$\int x\,\mathrm{e}^x\,\delta_0$ is a constant function by the identity theorem, and we can set it equal to 0. Next note that

$$\langle \mathrm{e}^x\,\delta_0, \varphi\rangle = \langle \delta_0, \mathrm{e}^x\,\varphi\rangle = \mathrm{e}^0\,\varphi(0) = \varphi(0) = \langle \delta_0, \varphi\rangle$$

So $\int \mathrm{e}^x\,\delta_0 = \int \delta_0 = H$. The particular solution is given by

$$y_p(x) = x\,\mathrm{e}^{-x}\,H(x)$$

The general solution is given by

$$y(x) = x\,\mathrm{e}^{-x}\,H(x) + (c_1 + c_2 x)\,\mathrm{e}^{-x}$$

which is a regular distribution given by a function with a jump discontinuity at $x = 0$. □

Question 2

The principal logarithm is defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$\log z := \log|z| + \mathrm{i}\mathrm{Arg}(z), \quad \mathrm{Arg}(z) \in (-\pi, \pi)$$

Define $\log(x+\mathrm{i}0)$ and $\log(x-\mathrm{i}0)$ for each $\varphi \in \mathscr{D}(\mathbb{R})$ by the rules

$$\langle \log(x \pm \mathrm{i}0), \varphi\rangle := \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \log(x \pm \mathrm{i}\varepsilon)\varphi(x)\,\mathrm{d}x$$

(a) Show that $\log(x \pm \mathrm{i}0)$ hereby are distributions on $\mathbb{R}$.

Now let $k \in \mathbb{N}$ and define the distributions $(x+\mathrm{i}0)^{-k}$ and $(x-\mathrm{i}0)^{-k}$ as

$$(x \pm \mathrm{i}0)^{-k} := \frac{(-1)^{k-1}}{(k-1)!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\log(x \pm \mathrm{i}0) \quad \text{in } \mathscr{D}'(\mathbb{R})$$

(b) Show that for each $\varphi \in \mathscr{D}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j \in \{0, \dots, k\}$ we have

$$\left\langle (x \pm \mathrm{i}0)^{-k}, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k}\,\mathrm{d}x$$

(c) Prove that $\log(x+\mathrm{i}0) - \log(x-\mathrm{i}0) = 2\pi\mathrm{i}\widetilde{H}$, where H is the Heaviside function. Deduce the Plemelj-Sokhotsky jump relations:

$$(x+\mathrm{i}0)^{-k} - (x-\mathrm{i}0)^{-k} = 2\pi\mathrm{i}\frac{(-1)^k}{(k-1)!}\delta_0^{(k-1)}$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.

(d) Show that

$$x(x \pm \mathrm{i}0)^{-1} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R})$$

Deduce that

$$(x+\mathrm{i}0)^{-1}(x\delta_0) = 0 \neq \delta_0 = \left((x+\mathrm{i}0)^{-1}x\right)\delta_0$$

Next, show, for instance by using the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ on the case $k = 1$ iteratively, that

$$x^k(x \pm \mathrm{i}0)^{-k} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R})$$

holds for each $k \in \mathbb{N}$.

Proof. (a) We need to show that $\log(x \pm \mathrm{i}0) \in \mathscr{D}'(\mathbb{R})$. First we show that $\log(x+\mathrm{i}0) \in \mathscr{D}'(\mathbb{R})$.

- $\log(x+\mathrm{i}0)$ is well-defined:

We have

$$\left|\log(x+\mathrm{i}\varepsilon)\right| = \left|\log\left(\sqrt{x^2+\varepsilon^2}\right)+\mathrm{i}\operatorname{Arg}(x+\mathrm{i}\varepsilon)\right| \leqslant \pi + \frac{1}{2}\left|\log(x^2+\varepsilon^2)\right|$$

Then

$$\left|\langle \log(x+\mathrm{i}\varepsilon),\varphi\rangle\right| \leqslant \sup_{x\in\mathbb{R}}|\varphi(x)| \lim_{\varepsilon\searrow 0}\int_{\operatorname{supp}\varphi}\left(\pi+\frac{1}{2}\left|\log(x^2+\varepsilon^2)\right|\right)\mathrm{d}x$$

Note that $\left|\log(x^2+\varepsilon^2)\right|$ is bounded on $\operatorname{supp}\varphi \setminus [-1,1]$. And

$$\int_{-1}^{1}\left|\log(x^2+\varepsilon^2)\right|\mathrm{d}x \leqslant 2+4\left|\int_0^1 \log x\,\mathrm{d}x\right| = 2+4<\infty$$

Hence $\left|\langle \log(x+\mathrm{i}\varepsilon),\varphi\rangle\right| < \infty$. It is well-defined.

