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Problem Sheet 2

Quantum Field Theory

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Question 1. Manipulating the Klein-Gordan Lagrangian density.

All of these manipulations can be done purely in the continuum. Let

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\omega^2\phi^2$$

be the Lagrangian Density. Consider a spatial region V_Σ bounded by the surface Σ and consider the evolution of ϕ from $\phi_0(\boldsymbol{x})$ at $t = 0$ to $\phi_1(\boldsymbol{x})$ at $t = T$ so that

$$L = \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\mathcal{L}, \qquad \text{and} \qquad S = \int_0^T L\,\mathrm{d}t$$

a) By considering the variation $\phi(x) = \phi_{\mathrm{cl}}(x) + \delta\phi(x)$, with $\delta\phi(\boldsymbol{x}, 0) = \delta\phi(\boldsymbol{x}, T) = 0$ at the endpoints, show that

$$\delta S = \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\delta\phi\left(-(\partial_0)^2 + \boldsymbol{\nabla}^2 - \omega^2\right)\phi_{\mathrm{cl}} - \int_0^T \mathrm{d}t \int_\Sigma \delta\phi\widehat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_{\mathrm{cl}}\,\mathrm{d}A$$

where $\widehat{\boldsymbol{n}}$ is the unit normal vector to the surface Σ and $\mathrm{d}A$ is the element of area.

b) To deduce the KG equation we need the boundary term to vanish. Suppose Σ is the surface of a sphere of radius R; assuming that $|\delta\phi| <$ constant, and that $\phi_{\mathrm{cl}} \sim R^{-\eta}$ at large R how large does η have to be in order for the boundary term to vanish as $R \to \infty$? In practice for a massive scalar with a delta function source at the origin $\phi_{\mathrm{cl}} \sim R^{-1}\exp(-mR)$ so there is no problem here. When would there be a potential problem?

c) Define the canonical momentum field by

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}$$

Find an expression for the Hamiltonian Density which is defined by

$$\mathcal{H}(\pi, \phi) = \pi\partial_0\phi - \mathcal{L}$$

(Beware, $\mathcal{H}$ can contain $\boldsymbol{\nabla}\phi$ but not $\partial_0\phi$.)

d) What are the Hamiltonian equations of motion in this case? Show that they lead to the KG equation for ϕ. Take care: the Hamiltonian equation of motion are just that — they use the Hamiltonian. *not* the Hamiltonian density!

Proof. In convention, the greek letters $\mu, \nu, \ldots$ ranges from 0 to 3, and the latin letters $a, b, \ldots$ ranges from 1 to 3. The signature of the Minkowski spacetime is $(+---)$.

a) The variation of the Lagrangian density is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = -\omega^2\phi_{\mathrm{cl}}\delta\phi + \partial^\mu\phi_{\mathrm{cl}}\partial_\mu(\delta\phi)$$

The variation of the action is given by

$$\begin{aligned}
\delta S &= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\delta\mathcal{L} = \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\left(-\omega^2\phi_{\mathrm{cl}}\delta\phi + \partial^\mu\phi_{\mathrm{cl}}\partial_\mu(\delta\phi)\right) \\
&= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\left(-\omega^2\phi_{\mathrm{cl}}\delta\phi + \partial_0\phi_{\mathrm{cl}}\partial_0(\delta\phi) - \boldsymbol{\nabla}\phi_{\mathrm{cl}}\cdot\boldsymbol{\nabla}\delta\phi\right) \\
&= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\left(-\omega^2\phi_{\mathrm{cl}}\delta\phi - \partial_\mu\partial^\mu\phi_{\mathrm{cl}}\delta\phi\right) - \int_0^T \mathrm{d}t \int_\Sigma \delta\phi\boldsymbol{\nabla}\phi_{\mathrm{cl}}\cdot\widehat{\boldsymbol{n}}\,\mathrm{d}A + \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\partial_0\phi\delta\phi\Big|_{\delta\phi(\boldsymbol{x},0)}^{\delta\phi(\boldsymbol{x},T)} \\
&= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\delta\phi\left(-\omega^2 - \partial_0^2 + \nabla^2\right)\phi_{\mathrm{cl}} - \int_0^T \mathrm{d}t \int_\Sigma \delta\phi\boldsymbol{\nabla}\phi_{\mathrm{cl}}\cdot\widehat{\boldsymbol{n}}\,\mathrm{d}A
\end{aligned}$$

b) Suppose that $\delta\phi$ is bounded by M. We have

$$\left|\int_0^T \mathrm{d}t \int_\Sigma \delta\phi \boldsymbol{\nabla}\phi_{\mathrm{cl}} \cdot \widehat{\boldsymbol{n}}\, \mathrm{d}A\right| \leqslant 4\pi R^2 TM \sup_{\boldsymbol{x}\in\Sigma, t\in[0,T]} \|\boldsymbol{\nabla}\phi_{\mathrm{cl}}(\boldsymbol{x},t)\|$$

