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Problem Sheet 3

C2.2: Homological Algebra

Overall mark: β+

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Section A: Introductory

Question 1

Prove that for $A, B \in \mathbb{Z}$-Mod, $\forall\, i > 1$, $\mathrm{Ext}^i_{\mathbb{Z}}(A,B) = 0 = \mathrm{Tor}^{\mathbb{Z}}_i(A,B)$.

Proof. The one-line answer is that $\mathbb{Z}$ *is a PID so that it has Krull's dimension 1.* ✓

Let $\pi : \mathbb{Z}^{\oplus A} \to A$ be the projection epimorphism. Then we have a short exact sequence

$$0 \longrightarrow \ker\pi \longrightarrow \mathbb{Z}^{\oplus A} \xrightarrow{\ \pi\ } A \longrightarrow 0$$

Since $\mathbb{Z}$ is a PID, by Question 6 of Sheet 2, every submodule of the free module $\mathbb{Z}^{\oplus A}$ is free. Hence $\ker\pi$ is free. The sequence above is a free resolution of A. ✓

We note that $\mathrm{Tor}_i(A,B) = H_i(A_\bullet \otimes_{\mathbb{Z}} B)$ where $A_0 = \mathbb{Z}^{\oplus A}$, $A_1 = \ker\pi$, and $A_i = 0$ for $i > 1$. Hence $\mathrm{Tor}^{\mathbb{Z}}_i(A,B) = 0$ for $i > 1/$

Similarly, $\mathrm{Ext}^i_{\mathbb{Z}}(A,B) = H^i(\mathrm{Hom}_{\mathbb{Z}}(A_\bullet, B))$. We have $\mathrm{Ext}^i_{\mathbb{Z}}(A,B) = 0$ for $i > 1$. ✓ □

Section B: Core

Before going into Question 2 and 3, we purpose the following lemma about the Hom functor, which will be useful later; Do you mean "propose"?

Lemma 1

1. For $a \geqslant b, c \geqslant 0$, $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c) \cong \mathbb{Z}/2^{\min\{b,c\}}$.
2. For $n \geqslant 2$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = 0$.

Proof. ✓

1. Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$. φ is uniquely determined by $\varphi(1) \in \mathbb{Z}/2^c$. If $b \geqslant c$, then $\varphi(1)$ could be any element in c. Hence $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$ is bijective to $\mathbb{Z}/2^c$. It is easy to check that this is in fact a $\mathbb{Z}/2^a$-module isomorphism. On the other hand, if $b \leqslant c$, then $2^b\varphi(1) = \varphi(2^b) \in \mathbb{Z}/2^c$. Hence $\varphi(1)$ lies in the unique subgroup of $\mathbb{Z}/2^c$ which is isomorphic to $\mathbb{Z}/2^b$. It is easy to check that we have a $\mathbb{Z}/2^a$-module isomorphism $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c) \cong \mathbb{Z}/2^b$.
2. Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$. Then $n\varphi(1) = \varphi(n) = \varphi(0) = 0$. Since $\mathbb{Z}$ is torsion-free, $\varphi(1) = 0$ and hence $\varphi = 0$. We have $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = 0$. □

β+ **Question 2**

Compute the following Ext, Tor groups:

(a) $\mathrm{Tor}^{\mathsf{k}[x]}_*\left(\dfrac{\mathsf{k}[x]}{x-a}, \dfrac{\mathsf{k}[x]}{x-b}\right)$ for $a, b \in \mathsf{k}$ a field.

(b) $\mathrm{Tor}^{\mathbb{Z}}_*\left(\dfrac{\mathbb{Z}}{a}, \dfrac{\mathbb{Z}}{b}\right)$ for $a, b \in \mathbb{Z}$.

(c) $\mathrm{Ext}^*_{\mathbb{Z}/4}\left(\dfrac{\mathbb{Z}}{2}, \dfrac{\mathbb{Z}}{2}\right)$.

(d) $\mathrm{Ext}^*_{\mathbb{Z}/2^a}\left(\dfrac{\mathbb{Z}}{2^b}, \dfrac{\mathbb{Z}}{2^c}\right)$ for $a \geqslant b \geqslant c$.

(e) $\mathrm{Ext}^*_{\mathsf{k}[x,y]/(x^2,xy,y^2)}(\mathsf{k},\mathsf{k})$.

