3: A^+
4: B^+
5: A^-
6: A
7: –
8: –

A^-

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Problem Sheet 3

C3.4: Algebraic Geometry

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Section A: Introductory

Question 1. Degree of a conic, explicitly

Let $C = \{x_0^2 + x_1^2 + x_2^2 = 0\} \subseteq \mathbb{P}^2_{x_0,x_1,x_2}$. Consider lines

$$L = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\} \subseteq \mathbb{P}^2_{x_0,x_1,x_2}$$

(a) Find the maximum number of intersection points between C and L.

(b) Find also the set of coefficients (a_0, a_1, a_2) so that the intersection $L \cap C$ *does not* consist of the maximal number of intersection points.

Question 2. Basics on dimension

(a) Show that dimension is an invariant of the isomorphism class of a projective variety.

(b) Show that, if $X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.

Section B: Core

Question 3. Another set of equations for the twisted cubic curve

(a) Let $Q = \{x_0x_2 = x_1^2\} \subseteq \mathbb{P}^3$ and $F = \{x_0x_3^2 - 2x_1x_2x_3 + x_2^3 = 0\} \subseteq \mathbb{P}^3$. Prove that $C = Q \cap F \subseteq \mathbb{P}^3$ is the twisted cubic curve, the image of the third Veronese embedding $v_3(\mathbb{P}^1) \subseteq \mathbb{P}^3$.

Hint: multiply the second equation with x_2 and use the first equation to put it in the form of a perfect square.

(b) Does this mean that the ideal $\mathbb{I}(v_3(\mathbb{P}^1))$ can in fact be generated by two elements? (Compare with Sheet 2, Question 3.)

Proof. (a) Let $C = v_3(\mathbb{P}^1) = \{[t^3 : t^2s : ts^2 : s^3] : [t : s] \in \mathbb{P}^1\} = \mathbb{V}(F_0, F_1, F_2) \subseteq \mathbb{P}^3$ be the twisted cubic curve. where F_0, F_1, F_2 are the homogeneous polynomials as given in Question 3 of Sheet 2:

$$F_0(x_0,x_1,x_2,x_3) = x_0x_2 - x_1^2, \quad F_1(x_0,x_1,x_2,x_3) = x_0x_3 - x_1x_2, \quad F_2(x_0,x_1,x_2,x_3) = x_1x_3 - x_2^2$$

For $[z_0 : z_1 : z_2 : z_3] \in Q \cap F$, we have

$$\begin{cases} z_0z_2 = z_1^2 \\ z_0z_3^2 - 2z_1z_2z_3 + z_2^3 = 0 \end{cases}$$

Multiply the second equation with z_2 and substitute the first equation into it:

$$0 = z_1^2z_3^2 - 2z_1z_2^2z_3 + z_2^4 = (z_2^2 - z_1z_3)^2$$

Hence $F_2(z_0,z_1,z_2,z_3) = z_1z_3 - z_2^2 = 0$. Multiply the second equation with z_0 and substitute the first equation into it:

$$0 = z_0^2z_3^2 - 2z_0z_1z_2z_3 + z_1^2z_2^2 = (z_0z_3 - z_1z_2)^2$$

Hence $F_1(z_0,z_1,z_2,z_3) = z_0z_3 - z_1z_2 = 0$. It is clear that $F_0(z_0,z_1,z_2,z_3) = 0$. Hence $[z_0 : z_1 : z_2 : z_3] \in C$.

On the other hand, for $[t^3 : t^2s : ts^2 : s^3] \in C$, we can directly verify that $[t^3 : t^2s : ts^2 : s^3] \in Q \cap F$. Hence we deduce that $C = Q \cap F = v_3(\mathbb{P}^1)$.

(b) By the projective Nullstellensatz and part (a), we have

$$\sqrt{\langle x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2\rangle} = \mathbb{I}(\nu_3(\mathbb{P}^1)) = \sqrt{\langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle}$$

The pull-back of the Veronese map ν_3 gives a ring isomorphism from $\dfrac{\Bbbk[x_0,x_1,x_2,x_3]}{\langle x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2\rangle}$ to $\Bbbk[t]$, which is an integral domain. Hence $\langle x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2\rangle$ is a prime ideal. But $\langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle$ is not radical, since

$$(x_0x_3 - x_1x_2)^2 = x_0(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) - x_2^2(x_0x_2 - x_1^2) \in \langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle$$

and $x_0x_3 - x_1x_2 \notin \langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle$. So

$$\mathbb{I}(\nu_3(\mathbb{P}^1)) = \sqrt{\langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle} \neq \langle x_0x_2 - x_1^2,\ x_0x_3^2 - 2x_1x_2x_3 + x_2^2\rangle$$

A+

We cannot say that $\mathbb{I}(\nu_3(\mathbb{P}^1))$ is generated by two elements from (a). (In Question 3 of Sheet 2 we have shown that any two of $\{x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2\}$ cannot generate $\mathbb{I}(\nu_3(\mathbb{P}^1))$.) □

Great!