If $\phi_{\mathrm{cl}} \sim R^{-\eta}$, then $\|\boldsymbol{\nabla}\phi_{\mathrm{cl}}\| \sim R^{-(\eta+1)}$. The boundary integral

$$\left|\int_0^T \mathrm{d}t \int_\Sigma \delta\phi \boldsymbol{\nabla}\phi_{\mathrm{cl}} \cdot \widehat{\boldsymbol{n}}\, \mathrm{d}A\right| \sim R^{-(\eta-1)}$$

If the boundary term needs to vanish as $R \to \infty$, we must have $\eta > 1$.

c) The canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial_0 \varphi$$

Therefore the Hamiltonian density

$$\mathcal{H}(\pi, \phi) = \pi\partial_0\phi - \mathcal{L} = \pi^2 - \frac{1}{2}\left(\pi^2 - \partial_a\phi\partial^a\phi - \omega^2\phi^2\right) = \frac{1}{2}(\pi^2 + \|\boldsymbol{\nabla}\phi\|^2 + \omega^2\phi^2) \qquad (*)$$

d) The Hamiltonian is given by

$$H[\pi,\phi] = \int_{\mathbb{R}^3} \mathcal{H}\, \mathrm{d}^3\boldsymbol{x} = \int_{\mathbb{R}^3} \frac{1}{2}(\pi^2 + \nabla^2\phi + \omega^2\phi)\, \mathrm{d}^3\boldsymbol{x}$$

We derive the Hamiltonian canonical equations in the general case.

From the Lagrangian formalism, by demanding $\delta S = 0$ we obtain the Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

The variation of the Hamiltonian is given by

$$\begin{aligned}
\delta H &= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, (\partial_0\phi\delta\pi + \pi\delta(\partial_0\phi) - \delta\mathcal{L}) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \left((\partial_0\phi\delta\pi + \pi\delta(\partial_0\phi)) - \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \pi\delta(\partial_0\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\partial_a(\delta\phi)\right)\right) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \left(\partial_0\phi\delta\pi + \left(-\frac{\partial\mathcal{L}}{\partial\phi} + \frac{\partial}{\partial x^a}\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right)\delta\phi\right) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, (\partial_0\phi\delta\pi - \partial_0\pi\delta\phi) \qquad \text{(substituting the Euler-Lagrange equation)}
\end{aligned}$$

But we also have

$$\begin{aligned}
\delta H[\pi,\phi] &= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \left(\frac{\partial\mathcal{H}}{\partial\pi}\delta\pi + \frac{\partial\mathcal{H}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{H}}{\partial(\partial_a\phi)}\partial_a(\delta\phi)\right) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \left(\frac{\partial\mathcal{H}}{\partial\pi}\delta\pi + \left(\frac{\partial\mathcal{H}}{\partial\phi} - \frac{\partial}{\partial x^a}\frac{\partial\mathcal{H}}{\partial(\partial_a\phi)}\right)\delta\phi\right)
\end{aligned}$$

By comparing the two expressions we obtain the canonical equations:

$$\partial_0\phi = \frac{\partial\mathcal{H}}{\partial\pi}, \qquad \partial_0\pi = -\frac{\partial\mathcal{H}}{\partial\phi} + \frac{\partial}{\partial x^a}\frac{\partial\mathcal{H}}{\partial(\partial_a\phi)}$$

We substitute $(*)$ into the second equation above:

$$\partial_0\pi = -\omega^2\phi + \partial^a\partial_a\phi$$

Since $\pi = \partial_0 \phi$, we recover the Klein-Gordan equation:

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\nabla^2 - \omega^2\right) \phi$$

□

Question 2. Manipulating the quantum scalar field.