In addition, the same proof also shows that

$$\lim_{\varepsilon\searrow 0}\int_K \left|\log(x+\mathrm{i}\varepsilon)\right|\mathrm{d}x$$

is bounded on any compact set $K\subseteq\mathbb{R}$.

true but I'm not sure that you've proven these limits actually exist.

- $\log(x+\mathrm{i}0)$ is linear:

  For $\varphi_1,\varphi_2\in\mathscr{D}(\mathbb{R})$ and $\lambda_1,\lambda_2\in\mathbb{R}$, we have

$$\begin{aligned}\langle \log(x+\mathrm{i}0),\lambda_1\varphi_1+\lambda_2\varphi_2\rangle &= \lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\log(x+\mathrm{i}\varepsilon)\left(\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)\right)\mathrm{d}x\\ &= \lambda_1\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\log(x+\mathrm{i}\varepsilon)\varphi_1(x)\,\mathrm{d}x+\lambda_2\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\log(x+\mathrm{i}\varepsilon)\varphi_2(x)\,\mathrm{d}x\\ &= \lambda_1\langle \log(x+\mathrm{i}0),\varphi_1\rangle+\lambda_2\langle\log(x+\mathrm{i}0),\varphi_2\rangle\end{aligned}$$

  Hence $\log(x+\mathrm{i}0)$ is a linear functional.

- $\log(x+\mathrm{i}0)$ is continuous:

  Suppose that $\{\varphi_n\}\subseteq\mathscr{D}(\mathbb{R})$ and $\varphi\in\mathscr{D}(\mathbb{R})$ such that $\varphi_n\to\varphi$ in $\mathscr{D}(\mathbb{R})$. Let $K\subseteq\mathbb{R}$ be a compact set such that $\operatorname{supp}\varphi_n,\operatorname{supp}\varphi\subseteq K$. We have

$$\begin{aligned}\left|\langle\log(x+\mathrm{i}0),\varphi_n\rangle-\langle\log(x+\mathrm{i}0),\varphi\rangle\right| &= \lim_{\varepsilon\searrow 0}\left|\int_{-\infty}^{\infty}\log(x+\mathrm{i}\varepsilon)(\varphi_n(x)-\varphi(x))\,\mathrm{d}x\right|\\ &\leqslant \|\varphi_n-\varphi\|_\infty \lim_{\varepsilon\searrow 0}\int_K\left|\log(x+\mathrm{i}\varepsilon)\right|\mathrm{d}x\\ &\leqslant \|\varphi_n-\varphi\|_\infty M(K)\to 0\end{aligned}$$

  as $\|\varphi_n-\varphi\|_\infty\to 0$. Hence $\log(x+\mathrm{i}0)$ is continuous.

We deduce that $\log(x+\mathrm{i}0)\in\mathscr{D}'(\mathbb{R})$. Similarly $\log(x-\mathrm{i}0)\in\mathscr{D}'(\mathbb{R})$.

what is an explicit form for these distrib.?

(b) First we note that there is a complex version of integration by parts. Suppose that $f(x)=u(x)+\mathrm{i}v(x)$ where u,v are differentiable real-valued functions. Then for $\varphi\in\mathscr{D}(\mathbb{R})$

$$\int_{\mathbb{R}}f(x)\varphi'(x)\,\mathrm{d}x=\int_{\mathbb{R}}u(x)\varphi'(x)\,\mathrm{d}x+\mathrm{i}\int_{\mathbb{R}}v(x)\varphi'(x)\,\mathrm{d}x=-\int_{\mathbb{R}}u'(x)\varphi(x)\,\mathrm{d}x-\mathrm{i}\int_{\mathbb{R}}v'(x)\varphi(x)\,\mathrm{d}x=-\int_{\mathbb{R}}f'(x)\varphi(x)\,\mathrm{d}x$$

