Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 4

Particle Physics I

B4: Subatomic Physics

70%

20 January, 2020

Question 4.1

The cross-section for the reaction $\pi^- p \to \pi^0 n$ shows a prominent peak when measured as a function of the π^- energy. The peak corresponds to the Δ resonance, which has a mass of 1232MeV, with $\Gamma = 120$MeV. The partial widths for the incoming and outgoing states are $\Gamma_i = 40$MeV, and $\Gamma_f = 80$MeV, respectively, for this reaction.

a) The pions can be represented by an isospin triplet ($I = 1$) and the nucleons form an isospin doublet $\left(I = \frac{1}{2}\right)$, while the Δ series of resonances have $I = \frac{3}{2}$.

By assuming that the isospin operators $I, I_3, I_\pm$ obey the same algebra as the quantum mechanical angular momentum operators $J, J_z, J_\pm$, explain that the ratio $\Gamma_i/\Gamma_f = \frac{1}{2}$.

(Remember that $\left|j_1 j_2 J M\right\rangle = \sum \left\langle j_1 j_2 m_1 m_2 \mid j_1 j_2 J M\right\rangle \left|j_1 j_2 m_1 m_2\right\rangle$ with the Clebsch-Gordan coefficients $c_{J,M,m_1,m_2} = \left\langle j_1 j_2 m_1 m_2 \mid j_1 j_2 J M\right\rangle$.)

b) At what pion beam energy will the cross-section be maximal for a stationary proton?

c) On the same plot draw the cross-sections as a function of the centre-of-mass energy for the processes $\pi^- p \to \pi^0 n$ and the one for $\pi^- p \to \pi^- p$ around the resonance, giving values for the variables in the Breit-Wigner formula where possible.

d) By considering the quark content of the intermediate states, discuss whether you would expect similar peaks in the cross-sections for the reactions (i) $K^- p \to$ products and (ii) $K^+ p \to$ products.

Solution. a) Consider the reaction

$$\pi^- + p \to \Delta \to \pi^0 + n$$

where the isospin eigenstates are given by $\pi^- = |1,-1\rangle$, $p = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$, $\Delta = \left|1, \frac{1}{2}; \frac{3}{2}, -\frac{1}{2}\right\rangle$, $\pi^0 = |1,0\rangle$, $n = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

By Fermi theory, the partial width $\Gamma_n \propto |\langle n|\delta H|i\rangle|^2 \propto |\langle n \mid i\rangle|^2$ since the Hamiltonian is invariant under isospin transformation. Therefore

$$\frac{\Gamma_i}{\Gamma_f} = \frac{|\langle \pi^- \otimes p \mid \Delta\rangle|^2}{|\langle \pi^0 \otimes n \mid \Delta\rangle|^2} = \left(\frac{C^{1,1/2}_{-1,1/2;3/2,-1/2}}{C^{1,1/2}_{0,-1/2;3/2,-1/2}}\right)^2$$

where $C^{j_1,j_2}_{m_1,m_2;j,m}$ is the Clebsch-Gordan coefficient. From the known results,

$$C^{1,1/2}_{-1,1/2;3/2,-1/2} = \sqrt{\frac{1}{3}}, \quad C^{1,1/2}_{0,-1/2;3/2,-1/2} = \sqrt{\frac{2}{3}}$$

2

Therefore we must have $\Gamma_i/\Gamma_f = 1/2$.

b) Breit-Wigner formula:

$$\sigma(i \to R \to f) \propto \frac{\Gamma_i \Gamma_f}{(E_i - E_R)^2 + \Gamma^2/4}$$

Therefore when the centre of mass energy E_i before scattering is equal to the rest mass of Δ, $E_R = m_\Delta$, the cross-section is maximal. Then

$$(m_p + E_{\pi^-})^2 - (E_{\pi^-}^2 - m_{\pi^-}^2) = m_\Delta^2 \implies E_{\pi^-} = \frac{m_\Delta^2 - m_p^2 - m_{\pi^-}^2}{2m_p} = 329.3 \text{ MeV}$$

2

The pion beam energy is 329.3 MeV.