Given that

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^\dagger_{-\boldsymbol{p}} + a_{\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{1}$$

$$\pi(\boldsymbol{x}) = i \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^\dagger_{-\boldsymbol{p}} - a_{\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{2}$$

$$H = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{3}$$

$$\boldsymbol{P} = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \boldsymbol{p} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{4}$$

a) Find $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ in terms of $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ by Fourier transforming (1) and (2). By substituting your results into (3) find H in terms of $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$.

b) Again by substituting your results for $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ from part a) into (4) show that

$$\boldsymbol{P} = -\int \mathrm{d}^3 \boldsymbol{x} \pi(\boldsymbol{x}) \boldsymbol{\nabla} \phi(\boldsymbol{x})$$

c) By explicit calculation find the eigenvalues of $(H, \boldsymbol{P})$ for the two particle state $|\boldsymbol{p}, \boldsymbol{p}'\rangle = a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}'} |0\rangle$.

d) Find $\langle 0|\phi(\boldsymbol{x})\phi(\boldsymbol{y})|\boldsymbol{p}, \boldsymbol{p}'\rangle$.

Proof. a) (Taking the Fourier transform of some operator-valued integral really makes no sense in mathematics...at least in Fourier analysis. Probably we need something like spectral resolutions. From now on I just pretend that *the operators are just scalars and vectors in* $\mathbb{R}^3$.)

Recall that the Fourier transform in $\mathbb{R}^3$ is defined by

$$\mathcal{F}[f](\boldsymbol{p}) := \int_{\mathbb{R}^3} \mathrm{d}^3 \boldsymbol{x}\, f(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}, \qquad f \in \mathrm{L}^1(\mathbb{R}^3)$$

with the inversion given by

$$\mathcal{F}^{-1}[f](\boldsymbol{x}) := \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} f(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}, \qquad f \in \mathrm{L}^1(\mathbb{R}^3)$$

such that $\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{id}$ over $\mathrm{L}^1(\mathbb{R}^3)$. Then we note that

$$\phi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^\dagger_{-\boldsymbol{p}} + a_{\boldsymbol{p}}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = \mathcal{F}^{-1}\left[\frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^\dagger_{-\boldsymbol{p}} + a_{\boldsymbol{p}}\right)\right]$$

$$\pi(\boldsymbol{x}) = \mathrm{i} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^\dagger_{-\boldsymbol{p}} - a_{\boldsymbol{p}}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = \mathcal{F}^{-1}\left[\mathrm{i}\sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^\dagger_{-\boldsymbol{p}} - a_{\boldsymbol{p}}\right)\right]$$

Therefore

$$\overline{\phi}(\boldsymbol{p}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^\dagger_{-\boldsymbol{p}} + a_{\boldsymbol{p}}\right) \qquad \overline{\pi}(\boldsymbol{p}) = \mathrm{i}\sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^\dagger_{-\boldsymbol{p}} - a_{\boldsymbol{p}}\right)$$

Hence we can solve the creation & annihilation operators:

$$a_{\boldsymbol{p}} = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(\boldsymbol{p}) + \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(\boldsymbol{p}) \qquad\qquad a_{\boldsymbol{p}}^\dagger = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(-\boldsymbol{p}) - \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(-\boldsymbol{p})$$

(We used the energy-momentum relation which implies that $E_{\boldsymbol{p}} = E_{-\boldsymbol{p}}$.)

Substituting the results into (3):

$$\begin{aligned}
H &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, E_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, E_{\boldsymbol{p}} \left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(-\boldsymbol{p}) - \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(-\boldsymbol{p})\right)\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(\boldsymbol{p}) + \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(\boldsymbol{p})\right) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}\left(E_{\boldsymbol{p}}^2\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p}) - \mathrm{i}E_{\boldsymbol{p}}\left(\overline{\pi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) - \overline{\phi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p})\right)\right) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}\left(E_{\boldsymbol{p}}^2\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p}) - \mathrm{i}E_{\boldsymbol{p}}\left[\overline{\pi}(-\boldsymbol{p}), \overline{\phi}(\boldsymbol{p})\right]\right)
\end{aligned}$$

In the last step we swap $\boldsymbol{p}$ with $-\boldsymbol{p}$. In order to handle the cross terms, we consider the canonical quantisation given by