Proof. (a) We first find a free resolution $P_\bullet \to A$ for $A := \mathsf{k}[x]/\langle x-a\rangle$. This is easy:

$$0 \longrightarrow \mathsf{k}[x] \xrightarrow{\cdot(x-a)} \mathsf{k}[x] \longrightarrow \frac{\mathsf{k}[x]}{\langle x-a\rangle} \longrightarrow 0$$

By tensoring $B := \mathsf{k}[x]/\langle x-b\rangle$ to the free resolution, we obtain the chain complex:

$$0 \longrightarrow \frac{\mathsf{k}[x]}{\langle x-b\rangle} \xrightarrow{\partial_1} \frac{\mathsf{k}[x]}{\langle x-b\rangle} \longrightarrow 0$$

Therefore

$$\operatorname{Tor}_n^{\mathsf{k}[x]}(A,B) = \begin{cases} \operatorname{coker}\partial_1, & n=0 \\ \ker\partial_1, & n=1 \\ 0, & n>1 \end{cases}$$

- If $a=b$, then the map $\partial_1 : \frac{\mathsf{k}[x]}{\langle x-b\rangle} \to \frac{\mathsf{k}[x]}{\langle x-b\rangle}$ is zero. We have

$$\operatorname{Tor}_n^{\mathsf{k}[x]}(A,B) = \begin{cases} \frac{\mathsf{k}[x]}{\langle x-a\rangle}, & n=0,1 \\ 0, & n>1 \end{cases}$$

- If $a \neq b$, then $\langle x-a\rangle$ and $\langle x-b\rangle$ are coprime ideals of $\mathsf{k}[x]$. Hence ∂_1 is bijective. We have

$$\operatorname{Tor}_n^{\mathsf{k}[x]}(A,B) = 0, \quad n \in \mathbb{N}$$

(b) $A := \mathbb{Z}/a$ has a free resolution:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \longrightarrow \mathbb{Z}/a \longrightarrow 0$$

Tensoring $B := \mathbb{Z}/b$:

$$0 \longrightarrow \mathbb{Z}/b \xrightarrow{\partial_1} \mathbb{Z}/b \longrightarrow 0$$

Let $d = \gcd(a,b)$, $a = pd$ and $b = qd$. Then

$$\ker\partial_1 = \{\overline{n} : b \mid an\} = \{\overline{n} : q \mid n\} = q\mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$

and

$$\operatorname{coker}\partial_1 = \frac{\mathbb{Z}/b\mathbb{Z}}{\operatorname{im}\partial_1} = \frac{\mathbb{Z}/b\mathbb{Z}}{(a\mathbb{Z}+b\mathbb{Z})/b\mathbb{Z}} \cong \frac{\mathbb{Z}}{a\mathbb{Z}+b\mathbb{Z}} = \mathbb{Z}/d\mathbb{Z}$$

Hence

$$\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a,\mathbb{Z}/b) = \begin{cases} \mathbb{Z}/\gcd(a,b)\mathbb{Z}, & n=0,1 \\ 0, & n>1 \end{cases}$$

In particular, if a and b are coprime, then $\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a,\mathbb{Z}/b) = 0$ for all $n \in \mathbb{N}$.

(c) We take a free resolution of $\mathbb{Z}/2$: this map isn't injective, so this isn't a resolution.

$$0 \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

Applying the contravariant functor $\operatorname{Hom}_{\mathbb{Z}/4}(-,\mathbb{Z}/2)$ to the free resolution:

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4,\mathbb{Z}/2) \xrightarrow{-\circ 2} \operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4,\mathbb{Z}/2) \longrightarrow 0$$

Note that $\operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4,\mathbb{Z}/2) \cong \mathbb{Z}/2$. So it is equivalent to

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow 0$$

Hence

$$\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2,\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2, & n=0,1\\ 0, & n>1\end{cases}$$ ✗

(d) We take a free resolution of $\mathbb{Z}/2^b$:

again, this isn't injective.

$$0\longrightarrow \mathbb{Z}/2^a \xrightarrow{\cdot 2^b} \mathbb{Z}/2^a \longrightarrow \mathbb{Z}/2^b \longrightarrow 0$$

Applying the contravariant functor $\mathrm{Hom}_{\mathbb{Z}/2^a}(-,\mathbb{Z}/2^c)$:

$$0\longrightarrow \mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^a,\mathbb{Z}/2^c)\xrightarrow{-\circ 2^b}\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^a,\mathbb{Z}/2^c)\longrightarrow 0$$