(This theory may not be vaild, as we see that $E_{\pi^-} \gg m_{\pi^-} = 139.6$ MeV, which suggests that the relativistic effect is not negligible. However, the Breit-Wigner formula is completely non-relativistic.)

c) For the two reactions,

40 x 80

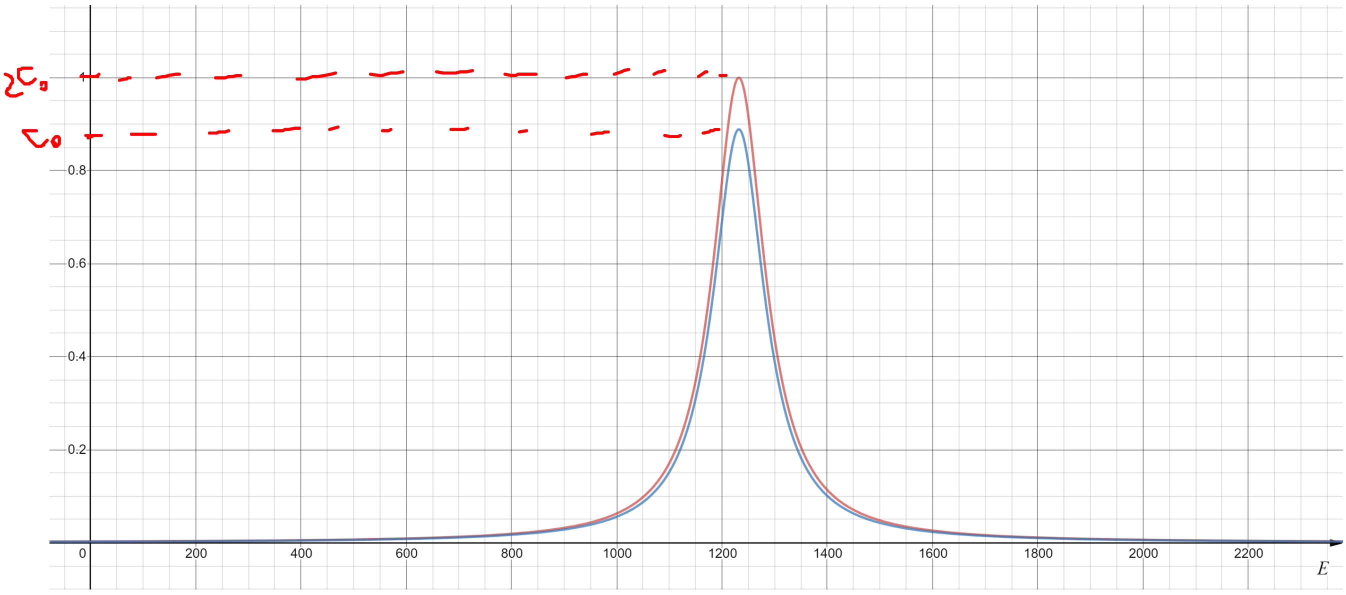
$$\sigma(i \to R \to f) = \frac{\pi}{k_i^2} \frac{\Gamma_i \Gamma_f}{(E_i - E_R)^2 + \Gamma^2/4} = \frac{\pi}{k_i^2} \frac{3200}{(E_i - 1232)^2 + 3600}$$

40 x 40 $(\Gamma_i \Gamma_i)$

$$\sigma(i \to R \to i) = \frac{\pi}{k_i^2} \frac{\Gamma^2/4}{(E_i - E_R)^2 + \Gamma^2/4} = \frac{\pi}{k_i^2} \frac{3600}{(E_i - 1232)^2 + 3600}$$

The cross section for the second process is exactly 9/8 of that of the first process for any energy E_i.

The plot of the two functions:



1

7/8

d) The quark combinations for the particles are $K^- = s\bar{u}$, $K^+ = u\bar{s}$ and $p = uud$. We see that $K^- + p$ can form an intermediate bound state consisting of uds, whereas $K^+ + p$ cannot form a similar intermediate bound state. 2 ☐

Question 4.2

a) What is a symmetry in physics? What is conservation of parity, and what is parity violation?

b) K^+ is found to decay to $\pi^+\pi^0$, as well as $\pi^+\pi^+\pi^-$. Explain why this can only be explained if parity is violated in the weak interaction (if you assume that angular momentum conservation holds).

c) ^{60}Co nuclei $\left(J^P = 5^+\right)$ are polarised by immersing them at low temperature in a magnetic field. When these nuclei β-decay to ^{60}Ni$\left(4^+\right)$ more electrons are emitted opposite to the aligning B field than along it (Reported in C.S.Wu et al., Phys. Rev. 105, 1413(1957)). Explain why this demonstrates parity violation in the weak interaction.

d) The intensity of the electrons emitted in the decay is found to be consistent with

$$I(v,\theta) = 1 - \frac{v}{c}\cos\theta$$

where v is the magnitude of the electron velocity and θ the angle between its direction and the direction of the ^{60}Co spin. What can you deduce from this observation about the possible orientation of the spin of an ultra-relativistic (i.e. massless) lepton involved in a weak interaction in relation to the direction of its movement?

e) What consequence does this have for the decay $\pi^\pm \to \mu^\pm \nu$?