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{x}')$$

which implies that

$$[\mathcal{F}[\phi](\boldsymbol{p}), \mathcal{F}'[\pi](\boldsymbol{p}')] = \mathcal{F}\circ\mathcal{F}'[\mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{x}')] = \mathcal{F}[\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}}] = \mathrm{i}(2\pi)^3\delta(\boldsymbol{p} + \boldsymbol{p}')$$

Therefore $[\overline{\pi}(-\boldsymbol{p}), \overline{\phi}(\boldsymbol{p})] = -\mathrm{i}(2\pi)^3\delta(0)$. Substitute this back to the expression of Hamiltonian:

$$H = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}\left(E_{\boldsymbol{p}}^2\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p})\right) + \int_{\mathbb{R}^3} E_{\boldsymbol{p}}\delta(0)\,\mathrm{d}^3\boldsymbol{p}$$

The second term corresponds to an infinitely large zero-point energy, which can be neglected as we are doing physics. Therefore we have

$$H = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}\left(E_{\boldsymbol{p}}^2\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p})\right)$$

We expand the Fourier transforms:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}E_{\boldsymbol{p}}^2\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \frac{1}{2}E_{\boldsymbol{p}}^2 \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}')\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}') \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left(\omega^2 + \boldsymbol{p}^2\right)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x} - \boldsymbol{x}')} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}')\mathcal{F}^{-1}\left[\left(\omega^2 + \boldsymbol{p}^2\right)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}\right] \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}')\left(\omega^2\mathcal{F}^{-1}[\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}] - \nabla^2\mathcal{F}^{-1}[\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}]\right) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}')\left(\omega^2 - \nabla^2\right)\delta(\boldsymbol{x} - \boldsymbol{x}') \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}'\, \phi(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}')\left(\omega^2 - \nabla^2\right)\phi(\boldsymbol{x}') \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \frac{1}{2}\phi(\boldsymbol{x})\left(\omega^2 - \nabla^2\right)\phi(\boldsymbol{x}) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \frac{1}{2}\left(\omega^2\phi(\boldsymbol{x}) + \|\boldsymbol{\nabla}\phi(\boldsymbol{x})\|^2\right)
\end{aligned}$$

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\pi(\boldsymbol{x}')\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}$$
$$= \frac{1}{2}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\pi(\boldsymbol{x}')\mathcal{F}^{-1}\left[\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}\right]$$
$$= \frac{1}{2}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\phi(\boldsymbol{x})\phi(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$= \int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\,\frac{1}{2}\pi(\boldsymbol{x})^2$$

In summary, the Hamiltonian is given by

$$H = \int_{\mathbb{R}^3}\frac{1}{2}\left(\pi(\boldsymbol{x})^2 + \omega^2\phi(\boldsymbol{x})^2 + \left\|\boldsymbol{\nabla}\phi(\boldsymbol{x})\right\|^2\right)\mathrm{d}^3\boldsymbol{x}$$

b) Similar to the calculation for the Hamiltonian, we have

$$\boldsymbol{P} = \int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}$$
$$= \int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(-\boldsymbol{p}) - \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(-\boldsymbol{p})\right)\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(\boldsymbol{p}) + \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(\boldsymbol{p})\right)$$
$$= \int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\boldsymbol{p}\left(E_{\boldsymbol{p}}\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \frac{1}{E_{\boldsymbol{p}}}\overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p}) - \mathrm{i}\left(\overline{\pi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) - \overline{\phi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p})\right)\right)$$
$$= \int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\boldsymbol{p}\left(E_{\boldsymbol{p}}\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) + \frac{1}{E_{\boldsymbol{p}}}\overline{\pi}(-\boldsymbol{p})\overline{\pi}(\boldsymbol{p}) - \mathrm{i}\left\{\overline{\pi}(-\boldsymbol{p}),\overline{\phi}(\boldsymbol{p})\right\}\right)$$

In the last step, we obtain the anti-commutator $\left\{\overline{\pi}(-\boldsymbol{p}),\overline{\phi}(\boldsymbol{p})\right\}$ when we swap $\boldsymbol{p}$ with $-\boldsymbol{p}$. Note that

$$\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\boldsymbol{p}\left(E_{\boldsymbol{p}}\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p})\right) = \int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}(-\boldsymbol{p})\left(E_{-\boldsymbol{p}}\overline{\phi}(\boldsymbol{p})\overline{\phi}(-\boldsymbol{p})\right) = -\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2}\boldsymbol{p}\left(E_{\boldsymbol{p}}\overline{\phi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p})\right)$$