Note that $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^a,\mathbb{Z}/2^c)\cong\mathbb{Z}/2^c$. So it is equivalent to

$$0\longrightarrow \mathbb{Z}/2^c \xrightarrow{0} \mathbb{Z}/2^c \longrightarrow 0$$

Hence

$$\mathrm{Ext}^n_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b,\mathbb{Z}/2^c)=\begin{cases}\mathbb{Z}/2^c\mathbb{Z}, & n=0,1\\ 0, & n>1\end{cases}$$ ✗

(e) Let $R:=\dfrac{\mathsf{k}[x,y]}{\langle x^2,xy,y^2\rangle}$ for simplicity. First we note that k has a unique R-module structure given by $x\cdot 1=y\cdot 1=0$. This is because $0=x^2\cdot 1=(x\cdot 1)^2$ in k and k is a domain implies that $x\cdot 1=0$. Similarly $y\cdot 1=0$.

We construct a free resolution for k as follows. Consider the projection $\pi:R\to\mathsf{k}$ given by $x,y\mapsto 0$. We have $\ker\pi=\langle x,y\rangle\lhd R$. Next, choose $\varphi:R^2\to R$ given by $(1,0)\mapsto x$, $(0,1)\mapsto y$. Then $\mathrm{im}\,\varphi=\langle x,y\rangle=\ker\pi$. Note that

$$\varphi((a+bx+cy,d+ex+fy))=ax+bx^2+cxy+dy+exy+fy^2=ax+dy$$

Hence $\ker\varphi=\langle x,y\rangle\oplus\langle x,y\rangle=\ker\pi\oplus\ker\pi$. Inductively we can construct an infinite sequence:

$$\cdots\longrightarrow R^4\xrightarrow{(\varphi,\varphi)}R^2\xrightarrow{\varphi}R\xrightarrow{\pi}\mathsf{k}\longrightarrow 0$$

Apply the functor $\mathrm{Hom}_R(-,\mathsf{k})$:

$$0\longrightarrow\mathrm{Hom}_R(\mathsf{k},\mathsf{k})\xrightarrow{-\circ\pi}\mathrm{Hom}_R(R,\mathsf{k})\xrightarrow{-\circ\varphi}\mathrm{Hom}_R(R^2,\mathsf{k})\xrightarrow{-\circ(\varphi,\varphi)}\mathrm{Hom}_R(R^4,\mathsf{k})\longrightarrow\cdots$$

It is clear that $\mathrm{Hom}_R(R^n,\mathsf{k})\cong\mathsf{k}^n$, Also, $\mathrm{Hom}_R(\mathsf{k},\mathsf{k})\cong\mathsf{k}$, because $\psi\in\mathrm{Hom}_R(\mathsf{k},\mathsf{k})$ is uniquely determined by $\psi(1)\in\mathsf{k}$. Then we look at the induced maps.

Since $\pi|_{\mathsf{k}}=\mathrm{id}_{\mathsf{k}}$, $\pi:R\to\mathsf{k}$ induces the identity map on k.

Consider the induced map $-\circ\varphi:\mathrm{Hom}_R(R,\mathsf{k})\to\mathrm{Hom}_R(R,\mathsf{k}^2)$ of $\varphi:R^2\to R$. For $\psi\in\mathrm{Hom}_R(R,\mathsf{k})$,

$$\psi\circ\varphi(a+bx+cy,d+ex+fy)=\psi(ax+dy)=0$$

Hence $-\circ\varphi=0$. We have the (augmented) chain complex:

$$0\longrightarrow\mathsf{k}\xrightarrow{\mathrm{id}}\mathsf{k}\xrightarrow{0}\mathsf{k}^2\xrightarrow{0}\mathsf{k}^4\longrightarrow\cdots$$

Taking the cohomology, we obtain the Ext modules:

$$\mathrm{Ext}^n_R(\mathsf{k},\mathsf{k})=\mathsf{k}^{2^n},\quad n\in\mathbb{N}$$ ✓ Very nice! □

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Question 3

Compute all terms and maps in the following long exact sequences:

i) $\cdots \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \longrightarrow \operatorname{Ext}^{i+1}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \longrightarrow \cdots$

ii) $\cdots \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/4) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2) \longrightarrow \operatorname{Ext}^{i+1}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2) \longrightarrow \cdots$

associated with the short exact sequence $0 \longrightarrow \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \longrightarrow 0$ in $\mathbb{Z}$-Mod.

iii) $\cdots \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/4,\mathbb{Z}/4) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \longrightarrow \operatorname{Ext}^{i+1}_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \longrightarrow \cdots$

associated with the short exact sequence $0 \longrightarrow \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \longrightarrow 0$ in $\mathbb{Z}/8$-Mod.

iv) $\cdots \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/4,\mathbb{Z}/2^b) \longrightarrow \operatorname{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \longrightarrow \operatorname{Ext}^{i+1}_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \longrightarrow \cdots$

associated with the short exact sequence $0 \longrightarrow \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \longrightarrow 0$ in $\mathbb{Z}/2^a$-Mod where $a > b \geqslant 2$.

