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Problem Sheet 2

ASO: Calculus of Variations

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Question 1

It is required to find an extremal of the functional

$$\int_a^b F(x, y(x), y'(x), y''(x))\, \mathrm{d}x$$

among all smooth functions $y(x)$ which satisfy the boundary conditions

$$y(a) = y(b) = 0$$

Show that such an extremal must be a solution of the differential equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} = 0$$

and must satisfy the natural boundary conditions

$$\frac{\partial F}{\partial y''} = 0 \text{ at } x = a \text{ and } x = b.$$

Proof. Let y be a minimizer of the functional $S[y] = \int_a^b F(x, y(x), y'(x), y''(x))\, \mathrm{d}x$ and η be a smooth function such that $\eta(a) = \eta(b) = 0$. Then $y + \alpha\eta$ satisfies the same constraints as y for any $\alpha \in \mathbb{R}$. In particular we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} J[y + \alpha\eta]\bigg|_{\alpha=0} = 0$$

✓

Expanding the expression:

$$\int_a^b \left(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'} + \eta''\frac{\partial F}{\partial y''}\right)\, \mathrm{d}x = 0$$

By integration by parts, we have:

$$\int_a^b \eta\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''}\right)\, \mathrm{d}x + \left(\eta\frac{\partial F}{\partial y'} + \eta'\frac{\partial F}{\partial y''} - \eta\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)_{x=a}^{x=b} = 0$$

Since $\eta(a) = \eta(b) = 0$, the equation simplifies to

$$\int_a^b \eta\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''}\right)\, \mathrm{d}x + \eta'\frac{\partial F}{\partial y''}\bigg|_{x=a}^{x=b} = 0$$

By Lemma 2.3 in the notes, we deduce that

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} = 0 \qquad \frac{\partial F}{\partial y''}(a) = \frac{\partial F}{\partial y''}(b) = 0$$

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□

Question 2

An elastic beam has vertical displacement $y(x)$, $x \in [0, l]$. (The x-axis is horizontal and the y-axis is vertical and directed upwards.) The ends of the beam are supported, that is, $y(0) = y(l) = 0$, and the displacement minimizes the energy

$$\int_0^l \left(\frac{1}{2}D\left(y''(x)\right)^2 + \rho g y(x)\right)\, \mathrm{d}x,$$

where D, ρ and g are positive constants. Write down the differential equation and the boundary conditions that $y(x)$ must satisfy and show that

$$y(x) = -\frac{\rho g}{24D} x(l-x)\left(l^2 + x(l-x)\right)$$

Proof. Let $\mathcal{L} = \frac{1}{2}D\left(y''(x)\right)^2 + \rho g y(x)$ and $S[y] = \int_a^b \mathcal{L}\,\mathrm{d}x$. By Question 1, the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathcal{L}}{\partial y''} = 0$$

Hence

$$Dy'''' + \rho g = 0$$

with boundary conditions

$$y(0) = y(l) = 0 \qquad Dy''(0) = Dy''(l) = 0$$

Integrating four times we obtain

$$y(x) = -\frac{\rho g}{24D}x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$$

Substituting the boundary conditions

$$C_0 = 0, \qquad -\frac{\rho g}{24D}l^4 + C_3l^3 + C_2l^2 + C_1l + C_0 = 0, \qquad 2C_2 = 0, \qquad -\frac{\rho g}{2D}l^2 + 6C_3l + 2C_2 = 0.$$

Then $C_3 = \frac{\rho g}{12D}l$, $C_1 = \frac{\rho g}{24D}l^3 - \frac{\rho g}{12D}l^3 = -\frac{\rho g}{24D}l^3$. The solution y is given by

$$y(x) = -\frac{\rho g}{24D}x^4 + \frac{\rho g}{12D}lx^3 - \frac{\rho g}{24D}l^3x = -\frac{\rho g}{24D}x(x^3 - 2lx^2 + l^3) = -\frac{\rho g}{24D}x(l-x)\left(l^2 + x(l-x)\right)$$