Question 4. Projecting a space curve

Let $Q_0 = \{x_0x_3 = x_1^2\} \subseteq \mathbb{P}^3$ and $Q_1 = \{x_1x_3 = x_2^2\} \subseteq \mathbb{P}^3$. Consider the projective variety $C = Q_0 \cap Q_1 \subseteq \mathbb{P}^3$. Show that the formula $\pi : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : x_2]$ defined on a Zariski open subset of C can be extended to a projective morphism

$$\pi : C \to \mathbb{P}^2$$

Show that π maps C isomorphically to the projective variety (plane curve)

$$D = \{x_0x_2^2 = x_1^3\} \subseteq \mathbb{P}^2$$

Hint: to extend π, try to express the ratios $x_0 : x_1 : x_2$ in a different way using the equations of C.

Proof. Let $[z_0 : z_1 : z_2 : z_3] \in C$. If $z_3 \neq 0$, we set $z_3 = 1$. Then $z_0z_3 = z_1^2$ and $z_1z_3 = z_2^2$ imply that $z_0 = z_1^2 = z_2^4$. Hence $[z_0 : z_1 : z_2 : z_3] = [z_2^4 : z_2^2 : z_2 : 1]$. We have, for $z_2 \neq 0$

$$\pi([z_2^4 : z_2^2 : z_2 : 1]) = [z_2^4 : z_2^2 : z_2] = [z_2^3 : z_2 : 1] \in D$$

For $z_2 = 0$, $[0:0:0:1]$ is not in the domain of π. But the above expression suggests that we extend π by defining

$$\pi([0:0:0:1]) = [0:0:1] \in D$$

Next we check that $\pi : C \to D \subseteq \mathbb{P}^2$ is a projective isomorphism. It is clear from the definition that π is injective. For surjectivity, suppose that $[z_0 : z_1 : z_2] \in D$.

- If $z_0 = 0$, then $z_1 = 0$. Hence $[z_0 : z_1 : z_2] = [0:0:1]$. We have $\pi([0:0:0:1]) = [0:0:1] \in \operatorname{im}\pi$;
- If $z_2 = 0$, then $z_1 = 0$. Hence $[z_0 : z_1 : z_2] = [1:0:0]$. We have $\pi([1:0:0:0]) = [1:0:0] \in \operatorname{im}\pi$;
- If $z_0z_2 \neq 0$, then $z_1 \neq 0$. We set $z_2 = 1$. hence $[z_0 : z_1 : z_2] = [z_1^3 : z_1 : 1] = \pi([z_1^4 : z_1^2 : z_1 : 1]) \in \operatorname{im}\pi$.

So π is surjective.

Finally we check that π is a projective morphism. On the open sets $U_0 \cap C, U_1 \cap C, U_2 \cap C$, π is a well-defined

You've shown $\pi : C \to D$ is a well-defined projective morphism, but to show π is an 'iso' it isn't enough to show π is bijective!

e.g. $\mathbb{A}^1_t \longrightarrow \{y^3 = x^2\} \subseteq \mathbb{A}^2_{x,y}$

$t \mapsto (t^3, t^2)$

is a bijective morphism which isn't an isomorphism.

projective morphism by definition. On $U_3 \cap C$, we have

$$\pi([z_0 : z_1 : z_2 : 1]) = [z_2^3 : z_2 : 1]$$

B+

So it is a projective morphism. □

Question 5. Dimension, degree and Hilbert polynomial

(a) Show carefully that if X is a reducible projective variety with equidimensional irreducible components X_i, then $\deg X = \sum_i \deg X_i$.

(b) Compute the degree $\deg \nu_d(\mathbb{P}^1)$ directly from the definition.

(c) Let F be a homogeneous irreducible polynomial of degree d in $k[x_0, \ldots, x_n]$, and let $X = \mathbb{V}(F) \subseteq \mathbb{P}^n$. Find the Hilbert polynomial of X. Deduce that, as expected, $\dim X = n-1$ and $\deg X = d$.