Proof. i) First we write down a free resolution for $\mathbb{Z}/2$. By Horseshoe Lemma, we obtain the split short exact sequence of resolutions.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to \mathbb{Z}/2 & \to 0 \\
 & & & & \downarrow f & & \\
 & & & & \mathbb{Z}/4 & & \\
 & & & & \downarrow g & & \\
0 \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to \mathbb{Z}/2 & \to 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to & \mathbb{Z}/2 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow f & \\
0 \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}/4 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow g & \\
0 \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to & \mathbb{Z}/2 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

We apply the dual functor $(-)^\vee := \operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to the diagram. We have $(\mathbb{Z}^n)^\vee = \mathbb{Z}^n$ and $(\mathbb{Z}/n)^\vee = 0$. Therefore we have a short exact sequence of the (row) complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & \mathbb{Z}^2 & \xrightarrow{(\cdot 2,\cdot 2)} & \mathbb{Z}^2 & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}
$$

For simplicity we let $f : \mathbb{Z} \to \mathbb{Z}$ be the multiplication by 2. The Snake Lemma gives a long exact sequence

$$
\begin{array}{ccccccc}
0 \longrightarrow & \ker f & \longrightarrow & \ker(f,f) & \longrightarrow & \ker f & \\
 & \operatorname{coker} f & \longrightarrow & \operatorname{coker}(f,f) & \longrightarrow & \operatorname{coker} f & \longrightarrow 0
\end{array}
$$

which is the long exact sequence of the Ext groups:

$$
\begin{array}{lccccc}
n=0 & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
n=1 & \mathbb{Z}/2 & \xrightarrow{\binom{1}{0}} & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xrightarrow{(0\ 1)} & \mathbb{Z}/2 \\
n=2 & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

ii) We write down the an injective resolution for $\mathbb{Z}/2$.

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{1\mapsto 1/2} \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

By Horseshoe Lemma, we obtain the split short exact sequence of resolutions.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{1\mapsto 1/2} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & f\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Q}/\mathbb{Z}\oplus\mathbb{Q}/\mathbb{Z} & \xrightarrow{(\cdot 2,\cdot 2)} & \mathbb{Q}/\mathbb{Z}\oplus\mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & g\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{1\mapsto 1/2} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

To get the Ext groups we apply the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},-)$. But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},-)$ is naturally isomorphic to the identity functor on $\mathbb{Z}$-Mod. So we are left with the same diagram. In particular, the row complexes are exact. Therefore the Ext groups are zero for all $n \in \mathbb{N}$.

iv) We write down a projective resolution for $\mathbb{Z}/2$. this map isn't injective, so this isn't a resolution.

$$0 \longrightarrow \mathbb{Z}/2^a \xrightarrow{\cdot 2} \mathbb{Z}/2^a \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

Apply the Horseshoe Lemma:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2^a & \xrightarrow{\cdot 2} & \mathbb{Z}/2^a & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & (\mathbb{Z}/2^a)^2 & \xrightarrow{(\cdot 2,\cdot 2)} & (\mathbb{Z}/2^a)^2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & \mathbb{Z}/2^a & \xrightarrow{\cdot 2} & \mathbb{Z}/2^a & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Apply the functor $\mathrm{Hom}_{\mathbb{Z}/2^a}(-,\mathbb{Z}/2^b)$. Since $a \geqslant b \geqslant 2$, we have $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \cong \mathbb{Z}/2$, $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/4,\mathbb{Z}/2^b) \cong \mathbb{Z}/4$, $\mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2^a,\mathbb{Z}/2^b) \cong \mathbb{Z}/2^b$, and $\mathrm{Hom}_{\mathbb{Z}/2^a}((\mathbb{Z}/2^a)^2,\mathbb{Z}/2^b) \cong (\mathbb{Z}/2^b)^2$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2^b & \xrightarrow{\cdot 2} & \mathbb{Z}/2^b & \longrightarrow & 0 \\
 & & g\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & (\mathbb{Z}/2^b)^2 & \xrightarrow{(\cdot 2,\cdot 2)} & \mathbb{Z}/2^b & \longrightarrow & 0 \\
 & & f\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2^b & \xrightarrow{\cdot 2} & \mathbb{Z}/2^b & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $f : \mathbb{Z}/2^b \to \mathbb{Z}/2^b$ be the multiplication by 2. We have $\ker f = \mathbb{Z}/2^{b-1}$ and $\operatorname{coker} f = \mathbb{Z}/2^{b-1}$. The Snake Lemma gives a long sequence