Solution. a) Roughly speaking, a symmetry of a physical system is a dynamical property that is invariant under certain transformations. In Quantum Mechanics, the parity is defined as a Hermitian operator $\hat{P}$ such that $\hat{P}\Psi(\mathbf{x},t) = \Psi(-\mathbf{x},t)$. A system has conservation of parity if $\hat{P}$ commutes with the Hamiltonian $\hat{H}$. If the parity is not conserved in an interaction, then it is a case of parity violation. 3

b) K^+, π^0 and $\pi^\pm$ are all mesons with $J^P = 0^-$. By conservation of angular momentum, the composite system of the decay products $\pi^+ + \pi^0$ has parity

$$P = (-1)\times(-1)\times(-1)^L = 1$$

argument for L ? 1

Therefore the process $K^+ \to \pi^+ + \pi^0$ has parity violation.

c) The magnetic field $\mathbf{B}$ is a 2-form, or an axial vector, and hence does not change direction under a parity transformation. The velocity of the electron, however, is a polar vector, and hence is reversed under a parity transformation. Therefore a preference of electron emission in the opposite direction of $\mathbf{B}$ suggests a parity violation during the decay. 1

d)

☐

5/9

Question 4.3

Write down the valence quark content for each of the different particles in the reactions below and check that the conservation laws of electric charge, flavour, strangeness and baryon number are satisfied throughout.

$$
\begin{array}{llll}
(1) & \pi^- + p & \rightarrow & K^0 + \Lambda \\
(2) & K^- + p & \rightarrow & K^0 + \Xi^0 \\
(3) & \Xi^- + p & \rightarrow & \Lambda + \Lambda \\
(4) & K^- + p & \rightarrow & K^+ + K^0 + \Omega^-
\end{array}
$$

Draw a quark flow diagram for the last reaction.

Solution. (I am not sure if the flavour refers to the number of each type of quarks in the reaction.)

		π^-	p	$\rightarrow$	K^0	Λ
	Quarks	$d\bar{u}$	uud		$d\bar{s}$	uds
(1)	C	-1	1		0	0
	B	0	1		0	1
	S	0	0		1	-1

The electric charge, baryon number, strangeness, and flavour are all conserved.

2

		K^-	p	$\rightarrow$	K^0	Ξ^0
	Quarks	$s\bar{u}$	uud		$d\bar{s}$	uss
(2)	C	-1	1		0	0
	B	0	1		0	1
	S	-1	0		1	-2

The electric charge, baryon number, strangeness, and flavour are all conserved.

2

		Ξ^-	p	$\rightarrow$	Λ	Λ
	Quarks	ssd	uud		uds	uds
(3)	C	-1	1		0	0
	B	1	1		1	1
	S	-2	0		-1	-1

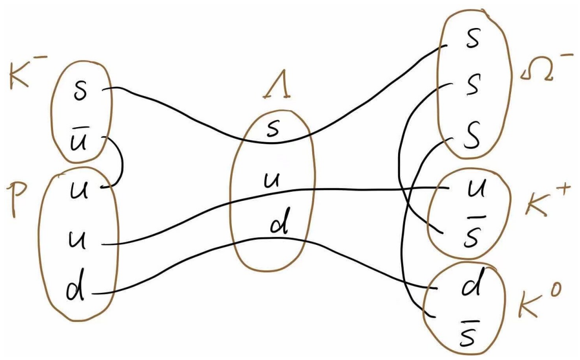
The electric charge, baryon number, strangeness, and flavour are all conserved.