✓

□

Question 3

Find the extremal corresponding to

$$\int_{-1}^{1} y\,\mathrm{d}x$$

when subject to $y(-1) = y(1) = 0$ and

$$\int_{-1}^{1}(y^2 + y'^2)\,\mathrm{d}x = 1$$

Solution. Let $I[y] = \int_{-1}^{1} y\,\mathrm{d}x = \int_{-1}^{1} F\,\mathrm{d}x$ and $J[y] = \int_{-1}^{1}(y^2 + y'^2)\,\mathrm{d}x = \int_{-1}^{1} G\,\mathrm{d}x$. Given the constraint $J[y] \equiv 1$, by the method of Lagrange multipliers we can write the Euler-Lagrange equation as follows

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}(F - \lambda G) - \frac{\partial}{\partial y}(F - \lambda G) = 0$$

We obtain that

$$-2\lambda y'' - (1 - 2\lambda y) = 0 \quad \Longrightarrow \quad y'' - y = -\frac{1}{2\lambda}$$

The general solution to the differential equation is

$$y(x) = A\sinh x + B\cosh x + \frac{1}{2\lambda}$$

✓

The boundary conditions are $y(-1) = y(1) = 0$. In addition we need to obtain another constraint for y by computing G and $J[y]$:

$$G(y,y') = y^2 + y'^2 = \left(A\sinh x + B\cosh x + \frac{1}{2\lambda}\right)^2 + (A\cosh x + B\sinh x)^2 =$$
$$(A^2+B^2)\cosh 2x + 2AB\sinh 2x + \frac{1}{\lambda}(A\sinh x + B\cosh x) + \frac{1}{4\lambda^2}$$

Therefore

$$J[y] = \int_{-1}^{1} G(y,y')\,\mathrm{d}x = \int_{-1}^{1}\left((A^2+B^2)\cosh 2x + \frac{B}{\lambda}\cosh x + \frac{1}{4\lambda^2}\right)\mathrm{d}x \qquad \text{(we use the fact that } \sinh x \text{ is odd)}$$
$$= \left(\frac{1}{2}(A^2+B^2)\sinh 2x + \frac{B}{\lambda}\sinh x + \frac{x}{4\lambda^2}\right)_{x=-1}^{x=1} = (A^2+B^2)\sinh 2 + \frac{2B}{\lambda}\sinh 1 + \frac{1}{2\lambda^2}$$

The other two constraints are

$$-A\sinh 1 + B\cosh 1 + \frac{1}{2\lambda} = 0 \qquad A\sinh 1 + B\cosh 1 + \frac{1}{2\lambda} = 0$$

The solution for the coefficients are $A = 0$, $B = \pm\dfrac{1}{\sqrt{1+\cosh 2 - \sinh 2}}$ and $1/2\lambda = -B\cosh 1$. Hence the solution for the original problem is

cosh/sinh(1), not cosh/sinh(2)

$$y(x) = \pm\frac{1}{\sqrt{1+\cosh 2 - \sinh 2}}(\cosh x - \cosh 1)$$

□

Question 4

(a) Suppose that $F : \mathbb{R}^7 \to \mathbb{R}$ is a C^2 function and that the C^2 function $u : \mathbb{R} \to \mathbb{R}$ gives a stationary value to the integral

$$\iiint_V F(x,y,z,u,u_x,u_y,u_z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

and satisfies $u = f$ on the smooth simple closed surface ∂V which bounds the open set V in $\mathbb{R}^3$, Show that u satisfies the Euler equation

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} + \frac{\partial}{\partial z}\frac{\partial F}{\partial u_z} = \frac{\partial F}{\partial u}$$

(b) Let $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}$. Find an extremal $u = u(x,y,z)$ for the problem of minimizing the integral

$$\iiint_V (u_x^2 + u_y^2 + u_z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

wehn subject to the constraints

$$\iiint_V u\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 4\pi$$

and $u = 1$ on the boundary of V.