$$0 \longrightarrow \ker f \longrightarrow \ker(f,f) \longrightarrow \ker f \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker}(f,f) \longrightarrow \operatorname{coker} f \longrightarrow 0$$

This is exactly the long exact sequence of the Ext modules:

$$\begin{array}{lccccc}
n=0 & \mathbb{Z}/2^{b-1} & \xrightarrow{\binom{1}{0}} & \mathbb{Z}/2^{b-1}\oplus\mathbb{Z}/2^{b-1} & \xrightarrow{(0\ 1)} & \mathbb{Z}/2^{b-1} \\
n=1 & \mathbb{Z}/2^{b-1} & \xrightarrow{\binom{1}{0}} & \mathbb{Z}/2^{b-1}\oplus\mathbb{Z}/2^{b-1} & \xrightarrow{(0\ 1)} & \mathbb{Z}/2^{b-1} \\
n=2 & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

It remains to compute the connecting map $\delta : \mathbb{Z}/2^{b-1} \to \mathbb{Z}/2^{b-1}$. But the exactness at $\mathrm{Ext}^0_{\mathbb{Z}/2^a}(\mathbb{Z}/2, \mathbb{Z}/2^b) \cong \mathbb{Z}/2^{b-1}$ forces $\delta = 0$.

iii) Take $a = 3$ and $b = 2$ in (iv). We obtain that

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$$\begin{array}{lccccc}
n=0 & \mathbb{Z}/2 & \xrightarrow{\binom{1}{0}} & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \xrightarrow{(0\ 1)} & \mathbb{Z}/2 \\
n=1 & \mathbb{Z}/2 & \xrightarrow{\binom{1}{0}} & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \xrightarrow{(0\ 1)} & \mathbb{Z}/2 \\
n=2 & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

□

Question 4

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Is $\prod_I : R\text{-Mod} \to R\text{-Mod}$ left exact or right exact? What is the derived functor?

Proof. First we have to make sense that $\prod_I$ is a functor. It is clear that *the source of* $\prod_I$ *is not* R-Mod. We make the following claim:

Is that clear? Isn't it M \mapsto \Pi_I(M)?
i.e. the I-fold product of M with itself?

Let I be a discrete category such that $\mathrm{Obj}(\mathsf{I}) = I$. Let $(R\text{-Mod})^I$ be the functor category, whose objects are functors $\mathsf{I} \to R\text{-Mod}$, and whose morphisms are natural transformations.

- $(R\text{-Mod})^I$ is an Abelian category.

  The objects of $(R\text{-Mod})^I$ are indexed families of R-modules $\{A_i\}_{i\in I}$. The morphisms of $(R\text{-Mod})^I$ are R-module homomorphisms of the indexed famillies $\{A_i \to B_i\}_{i\in I}$. $(R\text{-Mod})^I$ has a zero object $\{0_i\}_{i\in I}$. The biproduct in $(R\text{-Mod})^I$ is the direct sum of families of R-module $\{A_i \oplus B_i : i \in I\}$. Let $\{f_i\}_{i\in I}$ be a morphism in $(R\text{-Mod})^I$. It is easy to see that $\{\ker f_i\}_{i\in I}$ can be exihibited as the kernel of $\{f_i\}_{i\in I}$, and $\{\operatorname{coker} f_i\}_{i\in I}$ can be exihibited as the cokernel of $\{f_i\}_{i\in I}$. The kernels and the kernels of cokernels and the cokernels and the cokernels of kernels. We then conclude that $(R\text{-Mod})^I$ is an Abelian category.

  In addition, we note that the short exact sequences in $(R\text{-Mod})^I$ are of the form $\{0 \to A_i \to B_i \to C_i \to 0\}_{i\in I}$, where $0 \to A_i \to B_i \to C_i \to 0$ is a short exact sequence of R-modules for each $i \in I$.