2

		K^-	p	$\rightarrow$	K^0	K^+	Ω^-
	Quarks	$s\bar{u}$	uud		$d\bar{s}$	$u\bar{s}$	sss
(4)	C	-1	1		0	1	-1
	B	0	1		0	0	1
	S	-1	0		1	1	-3

2

The electric charge, baryon number, strangeness, and flavour are all conserved. The quark flow diagram is shown below:



1

9/9

□

Question 4.4

Consider the decay of the ρ^0 meson $(J^P = 1^-)$ in the following decay modes:

a) $\rho^0 \to \pi^0 + \gamma$

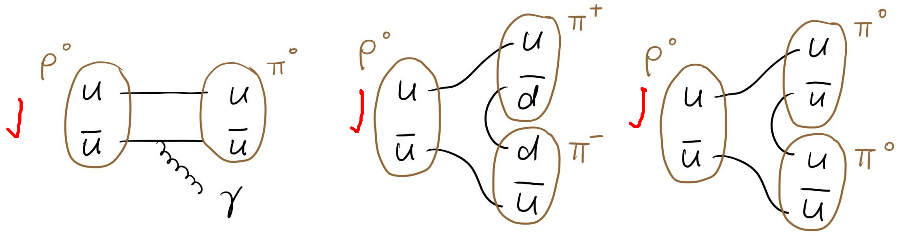
b) $\rho^0 \to \pi^+ + \pi^-$

c) $\rho^0 \to \pi^0 + \pi^0$.

Draw diagrams in each case to show the quark flow. Consider the symmetry of the wave-function required for $\pi^0 + \pi^0$ and explain why this decay mode is forbidden.

From consideration of the relative strength of the different fundamental forces, determine which of the other two decay modes will dominate.

Solution. The quark content for π^0 and ρ^0 are $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$. I am not sure how it can be represented in the quark flow diagram. In the following diagrams they are represented by a $u\bar{u}$. Choose the pion to be u-ubar and rho to be u-ubar, for example.



3

By conservation of angular momentum, the composite system of $\pi^0 + \pi^0$ has orbital angular momentum number $\ell = 1$. It is known that the spherical harmonic $Y_{\ell,m}(\theta,\varphi)$ is anti-symmetric for $\ell = 1$. Hence the wave function of the system $\pi^0 + \pi^0$ is anti-symmetric, which contradicts that π^0 is a boson. Therefore the decay mode $\rho^0 \to \pi^0 + \pi^0$ is forbidden. 1

J = 1 Conserved

If we look at the parity of the decay mode (a) and (b), we note that $\pi^0 + \gamma$ has no orbital angular momentum (since γ is spin-1) and hence has parity $P = (-1) \times (-1) = 1$. The parity conservation is violated. $\pi^+ + \pi^-$ has parity $P = (-1) \times (-1) \times (-1)^\ell = -1$. The parity is conserved. We deduce that decay mode (a) is a weak interaction, while decay mode (b) could be a strong interaction. Therefore (b) would be the dominating decay mode for ρ^0. □ 1

5/5

Question 4.5

The lightest group of baryons have $J^P = \frac{1}{2}^+$. A common representation of these baryons is to indicate their states in a coordinate system with the third component of the isospin and the strangeness (or the hypercharge) as the axes. Draw this diagram for this octet. What is the motivation for drawing the diagram in this way?

The following table gives the masses and lifetimes of the strange baryons (also called the "hyperons") in the $J^P = \frac{1}{2}^+$ octet.

Particle	Σ^+	Σ^0	Σ^-	Λ^0	Ξ^0	Ξ^-
Mass (MeV/c^2)	1189	1192	1197	1115	1315	1321
Lifetime	80ps	7.4×10^{-20} s	150ps	260ps	290ps	160ps

a) Explain how the properties of the baryons (spin, parity, masses) in this octet can be explained in the quark model. Which particle properties cannot be explained from the quark model?

b) Discuss what is different about the decay of Σ^0 compared to the other particles in the table and identify a plausible decay mode for it.

c) The neutron decays with a lifetime of 881 s. Why is this so much longer than for the above decays? The branching ratio for the semileptonic decay $\Lambda \to p e \bar{\nu}_e$ is 8.32×10^{-4}. Show that these numbers are compatible.

d) Even though all the hyperons decay in almost all cases into hadronic final states there is no theory which would allow us to compute the magnitude of the hadronic decay rate. However, if we look at the lifetimes of the Σ^+ and the Σ^-, what can we conclude about these decays?

e) How would you need to expand your diagram if you want to include baryons containing a charm quark?