So the first term is in fact zero. Similarly the second term is also zero. We obtain

$$\boldsymbol{P} = -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}\left\{\overline{\pi}(-\boldsymbol{p}),\overline{\phi}(\boldsymbol{p})\right\}$$
$$= -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}\left(-\left[\overline{\pi}(-\boldsymbol{p}),\overline{\phi}(\boldsymbol{p})\right] + 2\overline{\pi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p})\right)$$
$$= -\mathrm{i}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}\overline{\pi}(-\boldsymbol{p})\overline{\phi}(\boldsymbol{p}) - \delta(0)\int_{\mathbb{R}^3}\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{p}$$

But now we can happily claim that the second term is zero (perhaps in the sense of distributions) because there is an obvious symmetry $\boldsymbol{p}\mapsto-\boldsymbol{p}$.

Expand the Fourier tranform:

$$\boldsymbol{P} = -\mathrm{i}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\boldsymbol{p}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\phi(\boldsymbol{x}')\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')}$$
$$= -\mathrm{i}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\phi(\boldsymbol{x}')\mathcal{F}^{-1}[\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}]$$
$$= -\mathrm{i}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\phi(\boldsymbol{x}')(-\mathrm{i}\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}'))$$
$$= -\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}'\,\pi(\boldsymbol{x})\boldsymbol{\nabla}\phi(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$= -\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{x}\,\pi(\boldsymbol{x})\boldsymbol{\nabla}\phi(\boldsymbol{x})$$

c) First we calculate the commutator for the ladder operators.

$$\begin{aligned}[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger] &= \left[\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(\boldsymbol{p}) + \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(\boldsymbol{p}),\ \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\overline{\phi}(-\boldsymbol{p}') - \frac{\mathrm{i}}{\sqrt{2E_{\boldsymbol{p}}}}\overline{\pi}(-\boldsymbol{p}')\right] \\ &= \frac{\mathrm{i}}{2}[\overline{\pi}(\boldsymbol{p}), \overline{\phi}(-\boldsymbol{p}')] - \frac{\mathrm{i}}{2}[\overline{\phi}(\boldsymbol{p}), \overline{\pi}(-\boldsymbol{p}')] \\ &= (2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}')\end{aligned}$$

Next calculate the commutator of H and $a^\dagger$:

$$[H, a_{\boldsymbol{q}}^\dagger] = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, E_{\boldsymbol{p}}[a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger] = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, E_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger (2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{q}) = E_{\boldsymbol{q}} a_{\boldsymbol{q}}^\dagger$$

Hence

$$\begin{aligned}H\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle &= H a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger \left|0\right\rangle \\ &= \left(a_{\boldsymbol{p}}^\dagger H a_{\boldsymbol{p}'}^\dagger + E_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger\right)\left|0\right\rangle \\ &= \left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger H + E_{\boldsymbol{p}'} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger + E_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger\right)\left|0\right\rangle \\ &= (E_{\boldsymbol{p}} + E_{\boldsymbol{p}'})\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle\end{aligned}$$

Similar calculation for $\boldsymbol{P}$ shows that

$$[\boldsymbol{P}, a_{\boldsymbol{q}}^\dagger] = \boldsymbol{q} a_{\boldsymbol{q}}^\dagger$$

Hence

$$\boldsymbol{P}\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle = (\boldsymbol{p} + \boldsymbol{p}')\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle$$

d) Just calculate:

$$\begin{aligned}&\left\langle 0\right|\phi(\boldsymbol{x})\phi(\boldsymbol{y})\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}'}}}\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{x}+\boldsymbol{q}'\cdot\boldsymbol{y})}\left\langle 0\right|(a_{-\boldsymbol{q}}^\dagger + a_{\boldsymbol{q}})(a_{-\boldsymbol{q}'}^\dagger + a_{\boldsymbol{q}'})\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}'}}}\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{x}+\boldsymbol{q}'\cdot\boldsymbol{y})}\sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{p}'}}\left\langle 0\right|(a_{-\boldsymbol{q}}^\dagger + a_{\boldsymbol{q}})(a_{-\boldsymbol{q}'}^\dagger + a_{\boldsymbol{q}'})a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger\left|0\right\rangle \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}'}}}\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{x}+\boldsymbol{q}'\cdot\boldsymbol{y})}\sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{p}'}}\left\langle 0\right|a_{\boldsymbol{q}}a_{\boldsymbol{q}'}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger\left|0\right\rangle\end{aligned}$$