- **$\prod_I$ is a functor from $(R\text{-Mod})^I$ to R-Mod.**

  For $\{M_i\}_{i\in I} \in \mathrm{Obj}((R\text{-Mod})^I)$, $\prod_I$ maps $\{M_i\}_{i\in I}$ to the product R-module $\prod_{i\in I} R_i$. In fact, $\prod_I$ maps an object in $(R\text{-Mod})^I$ to its projective limit. By the universal property it is easy to check the functoriality.

We claim that $\prod_I$ is an exact functor. The proof of left exactness is relatively easy.

Fix M to be an R-module. Since $\mathrm{Hom}(M, -)$ is the right adjoint to $- \otimes_R M$, it preserves projective limits. In particular,

$$\mathrm{Hom}_R\left(M, \prod_{i\in I} A_i\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M, A_i)$$

Let $D\colon R\text{-Mod} \to (R\text{-Mod})^I$ be the diagonal functor. That is $D(A) = \{A_i\}_{i\in I}$, where $A_i = A$ for all $i \in I$. Then

$$\mathrm{Hom}_{(R\text{-Mod})^I}(D(M), \{A_i\}_{i\in I}) = \prod_{i\in I} \mathrm{Hom}_R(M, A_i) \cong \mathrm{Hom}_R\left(M, \prod_I \{A_i\}_{i\in I}\right)$$

Hence $\prod_I$ is the right adjoint to D. Therefore $\prod_I$ is left exact.

Since R-Mod has enough injectives, so is $(R\text{-Mod})^I$. Then the left exactness of $\prod_I$ implies that it has right derived functors. For $\{A_i\}_{i\in I}$ in $(R\text{-Mod})^I$, we take an injective resolution $\{M_i^\bullet\}_{i\in I}$. Then the n-th right derived functor of $\{A_i\}_{i\in I}$ is the cohomology

$$\mathrm{R}^n \prod_I \{A_i\}_{i\in I} := H^n\left(\prod_{i\in I} M_i^\bullet\right)$$

But cohomology commutes with products (why?) in R-Mod. So we have

$$\mathrm{R}^n \prod_I \{A_i\}_{i\in I} := H^n\left(\prod_{i\in I} M_i^\bullet\right) = \prod_{i\in I} H^n(M_i^\bullet) = 0, \quad n \geqslant 1$$

as $M_i^\bullet$ is exact at M_i^n for $n \geqslant 1$. Hence we have

$$\mathrm{R}^n \prod_I = \begin{cases} \prod_I, & n = 0 \\ 0, & n \geqslant 1 \end{cases}$$

Finally, the right exactness of $\prod_I$ can be deduced from that $\mathrm{R}^1 \prod_I = 0$. To see this, consider a short exact sequence in $(R\text{-Mod})^I$:

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0, \qquad i \in I$$

It induces a long exact sequence in R-Mod:

$$0 \longrightarrow \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i \longrightarrow \prod_{i\in I} C_i \longrightarrow \mathrm{R}^1 \prod_{i\in I} A_i \longrightarrow \mathrm{R}^1 \prod_{i\in I} B_i \longrightarrow \mathrm{R}^1 \prod_{i\in I} C_i \longrightarrow \cdots$$

Since $\mathrm{R}^1 \prod_{i\in I} A_i = 0$, it breaks into a short exact sequence

$$0 \longrightarrow \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i \longrightarrow \prod_{i\in I} C_i \longrightarrow 0$$

Hence $\prod_I$ is an exact functor. □

Section C: Optional

Question 5

Let k be a field, $R = k[x, y]$, $M := R/\langle x, y\rangle^2 \cong k \oplus kx \oplus ky$ as a k-module.

Consider the following R-Mod short exact sequences and compute the associated Tor long exact sequences.

$$0 \longrightarrow k \oplus k \longrightarrow M \longrightarrow k \longrightarrow 0$$

i) Long exact sequence from $M \otimes_R -$.

$$0 \longrightarrow k \longrightarrow \mathrm{Hom}_k(M, k) \longrightarrow k \oplus k \longrightarrow 0$$

ii) Long exact sequence from $M \otimes_R -$.

iii) Long exact sequence from $k \otimes_R -$.

$$0 \longrightarrow k^{\oplus 3} \longrightarrow \frac{M \oplus M}{\langle (y, -x) \rangle} \longrightarrow k \oplus k \longrightarrow 0$$

iv) Long exact sequence from $k \otimes_R -$.