Proof. (a) X is a finite union of its irreducible components because the polynomial ring $R := \mathrm{k}[x_0, ..., x_n]$ is Noetherian. Let $X = \bigcup_{i=1}^{n} X_i$. We use induction on n to prove that $\deg X = \sum_{i=1}^{n} \deg X_i$. Let $\dim X = d$. Let $Y_1 := \bigcup_{i=2}^{n} X_i$. Consider the ideals $I_1 = \mathbb{I}(X_1)$ and $I_2 = \mathbb{I}(X_2)$. We have an short exact sequence of graded R-modules

(For completeness: it's good to give the maps)

$$0 \longrightarrow R/(I_1 \cap I_2) \longrightarrow R/I_1 \oplus R/I_2 \longrightarrow R/(I_1 + I_2) \longrightarrow 0$$

which is just

$$0 \longrightarrow S(X_1 \cap Y_1) \longrightarrow S(X_1) \oplus S(Y_1) \longrightarrow S(X) \longrightarrow 0$$

By the definition of Hilbert functions, we have

$$h_{X_1 \cap Y_1} + h_X = h_{X_1} + h_{Y_1}$$

Since the Hilbert functions are eventually polynomials, the Hilbert polynomials satisfy

$$p_{X_1 \cap Y_1} + p_X = p_{X_1} + p_{Y_1}$$

We assumed that X_1 and Y_1 have pure dimensions, and $\dim X_1 = \dim Y_1 = d$. Then $\dim(X_1 \cap Y_1) < d$. Taking the leading order term ($\deg = d$) in the above equation, we have

How come?

$$\frac{1}{d!} \deg X = \frac{1}{d!} \deg X_1 + \frac{1}{d!} \deg Y_1$$

Hence $\deg X = \deg X_1 + \deg Y_1$. By induction, $\deg X = \sum_{i=1}^{n} \deg X_i$.

OK - I was expecting some justification why $\dim(X_1 \cap Y_1) < d$

(b) Let $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-th Veronese embedding of $\mathbb{P}^1$. Since ν_d is an isomorphism onto its image, we have $\dim \nu_d(\mathbb{P}^1) = 1$. Let L be a linear subvariety of dimension $n-1$. L is a hyperplane with $L = \mathbb{V}\left(\sum_{i=0}^{n} a_i x_i\right)$ where $[a_0 : \cdots : a_d] \in \mathbb{P}^d$. We shall count $\nu_d(\mathbb{P}^1) \cap L$. OK!

In the affine patch U_0, $x_0 \neq 0$. Consider the point $[1 : z_1 : \cdots : z_n] \in \nu_d(\mathbb{P}^1) \cap L$. Since

$$U_0 \cap \nu_d(\mathbb{P}^1) = \{[t^d : t^{d-1}s : \cdots : s^d] : [t : s] \in \mathbb{P}^1 \cap U_0\} = \{[1 : (s/t) : \cdots : (s/t)^d] : s/t \in \mathbb{C}\}$$

Substitute this into the equation of L, we have

$$\sum_{i=0}^{d} a_i (s/t)^i = 0$$

Can you say for which $[\cdots a:\cdots]$ this is true? Remember, you need to show that the set of such points is <u>open</u> in $\check{\mathbb{P}}^d$

For general $[a_0 : \cdots : a_d]$, the equation has d distinct roots $(s_1/t_1), \ldots, (s_d/t_d)$, corresponding to d distinct intersection points of $\nu_d(\mathbb{P}^1) \cap L$ in U_0.

Finally we consider $[z_0 : z_1 : \cdots : z_n] \in \nu_d(\mathbb{P}^1) \cap L$ with $z_0 = 0$. There is a unique such point $[z_0 : z_1 : \cdots : z_n] = [0 : 0 \cdots : 0 : 1]$ on $\nu_d(\mathbb{P}^1)$. Therefore "in general" $\nu_d(\mathbb{P}^1) \cap L = \varnothing$ in $\mathbb{P}^n \setminus U_0$. We conclude that "in general" $|\nu_d(\mathbb{P}^1) \cap L| = d$. Hence $\deg \nu_d(\mathbb{P}^1) = d$. You have the right idea, you just need to be more careful justifying <u>why</u> for generic $L \in \check{\mathbb{P}}^d$, $\#(\nu_d(\mathbb{P}^1) \cap L) = d$.

(c) The graded homogeneous coordinate ring of X is

$$S(X) = \mathsf{k}[x_0, \ldots, x_n] / \langle F(x_0, \ldots, x_n) \rangle$$

The grading of $\mathbb{I}(X)_m$ is $\mathsf{k}[x_0, \ldots, x_n]_{m-d}$ for $m \geqslant d$. Therefore the Hilbert function of X is, for $m \geqslant d$

$$h_X(m) = \dim_{\mathsf{k}} \mathsf{k}[x_0, \ldots, x_n]_m - \dim_{\mathsf{k}} \mathsf{k}[x_0, \ldots, x_n]_{m-d} = \binom{m+n}{m} - \binom{m+n-d}{m-d}$$

Hence the Hilbert polynomial is

$$p_X(m) = \binom{m+n}{m} - \binom{m+n-d}{m-d} = \frac{(m+1)\cdots(m+n)}{n!} - \frac{(m-d+1)\cdots(m-d+n)}{n!} \in \mathbb{Q}[m]$$

The leading order term of $p_X(m)$ is

$$\frac{1}{n!} m^{n-1} \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} (i-d) \right) = \frac{d}{(n-1)!} m^{n-1}$$

We know that

$$\frac{d}{(n-1)!} m^{n-1} = \frac{\deg X}{(\dim X)!} m^{\dim X}$$

Hence $\deg X = d$ and $\dim X = n-1$. ✓ Good A⁻ $\square$

Question 6. Affine and quasi-projective varieties

(a) Find an open affine cover of $\mathbb{A}^2 \setminus \{(0,0)\}$.