Second, note that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\log(x\pm\mathrm{i}\varepsilon)=\left.\frac{\mathrm{d}^k}{\mathrm{d}z^k}\log z\right|_{z=x\pm\mathrm{i}\varepsilon}=(-1)^{k-1}(k-1)!(x\pm\mathrm{i}\varepsilon)^{-k}$$

Then for $\varphi\in\mathscr{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle (x\pm\mathrm{i}0)^{-k},\varphi\right\rangle &= \left\langle\frac{(-1)^{k-1}}{(k-1)!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\log(x\pm\mathrm{i}0),\varphi\right\rangle=\left\langle\frac{(-1)^{k-1}}{(k-1)!}\log(x\pm\mathrm{i}0),(-1)^k\varphi^{(k)}\right\rangle\\ &= -\frac{1}{(k-1)!}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}\log(x\pm\mathrm{i}\varepsilon)\varphi^{(k)}(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{k+1}}{(k-1)!} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \log(x \pm \mathrm{i}\varepsilon)\varphi(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{\varphi(x)}{(x \pm \mathrm{i}\varepsilon)^k}\,\mathrm{d}x
\end{aligned}
$$

For sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that

$$
\left| \frac{\varphi(x)}{(x \pm \mathrm{i}\varepsilon)^k} \right| \leqslant C \left| \frac{\varphi(x)}{x^k} \right|
$$

Since $\varphi^{(j)}(0) = 0$ for $j \leqslant k$, we use the l'Hôptial's rule k times and find

$$
\lim_{x \to 0} \frac{\varphi(x)}{x^k} = \lim_{x \to 0} \frac{\varphi^{(k)}(x)}{k!} = 0
$$

Nice.

Hence $\left|\frac{\varphi(x)}{x^k}\right|$ is bounded near $x = 0$. Since $\operatorname{supp}\varphi$ is compact, the function is integrable over $\mathbb{R}$. By Dominated Convergence Theorem,

$$
\left\langle (x \pm \mathrm{i}0)^{-k}, \varphi \right\rangle = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{\varphi(x)}{(x \pm \mathrm{i}\varepsilon)^k}\,\mathrm{d}x = \int_{\mathbb{R}} \lim_{\varepsilon \searrow 0} \frac{\varphi(x)}{(x \pm \mathrm{i}\varepsilon)^k}\,\mathrm{d}x = \int_{\mathbb{R}} \frac{\varphi(x)}{x^k}\,\mathrm{d}x
$$

(c) For $\varepsilon > 0$,

$$
\log(x \pm \mathrm{i}\varepsilon) = \frac{1}{2}\log(x^2 + \varepsilon^2) \pm \mathrm{i}\arctan\frac{\varepsilon}{x}
$$

Then

$$
\log(x + \mathrm{i}\varepsilon) - \log(x - \mathrm{i}\varepsilon) = 2\mathrm{i}\arctan\frac{\varepsilon}{x}
$$

For $x > 0$, $\lim_{\varepsilon \searrow 0} 2\mathrm{i}\arctan\frac{\varepsilon}{x} = 0$; for $x < 0$, $\lim_{\varepsilon \searrow 0} 2\mathrm{i}\arctan\frac{\varepsilon}{x} = 2\pi\mathrm{i}$.

Then for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$
\begin{aligned}
\langle \log(x + \mathrm{i}0) - \log(x - \mathrm{i}0), \varphi \rangle &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \big(\log(x + \mathrm{i}\varepsilon) - \log(x - \mathrm{i}\varepsilon)\big)\varphi(x)\,\mathrm{d}x \\
&= 2\mathrm{i} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \arctan\frac{\varepsilon}{x}\varphi(x)\,\mathrm{d}x \\
&= 2\mathrm{i} \int_{\mathbb{R}} \lim_{\varepsilon \searrow 0} \arctan\frac{\varepsilon}{x}\varphi(x)\,\mathrm{d}x && \text{(bounded convergence theorem)} \\
&= 2\pi\mathrm{i} \int_{-\infty}^{0} \varphi(x)\,\mathrm{d}x = 2\pi\mathrm{i}\langle \widetilde{H}, \varphi \rangle
\end{aligned}
$$