We have

$$\begin{aligned}a_{\boldsymbol{q}}a_{\boldsymbol{q}'}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger &= a_{\boldsymbol{q}}[a_{\boldsymbol{q}'}, a_{\boldsymbol{p}}^\dagger]a_{\boldsymbol{p}'}^\dagger + a_{\boldsymbol{q}}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}'}a_{\boldsymbol{p}'}^\dagger \\ &= a_{\boldsymbol{q}}[a_{\boldsymbol{q}'}, a_{\boldsymbol{p}}^\dagger]a_{\boldsymbol{p}'}^\dagger + a_{\boldsymbol{q}}a_{\boldsymbol{p}}^\dagger[a_{\boldsymbol{q}'}, a_{\boldsymbol{p}'}^\dagger] + a_{\boldsymbol{q}}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger a_{\boldsymbol{q}'} \\ &= [a_{\boldsymbol{q}'}, a_{\boldsymbol{p}}^\dagger][a_{\boldsymbol{q}}, a_{\boldsymbol{p}'}^\dagger] + [a_{\boldsymbol{q}'}, a_{\boldsymbol{p}}^\dagger]a_{\boldsymbol{p}'}^\dagger a_{\boldsymbol{q}} + [a_{\boldsymbol{q}}, a_{\boldsymbol{p}}^\dagger][a_{\boldsymbol{q}'}, a_{\boldsymbol{p}'}^\dagger] + [a_{\boldsymbol{q}'}, a_{\boldsymbol{p}'}^\dagger]a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}} + a_{\boldsymbol{q}}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger a_{\boldsymbol{q}'}\end{aligned}$$

Hence

$$\left\langle 0\right|a_{\boldsymbol{q}}a_{\boldsymbol{q}'}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger\left|0\right\rangle = (2\pi)^6\left(\delta(\boldsymbol{q}' - \boldsymbol{p})\delta(\boldsymbol{q} - \boldsymbol{p}') + \delta(\boldsymbol{q} - \boldsymbol{p})\delta(\boldsymbol{q}' - \boldsymbol{p}')\right)$$

Substitute back to the first equation, we have

$$\left\langle 0\right|\phi(\boldsymbol{x})\phi(\boldsymbol{y})\left|\boldsymbol{p}, \boldsymbol{p}'\right\rangle = \mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{x}+\boldsymbol{p}'\cdot\boldsymbol{y})} + \mathrm{e}^{\mathrm{i}(\boldsymbol{p}'\cdot\boldsymbol{x}+\boldsymbol{p}\cdot\boldsymbol{y})}$$

□

Question 3. Calculating the unequal time commutator.

The Green's functions that we will meet in the lecture course can all be expressed more-or-less explicitly in terms of well-known special functions. For many purposes it is not necessary to do this but you should know how to go about it. In this question we will calculate

$$\Delta(x) = [\phi(x), \phi(0)] = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot x} - e^{ip\cdot x}\right)$$

a) Set $x^\mu = (t, \boldsymbol{x})$ and change to polar coordinates for the momentum integral, choosing the polar axis to be along the direction of $\boldsymbol{x}$. Integrate out the angular variables to show that

$$\Delta(x) = \frac{-i}{8\pi^2|\boldsymbol{x}|} \int_{-\infty}^{\infty} \frac{\rho d\rho}{E(\rho)} \operatorname{Re}\left(e^{-i(E(\rho)t - \rho|\boldsymbol{x}|)} - e^{i(E(\rho)t + \rho|\boldsymbol{x}|)}\right)$$

where $E(\rho) = +\sqrt{\rho^2 + m^2}$.

b) Now change to the *rapidity variables*: $t = s\cosh\tau$, $|\boldsymbol{x}| = s\sinh\tau$ and $\rho = m\sinh\phi$ (these are a very useful way of expressing kinematic quantities in a wide variety of applications). After some manipulations show that

$$\Delta(x) = \frac{-im}{4\pi^2 s} \int_{-\infty}^{\infty} \cosh\phi \left(\operatorname{Re} e^{ims\cosh\phi}\right) d\phi$$

What is notable about this result?

c) The remaining integral is related to the Bessel functions - find out exactly what the relationship is. If you are careful you will discover that we have made a subtle assumption in our manipulations - what is it?