Solution. □

Question 4.6

A good approximation of the potential in a $q\bar{q}$ system is

$$V(r) = -\frac{4}{3}\frac{\hbar c\alpha_s}{r} + kr$$

a) Sketch $V(r)$ against r. Show that for $r \ll r_0 = \sqrt{\hbar c\alpha_s/k}$ the $1/r$ term dominates.

b) Now let's assume we can ignore the linear term. Calculate values for α_s for two cases:

   i. The splitting between the $n = 2$ and $n = 1$ states in the Ψ system ($c\bar{c}$) is 588MeV and $m_c = 1870\text{MeV}/c^2$.

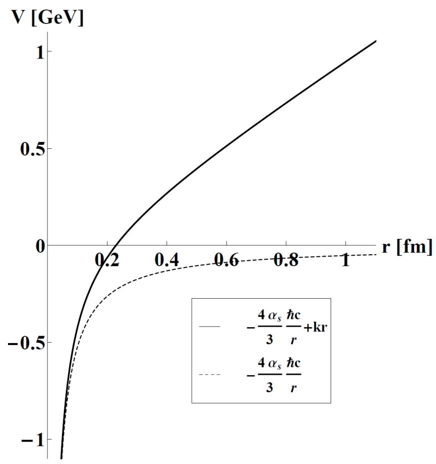
   ii. The splitting between the $n = 2$ and $n = 1$ states in the Υ system ($b\bar{b}$) is 563MeV, and $m_b = 5280\text{MeV}/c^2$.

   Why do these results differ?

c) Still ignoring the linear term, compute the Bohr radius a_0 for the J/Ψ and the Υ. The value of $\sqrt{k}$ is about $400\text{MeV}/\sqrt{\hbar c}$; is it a reasonable approximation to ignore the linear term in $V(r)$ to compute the energies of the lowest bound states?

d) Calculate the expectation value of the kinetic energy in the $n = 1$ states of the J/Ψ and the Υ. Comment on whether we are justified in using a non-relativistic approximation for these states; are relativistic corrections more significant in positronium or in these $q\bar{q}$ states?

Solution. a) We borrow the plot of $V(r)$ from the notes:



1

$$V(r) = -\frac{4}{3}\frac{\hbar c\alpha_s}{r}\left(1 + \frac{r^2}{r_0^2}\right) \sim -\frac{4}{3}\frac{\hbar c\alpha_s}{r} \quad \text{for } r \ll r_0$$

b) When $V(r) \propto -1/r$, the system behaves like a hydrogen atom. The Hamiltonian is given by

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{4\hbar c\alpha_s}{3r}$$

Therefore the energy levels are given by

$$E_n = -\frac{1}{n^2}\frac{8\mu c^2\alpha_s^2}{9} = -\frac{1}{n^2}\frac{4mc^2\alpha_s^2}{9} \implies E_2 - E_1 = \frac{1}{3}mc^2\alpha_s^2$$

For i., $\alpha_s = \sqrt{\frac{3(E_2 - E_1)}{m_c}} = 0.971$. For ii., $\alpha_s = \sqrt{\frac{3(E_2 - E_1)}{m_b}} = 0.566$. The results are different because the coupling

2

constant decreases strongly as the energy increases.

c) The Bohr radius is given by

$$a_0 = \frac{\hbar^2}{\mu}\frac{3}{4\hbar c\alpha_s} = \frac{3\hbar}{2mc\alpha_s}$$

For Ψ, $a_0 = 1.6\times10^{-16}$ m. For Υ, $a_0 = 9.9\times10^{-17}$ m.

For $\sqrt{k} \sim 400$ MeV/$\sqrt{\hbar c}$,

$$r_0 = \sqrt{\frac{\hbar c\alpha_s}{k}} \sim 4.9\cdot10^{-16}\sqrt{\alpha_s}\,\mathrm{m}$$

For $\alpha_s \sim 1$, the magnitude of a_0 and r_0 are comparable. Therefore it is not reasonable to neglect the linear term at the lowest bound state. 3

d) In the ground state, the expected kinetic energy is

$$T = -E_1 = \frac{4}{3}(E_2 - E_1)$$

For Ψ, $T = 784$ MeV. For Υ, $T = 751$ MeV. I think at this magnitude the non-relativistic approximation is still valid.

For the positronium, the rest mass of an electron is only 0.51 MeV. The relativistic correction is far more significant than that in the $q\bar{q}$ states. □

3

9/9