where $\widetilde{H}$ is the distribution such that $\langle \widetilde{H}, \varphi(x) \rangle = \langle H, \varphi(-x) \rangle$ for $\varphi \in \mathscr{D}(\mathbb{R})$. We deduce that $\log(x+\mathrm{i}0) - \log(x-\mathrm{i}0) = 2\pi\mathrm{i}\widetilde{H}$. Taking the distributional derivative k times, we have

$$
\frac{\mathrm{d}}{\mathrm{d}x^k}\big(\log(x + \mathrm{i}0) - \log(x - \mathrm{i}0)\big) 2\pi\mathrm{i}\widetilde{H}^{(k)}
$$

Using the result in (b) and the fact that $H' = \delta_0$, we have

$$
(x + \mathrm{i}0)^{-k} - (x - \mathrm{i}0)^{-k} = 2\pi\mathrm{i}\frac{(-1)^k}{(k-1)!}\delta_0^{(k-1)}
$$

(d) For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$
\langle x(x \pm \mathrm{i}0)^{-1}, \varphi \rangle = \langle (x \pm \mathrm{i}0)^{-1}, x\varphi(x) \rangle = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{x}{x \pm \mathrm{i}\varepsilon}\varphi(x)\,\mathrm{d}x
$$

The integrand is continuous and compactly supported. Hence by bounded convergence theorem we have

$$
\langle x(x \pm \mathrm{i}0)^{-1}, \varphi \rangle = \int_{\mathbb{R}} \lim_{\varepsilon \searrow 0} \frac{x}{x \pm \mathrm{i}\varepsilon}\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \langle 1, \varphi \rangle
$$

Its actually just that we need to be careful w. associativity!

Hence $x(x\pm i0)^{-1}=1$. We deduce that

$$(x+i0)^{-1}(x\delta_0)=0\neq\delta_0=\left((x+i0)^{-1}x\right)\delta_0$$

which suggests the product of two general distributions may not be well-defined.

We use induction on k to prove that $x^k(x\pm i0)^{-k}=1$. The base case $k=1$ is proven above. Suppose that $x^k(x\pm i0)^{-k}=1$. Then

$$0=x\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k(x\pm i0)^{-k}\right)=kx^k(x\pm i0)^{-k}-kx^{k+1}(x\pm i0)^{-(k+1)}\implies x^{k+1}(x\pm i0)^{-(k+1)}=x^k(x\pm i0)^{-k}=1$$

Hence $x^k(x\pm i0)^{-k}=1$ for all $k\geqslant 1$. □

Question 3

Let $g\in L^1_{\mathrm{loc}}(\mathbb{R})$ and assume that g is T-periodic for some $T>0$: $g(x+T)=g(x)$ holds for almost all $x\in\mathbb{R}$. Define for each $j\in\mathbb{N}$ the function

$$g_j(x)=g(jx),\quad x\in(0,1)$$

Prove that

$$g_j\to\frac{1}{T}\int_0^T g\,\mathrm{d}x\quad\text{in}\quad\mathscr{D}'(0,1)\quad\text{as}\quad j\to\infty$$

Proof. Let

$$f(x):=g(x)-\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t$$

and

$$F(x):=\int_0^x f(t)\,\mathrm{d}t=\int_0^x g(t)\,\mathrm{d}t-\frac{x}{T}\int_0^T g(t)\,\mathrm{d}t$$

Then F is absolutely continuous and T-periodic:

$$F(x+T)=\int_0^{x+T}g(t)\,\mathrm{d}t-\left(1+\frac{x}{T}\right)\int_0^T g(t)\,\mathrm{d}t=\int_T^{x+T}g(t)\,\mathrm{d}t-\frac{x}{T}\int_0^T g(t)\,\mathrm{d}t=\int_0^x g(t)\,\mathrm{d}t-\frac{x}{T}\int_0^T g(t)\,\mathrm{d}t=F(x)$$

In particular, F is bounded on $\mathbb{R}$, because $F(\mathbb{R})=F([0,T])$ is compact.