(b) Show that $\mathrm{GL}(n,k)$ is an affine variety, i.e. that it is isomorphic as a quasi-projective variety to a Zariski closed subset of an affine space.

(c) Let X be an affine variety and $f \in k[X]$. Show that f vanishes nowhere on X if and only if f is invertible in $k[X]$.

Proof. (a) $\mathbb{A}^2 \setminus \{0\}$ has an open cover

$$\mathbb{A}^2 \setminus \{0\} = U_0 \cap (U_1 \cup U_2) \cap \mathbb{P}^2 = (U_0 \cap U_1 \cap \mathbb{P}^2) \cup (U_0 \cap U_2 \cap \mathbb{P}^2)$$

i.e. $X_1 = D(x) \subseteq \mathbb{A}^2$ and $0 \neq f \in k[X] \Rightarrow D(f) = X - \mathbb{V}(f)$ is affine (from lectures)

We claim that $X_1 := U_0 \cap U_1 \subseteq \mathbb{P}^2$ and $X_2 := U_0 \cap U_2 \subseteq \mathbb{P}^2$ are affine. This is clear, as $X_1 \cong \{(x,y) : x \neq 0\} \subseteq \mathbb{A}^2$ and $X_2 \cong \{(x,y) : y \neq 0\} \subseteq \mathbb{A}^2$. So $\mathbb{A}^2 \setminus \{0\} = X_1 \cup X_2$ is an open affine cover of $\mathbb{A}^2 \setminus \{0\}$. OK

(b) We have

$$\mathrm{GL}_n(\mathsf{k}) = \left\{ M \in \mathbb{A}^{n^2} : \det M \neq 0 \right\} = \mathbb{A}^{n^2} \setminus \mathbb{V}(\det M)$$

where $\det M$ is a polynomial in the variables x_{ij}. By Lemma 10.1, $\mathrm{GL}_n(\mathsf{k})$ is an affine quasi-projective variety.

(c) Let $D_f := X \setminus \mathbb{V}(f)$. By Lemma 10.1, D_f is an affine quasi-projective variety, and the inclusion $\iota : D_f \hookrightarrow X$ has pull-back $\iota^* : \mathsf{k}[X] \to \mathsf{k}[D_f] \cong \mathsf{k}[X]_f$ given by $g \mapsto g/1$. We have

$$\begin{aligned} f \text{ vanishes nowhere on } X &\iff \mathbb{V}(f) \cap X = \{0\} \\ &\iff D_f = X \\ &\iff \iota^* : \mathsf{k}[X] \to \mathsf{k}[D_f] \cong \mathsf{k}[X]_f \text{ is an isomorphism} \\ &\iff 1/f \in \mathsf{k}[X] \\ &\iff f \text{ is invertible in } \mathsf{k}[X] \end{aligned}$$

✓ Nice!

□

Aside: this argument (unlike anything involving the Null'tz) carries over to the case of affine schemes:
if $f \in A$ then $A \to A_f$ corresponds to $D(f) = \mathrm{Spec}(A_f) \overset{\text{open}}{\subseteq} \mathrm{Spec}(A)$,
so $D(f) = \mathrm{Spec}(A) \Leftrightarrow A \xrightarrow{\simeq} A_f \Leftrightarrow f \in A^\times$

$\{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$

A

Section C: Optional

Question 7. The Veronese image

Find the Hilbert polynomial of the Veronese image $v_d(\mathbb{P}^n)$. Find the dimension and degree of this projective variety.

Question 8. Complete intersection of two hypersurfaces

(a) Let $f, g \in k[x_0, \ldots, x_n]$ be two essentially distinct irreducible homogeneous polynomials (i.e. not scalar multiples of each other) of degrees e, f such that the homogeneous ideal $I = \langle f, g \rangle$ is radical. Compute the Hilbert function of $X = \mathbb{V}(I) \subseteq \mathbb{P}^n$. Deduce the dimension and degree of X.

(b) Conclude that the twisted cubic $v_3(\mathbb{P}^1) \subseteq \mathbb{P}^3$ is not a *complete intersection*: its (radical) vanishing ideal cannot be generated by two homogeneous polynomials. Compare with Question .