Solution. (a) Consider any smooth function $\eta : \mathbb{R}^3 \to \mathbb{R}$ that satisfies $\eta = 0$ on ∂V. Similar to the 1-dimensional case, if u minimize the functional

$$I[u] = \iiint_V F(x,y,z,u,u_x,u_y,u_z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

the variation (along any smooth path):

$$\delta I = \frac{\mathrm{d}I}{\mathrm{d}\alpha}[u+\alpha\eta]\bigg|_{\alpha=0} = 0$$

We expand this expression:

$$\iint_V \left(\eta \frac{\partial F}{\partial u} + \frac{\partial \eta}{\partial x^i} \frac{\partial F}{\partial u_i} \right) \, \mathrm{d}\tau = 0$$

By chain rule:

$$\iint_V \left(\eta \frac{\partial F}{\partial u} + \frac{\partial}{\partial x^i} \left(\eta \frac{\partial F}{\partial u_i} \right) - \eta \frac{\partial}{\partial x^i} \frac{\partial F}{\partial u_i} \right) \, \mathrm{d}\tau = 0$$

If we denote the vector field $\boldsymbol{G} = \left(\frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_y}, \frac{\partial F}{\partial u_z} \right)$, then $\frac{\partial}{\partial x^i} \left(\eta \frac{\partial F}{\partial u_i} \right) = \boldsymbol{\nabla} \cdot (\eta \boldsymbol{G})$. By divergence theorem, we have:

$$\iint_V \eta \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x^i} \frac{\partial F}{\partial u_i} \right) \, \mathrm{d}\tau - \oiint_{\partial V} \eta \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{S} = 0$$

The surface integral is zero because $\eta = 0$ on ∂V. Since η is arbitrary, we deduce that

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x^i} \frac{\partial F}{\partial u_i} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} - \frac{\partial}{\partial z} \frac{\partial F}{\partial u_z} = 0$$

(b) Suppose that $F = u_x^2 + u_y^2 + u_z^2$ and $G = u$. By the method of Lagrange multipliers, the function $F - \lambda G$ satisfies the Euler-Lagrange equation. We have:

$$2(u_{xx} + u_{yy} + u_{zz}) = -\lambda \quad \Longrightarrow \quad \nabla^2 u = -\frac{\lambda}{2}$$

We observe that the constraint of u are spherically symmetry. So we shall look for a spherically symmetric solution $u = u(r)$. (Once the solution is determined, it is unique by the uniqueness theorem for Laplace's equations.)

In spherical coordinates, the Laplacian has the form

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\hat{L}^2}{\hbar^2}$$

Therefore, u satisfies the ODE

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}u}{\mathrm{d}r} \right) = -\frac{\lambda}{2}$$

By integrating twice we find that

$$u(r) = -\frac{\lambda}{12} r^2 + B r^{-1} + C$$

Since we want u to be bounded at $r = 0$, we put $B = 0$. The constraint $u(r = 1) = 1$ implies that $C = 1 + \frac{\lambda}{12} r^2$. And for the other constraint:

$$\begin{aligned}
&\iiint_V u \, \mathrm{d}x \mathrm{d}y \mathrm{d}z = 4\pi \\
\Longrightarrow \; &\int_0^1 u(r) r^2 \, \mathrm{d}r = 1 \\
\Longrightarrow \; &\int_0^1 \left(-\frac{\lambda}{12} r^4 + \left(1 + \frac{\lambda}{12} \right) r^2 \right) \, \mathrm{d}r = 1 \\
\Longrightarrow \; &-\frac{\lambda}{12} \cdot \frac{1}{5} + \frac{1}{3} \left(1 + \frac{\lambda}{12} \right) = 1 \\
\Longrightarrow \; &\lambda = 60
\end{aligned}$$