For $\varphi\in\mathscr{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle g_j-\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t,\varphi\right\rangle&=\int_{\mathbb{R}}\varphi(x)\left(g(jx)-\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t\right)\mathrm{d}x\\&=\int_{\mathbb{R}}f(jx)\varphi(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}}\frac{F'(jx)}{j}\varphi(x)\,\mathrm{d}x\\&=-\frac{1}{j}\int_{\mathbb{R}}F(jx)\varphi'(x)\,\mathrm{d}x\end{aligned}$$

Since F is bounded and φ' is compactly supported, there exists $M>0$ such that

$$\left|\int_{\mathbb{R}}F(jx)\varphi'(x)\,\mathrm{d}x\right|\leqslant M$$

for all $j\geqslant 1$. Then

$$\left|\left\langle g_j-\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t,\varphi\right\rangle\right|\leqslant\frac{M}{j}\to 0$$

as $j\to\infty$. We deduce that $g_j\to\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t$ in $\mathscr{D}'(\mathbb{R})$. □

Really nice arguement.

Question 4

Let $\theta \in \mathscr{D}(\mathbb{R})$.

(i) Explain how the convolution $\theta * u$ is defined for a general distribution $u \in \mathscr{D}'(\mathbb{R})$.

(ii) Prove that $\theta * u \in \mathrm{C}^\infty(\mathbb{R})$ when $u \in \mathscr{D}'(\mathbb{R})$.

(iii) Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that for a general distribution $u \in \mathscr{D}'(\mathbb{R})$ we have that

$$\rho_\varepsilon * u \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0$$

(iv) Show that for each $u \in \mathscr{D}'(\mathbb{R})$ we can find a sequence (u_j) in $\mathscr{D}(\mathbb{R})$ such that

$$u_j \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } j \to \infty$$

Proof. (i) $\theta * u$ is the distribution defined by

$$\langle \theta * u, \varphi \rangle = \langle u, \widetilde{\theta} * \varphi \rangle$$

for all $\varphi \in \mathscr{D}(\mathbb{R})$, where $\widetilde{\theta}(x) := \theta(-x)$. We can check that the definition agrees with the usual convolution for regular distributions $u(x) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$:

$$\begin{aligned}
\langle \theta * u, \varphi \rangle &= \int_{\mathbb{R}} u(x)(\widetilde{\theta} * \varphi)(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} u(x)\left(\int_{\mathbb{R}} \varphi(y)\widetilde{\theta}(x-y)\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi(y)\left(\int_{\mathbb{R}} u(x)\theta(y-x)\,\mathrm{d}x\right)\mathrm{d}y && \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} \varphi(y)(\theta * u)(y)\,\mathrm{d}y
\end{aligned}$$

✓

Then by the adjoint identity scheme, $\theta * u$ is a distribution for any $u \in \mathscr{D}'(\mathbb{R})$.

(ii) *Sorry that I did not have enough time to finish the rest of the question. In fact this is a bookwork question, whose complete proof is given in Lemma 4.12, Lemma 4.14 and Theorem 5.9. We don't really need the general form in Theorem 5.9. A variant of Lemma 4.12 is sufficient for (ii).*

No worries - we'll go through in class! (This sheet was super long...)

□

Question 5

Let

$$p(\partial) = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \quad (k \in \mathbb{N} \text{ and } c_\alpha \in \mathbb{C})$$

be a partial differential operator on $\mathbb{R}^n$ in the usual multi-index notation. For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$ show that the supports always obey the rule:

$$\operatorname{supp}(p(\partial)u) \subseteq \operatorname{supp}(u)$$

Give an example of a distribution $v \in \mathscr{D}'(\mathbb{R})$ such that the distributional derivative $v' \neq 0$ has compact support, but v itself hasn't. Next, show that also the singular supports satisfy the rule

$$\operatorname{sing.supp}(p(D)u) \subseteq \operatorname{sing.supp}(u)$$

and give an example of a distribution $u \in \mathscr{D}'(\mathbb{R}^2)$ and a partial differential operator $p(\partial)$ so

$$\operatorname{sing.supp}(u) = \mathbb{R}^2 \text{ and } \operatorname{sing.supp}(p(\partial)u) = \emptyset$$