For information on Bessel Functions use the book *Special Functions & Their Applications* by N.N. Lebedev which contains all the information you need. It is available as a pdf on the web.

Proof. a)

b) Using the rapidity variables, we have

$$E(\rho) = \sqrt{\rho^2 + m^2} = m\sqrt{1 + \sinh^2\varphi} = m\cosh\varphi$$

Then

$$\begin{aligned}\Delta(x) =& \frac{-\mathrm{i}}{8\pi^2 s\sinh\tau} \int_{\mathbb{R}} \frac{m^2\sinh\varphi\cosh\varphi \mathrm{d}\varphi}{m\cosh\varphi} \\ & \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}(ms\cosh\varphi\cosh\tau - ms\sinh\varphi\sinh\tau)} - \mathrm{e}^{\mathrm{i}(ms\cosh\varphi\cosh\tau + ms\sinh\varphi\sinh\tau)}\right) \\ =& \frac{-im}{8\pi^2 s\sinh\tau} \int_{\mathbb{R}} \sinh\varphi \operatorname{Re}\left(\mathrm{e}^{-ims\cosh(\varphi-\tau)} - \mathrm{e}^{ims\cosh(\varphi+\tau)}\right) \mathrm{d}\varphi\end{aligned}$$

□

Question 4. The Parity Operator.

Consider the same field operator and Hamiltonian as in Question 2, and define a new operator Q by

$$Q = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}}^\dagger - a_{-\boldsymbol{p}}^\dagger\right)$$

a) Show that

$$[Q, a_{\boldsymbol{q}} + s a_{-\boldsymbol{q}}] = -2(1-s)\left(a_{\boldsymbol{q}} - a_{-\boldsymbol{q}}\right)$$

where s is a constant.

b) Define $C_\lambda = e^{i\lambda Q}$ where λ is real. For operators A such that $[Q, A] = cA$ where c is a constant show that $A(\lambda) = C_\lambda A C_\lambda^\dagger$ satisfies

$$\frac{\partial A(\lambda)}{\partial \lambda} = icA(\lambda)$$

Solve this differential equation to find $A(\lambda)$ in terms of λ and A.

c) Using the result of parts a) and b) (hint: you can choose the value s, good choices help!) show that

$$C_{\frac{\pi}{4}} \phi(\boldsymbol{x}) C_{\frac{\pi}{4}}^\dagger = \phi(-\boldsymbol{x})$$

and deduce that therefore $C_{\frac{\pi}{4}}$ can be identified as the parity operator. What is $C_{\frac{\pi}{4}} H C_{\frac{\pi}{4}}^\dagger$?

Proof. a)

$$\begin{aligned}
[Q, a_{\boldsymbol{q}} + s a_{-\boldsymbol{q}}] &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}}\right)\left[a_{\boldsymbol{p}}^\dagger - a_{-\boldsymbol{p}}^\dagger,\ a_{\boldsymbol{q}} + s a_{-\boldsymbol{q}}\right] \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}}\right)\left([a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{q}}] + [a_{\boldsymbol{p}}^\dagger, s a_{-\boldsymbol{q}}] - [a_{-\boldsymbol{p}}^\dagger, a_{\boldsymbol{q}}] - [a_{-\boldsymbol{p}}^\dagger, s a_{-\boldsymbol{q}}]\right) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{p}\, (a_{\boldsymbol{p}} - a_{-\boldsymbol{p}})(1-s)\left(\delta(\boldsymbol{p}+\boldsymbol{q}) - \delta(\boldsymbol{p}-\boldsymbol{q})\right) \\
&= -2(1-s)\left(a_{\boldsymbol{q}} - a_{-\boldsymbol{q}}\right)
\end{aligned}$$

b) From the definition we note that Q is self-adjoint. $\{C_\lambda \colon \lambda \geqslant 0\}$ defines a one-parameter strongly continuous semi-group of unitary operators.[1]

$$C_\lambda = \mathrm{e}^{\mathrm{i}\lambda Q} \implies \frac{\mathrm{d}C_\lambda}{\mathrm{d}\lambda} = \mathrm{i}Q\,\mathrm{e}^{\mathrm{i}\lambda Q} = \mathrm{i}QC_\lambda$$