The solution is given by $u(r) = -5r^2 + 6$. In Cartesian coordinates, the solution is

$$u(x,y,z) = -5(x^2+y^2+z^2)+6.$$

□

Question 5

Let p be a positive real-valued function differentiable on the bounded interval $[a,b]$ and let q and r be positive real-valued continuous functions on $[a,b]$. Show that the extremals of

$$J[y] = \int_a^b \left(py'^2 + qy^2\right) \mathrm{d}x$$

subject to the constraint

$$\int_a^b ry^2 \, \mathrm{d}x = 1$$

must satisfy

$$(py')' + (-q+\lambda r)y = 0 \tag{A}$$

with $py' = 0$ at $x=a$ and $x=b$.

Show that if y_1 and y_2 are solutions to (A) for $\lambda = \lambda_1, \lambda_2$ respectively, where $\lambda_1 \neq \lambda_2$, then

$$\int_a^b ry_1y_2 \, \mathrm{d}x = 0 \tag{B}$$

Find the extremals of $\displaystyle\int_0^\pi y'^2 \, \mathrm{d}x$ subject to $\displaystyle\int_0^\pi y^2 \, \mathrm{d}x = 1$ and the corresponding values of λ. Verify that these extremals satisfy (B).

Solution. Let $F(x,y,y') = py'^2 + qy^2$ and $G(x,y) = ry^2$. By the method of Lagrange multipliers, the Euler-Lagrange equation of the problem is

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}(F-\lambda G) - \frac{\partial}{\partial y}(F-\lambda G) = 0$$

which is

$$2(py')' - 2qy + \lambda(2ry) = 0 \implies (py')' + (-q+\lambda r)y = 0$$

The differential equation satisfies the natural boundary condition:

$$\frac{\partial}{\partial y'}(F-\lambda G) = 2py' = 0 \text{ at } x=a \text{ and } x=b.$$

Let $L : y \mapsto -(py')' + qy$ be the Sturm-Liouville operator. By Part A Differential Equations II we know that L is a (fully) self-adjoint operator with respect to the inner product defined by integration:

$$\langle u(x)|v(x)\rangle := \int_a^b u(x)v(x) \, \mathrm{d}x$$

We have $Ly_1 = \lambda_1 ry_1$ and $Ly_2 = \lambda_2 ry_2$. Then

$$\lambda_1 \int_a^b ry_1y_2 \, \mathrm{d}x = \langle \lambda_1 ry_1 | ry_2\rangle = \langle Ly_1|y_2\rangle = \langle y_1|Ly_2\rangle = \langle y_1|\lambda_2 ry_2\rangle = \lambda_2 \int_a^b ry_1y_2 \, \mathrm{d}x$$

Since $\lambda_1 \neq \lambda_2$, we have $\displaystyle\int_a^b ry_1y_2 \, \mathrm{d}x = 0.$

For the extremals of $\int_0^\pi y'^2 \, \mathrm{d}x$ subject to $\int_0^\pi y^2 \, \mathrm{d}x = 1$, we observe that $p = 1$, $q = 0$ and $r = 1$. The differential equation satisfied by y is

$$y'' + \lambda y = 0, \qquad y'(0) = y'(\pi) = 0$$

As we require that y' vanishes at two distinct points, we require that $\lambda > 0$. The general solution of y is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

The boundary conditions imply that $B = 0$ and $\sqrt{\lambda}\pi \in \pi\mathbb{Z}$. Then $\lambda_n = n^2$ for $n \geqslant 1$ and $y_n(x) = \cos nx$.

For $m \neq n$:

$$\begin{aligned}
\int_0^\pi y_n y_m \, \mathrm{d}x &= \int_0^\pi \cos nx \cos mx \, \mathrm{d}x \\
&= \int_0^\pi \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right) \, \mathrm{d}x \\
&= \left(\frac{1}{2(m+n)}\sin(m+n)x + \frac{1}{2(m-n)}\sin(m-n)x\right)_0^\pi = 0
\end{aligned}$$

The orthogonality condition is satisfied as expected. □