Proof. For $x \notin \operatorname{supp}(u)$, there exists an open neighbourhood $U \subseteq \Omega$ of x such that $u|_U = 0$. For $\varphi \in \mathscr{D}'(U)$,

$$\langle p(\partial)u, \varphi\rangle = \left\langle \sum_{|\alpha|\leqslant k} c_\alpha \partial^\alpha u, \varphi \right\rangle = \left\langle u, \sum_{|\alpha|\leqslant k} c_\alpha (-1)^{|\alpha|}\partial^\alpha \varphi \right\rangle = 0$$

Hence $p(\partial)u|_U = 0$. $x \notin \operatorname{supp}(p(\partial)u)$. We deduce that $\operatorname{supp}(p(\partial)u) \subseteq \operatorname{supp}(u)$.

Let $B(x)$ be the standard bump function on $[-1,1]$, and let

$$v(x) = \int_0^x B(t)\,\mathrm{d}t + C$$

?

where $C > 5$ is a constant. Then v is a smooth function, and it defines a regular distribution $v \in \mathscr{D}'(\mathbb{R})$. Its distributional derivative v' is exactly the usual derivative $B(x)$. The support of v and v' as distributions are exactly the support as functions on $\mathbb{R}$. We have $\operatorname{supp}(v) = \mathbb{R}$ and $\operatorname{supp}(v') = \operatorname{supp}(B) = [-1,1]$. Therefore v' is compactly supported but v is not. Nice.

For $x \notin \operatorname{sing.supp}(u)$, there exists an open neighbourhood $U \subseteq \Omega$ of x such that $u|_U \in C^\infty(U)$. Then $p(\partial)u|_U \in C^\infty(U)$. Hence $x \notin \operatorname{sing.supp}(p(\partial)u)$. We deduce that $\operatorname{sing.supp}(p(\partial)u) \subseteq \operatorname{sing.supp}(u)$.

Consider the Dirichlet function on $\mathbb{R}^2$:

$$u(x,y) = \mathbf{1}_{\mathbb{Q}^2}$$

But $u = 0$ in L^1_{loc} & hence in $\mathcal{D}'$?

Then $u \in L^1_{\text{loc}}(\mathbb{R}^2)$, and it defines a regular distribution. Since u is nowhere continuous, $\operatorname{sing.supp}(u) = \mathbb{R}^2$. Consider the trivial differential operator $p(\partial) = 0$. Then $p(\partial)u = 0$ has singular support $\operatorname{sing.supp}(p(\partial)u) = \varnothing$. □

not true.

Question 6

Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically zero. Explain why the formula $f = \log|F|$ defines a distribution on $\mathbb{C}$ Prove that its distributional Laplacian equals

$$\Delta f = \sum_{j\in J} 2\pi m_j \delta_{z_j}$$

where $\{z_j : j \in J\}$ are the distinct zeros for F and $\{m_j : j \in J\}$ their multiplicities.

[*Hint: Use the Cauchy-Riemann operators to calculate the Laplacian.*]

Proof. From complex analysis we know that the zeros of F are isolated. Suppose that $z_j \in \mathbb{C}$ is a zero of F with multiplicity m_j. Then in some open disc $B(z_j, r)$, we have

$$F(z) = (z - z_j)^{m_j} F_j(z)$$

where F_j is holomorphic with $F_j(z_j) \neq 0$. In $B(z_j, r)$, we can choose a branch cut and define a holomorphic branch of logarithm in the cut disc. We have

$$\log|F| = m_j \log|z - z_j| + \log|F_j|$$

$\log|F_j|$ is locally bounded in the cut disc, and $\log|z - z_j|$ is locally integrable, as discussed in Question 2(a). Hence $\log|F|$, defined on a simply-connected subset of $\mathbb{C} \setminus F^{-1}(\{0\})$, is locally integrable. Thus $\log|F|$ is a regular distribution.

Sorry I did not have enough time to solve the question. I believe a good answer is given on `https://math.stackexchange.com/questions/3056814`. □