The Hermitian adjoint

$$\frac{\mathrm{d}C_\lambda^\dagger}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}C_\lambda}{\mathrm{d}\lambda}\right)^\dagger = -\mathrm{i}QC_\lambda^\dagger$$

Then

$$\frac{\mathrm{d}A(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}C_\lambda}{\mathrm{d}\lambda} A C_\lambda^\dagger + C_\lambda A \frac{\mathrm{d}C_\lambda^\dagger}{\mathrm{d}\lambda} = \mathrm{i}C_\lambda[Q, A]C_\lambda^\dagger = \mathrm{i}cC_\lambda A C_\lambda^\dagger = \mathrm{i}cA(\lambda)$$

With $A(0) = A$, the operator semi-group $A(\lambda)$ is given by

$$A(\lambda) = A\,\mathrm{e}^{\mathrm{i}c\lambda}$$

[1] By Hille-Yosida Theorem, Q must satisfy that Q is closed and densely defined, and there exists $m, \omega > 0$ such that $(\omega, +\infty) \cap \sigma(Q) = \varnothing$ and $\|(\lambda\,\mathrm{id} - Q)^{-n}\| \leqslant M(\lambda - \omega)^{-n}$ for any $\lambda > \omega$ and $n \in \mathbb{Z}_+$.

c) Take $s = 1$ in part (a). We have

$$[Q, a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}] = 0, \qquad [Q, a^\dagger_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}] = -[Q, a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}]^\dagger = 0$$

Take $s = -1$ in part (a). We have

$$[Q, a_{\boldsymbol{p}} - a_{-\boldsymbol{p}}] = -4(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}})$$

Taking the Hermitian adjoint

$$[Q, a^\dagger_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}] = -[Q, a_{\boldsymbol{p}} - a_{-\boldsymbol{p}}]^\dagger = 4(a^\dagger_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})$$

By part (b) we have

$$C_\lambda(a_{\boldsymbol{p}} + a_{-\boldsymbol{p}})C^\dagger_\lambda = a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}, \qquad C_\lambda(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}})C^\dagger_\lambda = \mathrm{e}^{-4\mathrm{i}\lambda}(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}})$$

Therefore we have

$$C_{\pi/4}a_{\boldsymbol{p}}C^\dagger_{\pi/4} = \frac{1}{2}\left((a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}) + \mathrm{e}^{-\mathrm{i}\pi}(a_{\boldsymbol{p}} - a_{-\boldsymbol{p}})\right) = a_{-\boldsymbol{p}}$$

and, taking the Hermitian adjoint

$$C_{\pi/4}a^\dagger_{-\boldsymbol{p}}C^\dagger_{\pi/4} = \left(C_{\pi/4}a_{-\boldsymbol{p}}C^\dagger_{\pi/4}\right)^\dagger = a^\dagger_{\boldsymbol{p}}$$

Put these results into the expression we are looking for:

$$\begin{aligned}
C_{\pi/4}\phi(\boldsymbol{x})C^\dagger_{\pi/4} &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \left(C_{\pi/4}a_{\boldsymbol{p}}C^\dagger_{\pi/4} + C_{\pi/4}a^\dagger_{-\boldsymbol{p}}C^\dagger_{\pi/4}\right) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \left(a_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right) \\
&= \phi(-\boldsymbol{x})
\end{aligned}$$

Hence $C_{\pi/4}$ is the parity operator for the free scalar field $\phi(\boldsymbol{x})$ in the Heisenberg picture.

For the Hamiltonian, we have

$$\begin{aligned}
C_{\pi/4}HC^\dagger_{\pi/4} &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}}C_{\pi/4}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}C^\dagger_{\pi/4} \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} \left(C_{\pi/4}a^\dagger_{\boldsymbol{p}}C^\dagger_{\pi/4}\right)\left(C_{\pi/4}a_{\boldsymbol{p}}C^\dagger_{\pi/4}\right) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}a_{-\boldsymbol{p}} \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} \\
&= H
\end{aligned}$$

Therefore the parity operator does not change the Hamiltonian. This is consistent with the spatial symmetry of reflection in a free scalar field theory. □