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Excellent work!

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Problem Sheet 4

C3.4: Algebraic Geometry

3 January, 2022

Section A: Introductory

Question 1

(a) Show that if $\mathrm{char}(k)$ does not divide d, then the hypersurface

$$\mathbb{V}\left(x_0^d + \ldots + x_n^d\right) \subseteq \mathbb{P}^n$$

is nonsingular.

(b) Under the same assumption on k, find the singular locus of the hypersurface

$$\mathbb{V}\left(x_0^d + \ldots + x_{n-1}^d\right) \subseteq \mathbb{P}^n$$

Question 2

Let X be an irreducible affine variety. Show that for any open set $U \subseteq X$ and point $p \in X$, the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ are subrings of the function field $k(X)$.

Hint. You need to explain first how to include $\mathcal{O}_X(U) \subseteq k(X)$ *and* $\mathcal{O}_{X,p} \subseteq k(X)$.

Section B: Core

Question 3

Let $X_1 = \mathbb{V}(xy(x-y)) \subseteq \mathbb{A}^2$ and $X_2 = \mathbb{V}(xy, yz, zx) \subseteq \mathbb{A}^3$.

(a) Decompose these varieties into irreducible components.

(b) By computing the dimension of tangent spaces at various points, show that X_1 and X_2 are not isomorphic.

Proof. (a) We have $X_1 = \mathbb{V}\big(xy(x-y)\big) = \mathbb{V}(x) \cup \mathbb{V}\big(y\big) \cup \mathbb{V}\big(x-y\big)$. Since the varieties are the vanishing loci of linear polynomials. This is an irreducible decomposition. ✓

For $(a,b,c) \in X_2$, we have $ab = bc = ca = 0$. Then two of a, b, c are zero. Then $X_2 = \mathbb{V}\big(xy, yz, zx\big) = \mathbb{V}\big(x, y\big) \cup \mathbb{V}\big(y, z\big) \cup \mathbb{V}(z, x)$. This is clearly an irreducible decomposition. ✓

(b) We note that the dimensional of the tangent space at some point is an intrinsic property of an algebraic variety, since $\dim \mathrm{T}_p X = \dim_k \mathfrak{m}_p / \mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the unique maximal ideal of the ring of germs of regular functions $\mathcal{O}_{X,p}$. In particular, if $f : X_1 \to X_2$ is an isomorphism, we must have $\dim \mathrm{T}_p X_1 = \dim \mathrm{T}_{f(p)} X_2$ for $p \in X_1$. ✓

Yes, though bear in mind $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the cotangent space

Both X_1 and X_2 are the union of three lines which intersect at the origin. For $p \in X_i \setminus \{0\}$, $\dim \mathrm{T}_p X_i = \dim_p X_i = 1$. ✓ We then compute $\dim \mathrm{T}_0 X_i$ for $i \in \{1, 2\}$.

The Jacobian of the generators of the ideal of X_1 at 0 is given by

$$J_0 = \begin{pmatrix} 2xy - y^2 & x^2 - 2xy \end{pmatrix}_{(x,y)=(0,0)} = 0$$

Then $\dim \mathrm{T}_0 X_1 = \dim \ker J_0 = 2$. ✓

i.e. $T_0 X_1$

The Jacobian of the generators of the ideal of X_2 at 0 is given by

$$J_0 = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}_{(x,y,z)=(0,0,0)} = 0$$

Then $\dim \mathrm{T}_0 X_2 = \dim \ker J_0 = 3$.

X_1 has a point with 2-dimensional tangent space, whereas no points in X_2 have 2-dimensional tangent space. Hence these two varieties cannot be isomorphic. ✓ Perfect! □

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Question 4

(a) Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that

$$\{(u, f(u)) : u \in U\} \subseteq U \times Y$$

is a closed subvariety. Define the graph Γ_F of F to be the (Zariski) closure of $\{(u, f(u)) : u \in U\} \subseteq X \times Y$. Show that the projection from the graph $\Gamma_F \to X$ is a birational equivalence.

(b) Define $F : \mathbb{A}^2 \to \mathbb{A}^1$ by $F(x, y) = \frac{y}{x}$. Find the equation defining $\Gamma_F \subseteq \mathbb{A}^3$.

Proof. (a) Consider the map $f \times \mathrm{id}_Y : U \times Y \to Y \times Y$. Since f is a regular map, f is a morphism of quasi-projective varieties and hence it is continuous. Then $f \times \mathrm{id}_Y$ is continuous. We have

Fixing $Y \subseteq \mathbb{P}^m_{y_i}$: f is given by $m+1$ regular fn's f_i, and so $\Gamma_f = \mathbb{V}(y_0 - f_0(x), \dots, y_m - f_m(x)) \subseteq U \times Y$

$$\{(u, f(u)) \colon u \in U\} = (f \times \mathrm{id}_Y)^{-1}(D) \quad ✓$$

where $D := \{(y, y) \colon y \in Y\}$. We claim that D is closed in $Y \times Y$. Let $Y \subseteq \mathbb{P}^n$. We consider the Segre embedding $Y \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^n \subseteq \mathbb{P}^{n^2+2n}$. Then we have

$$D = (Y \times Y) \cap (\mathbb{P}^n \times \mathbb{P}^n) \cap \mathbb{V}(\det M)$$

$\subseteq$) I'm happy with, but why does $\supseteq$) hold?

where $\det M$ is a polynomial in the matrix elements of $\mathbb{P}^{n^2+2n}$. (*We use the fact that* $\det(v^\top v) = 0$.) Hence D is closed in $Y \times Y$. We conclude that $\{(u, f(u)) \colon u \in U\}$ is closed in $U \times Y$.

Let $\gamma_F := \{(u, f(u)) \colon u \in U\}$ be the genuine graph and $\Gamma_F := \overline{\gamma_F}$ be the closed graph. The projection $\pi : \Gamma_F \to X$ is clearly a dominant rational map. We claim that $\mathrm{id}_X \times F \colon X \dashrightarrow X \times Y$ is an inverse to π. $\mathrm{id}_X \times F$ has a representative $(U, \mathrm{id}_X \times f)$. We have $\pi \circ (\mathrm{id}_X \times f) = \mathrm{id}_U$. The composition $(\mathrm{id}_X \times f) \circ \pi$ is defined on $\pi^{-1}(U) = \gamma_F$, on which clearly $(\mathrm{id}_X \times f) \circ \pi = \mathrm{id}_{\gamma_F}$. Since γ_F is dense in Γ_F, we have $(\mathrm{id}_X \times f) \circ \pi = \mathrm{id}_{\Gamma_F}$ as rational maps. Therefore π is a birational equivalence. ✓

(b) We claim that $\Gamma_F = \mathbb{V}(y - xz) \subseteq \mathbb{A}^3$. F is defined on $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$. From the definition we immediately note that

$$\gamma_F = \{(a, b, c) \colon a \neq 0,\ ac - b = 0\} = \mathbb{V}(y - xz) \cap \{x \neq 0\}$$

Then γ_F is open in $\mathbb{V}(y - xz)$. Since $\langle y - xz \rangle$ is prime, $\mathbb{V}(y - xz)$ is irreducible. Hence γ_F is dense in $\mathbb{V}(y - xz)$. We conclude that $\Gamma_F = \mathbb{V}(y - xz)$. ✓ □

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Question 5

Recall that the projective varieties $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic. Find a birational equivalence $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

$U_0 \to \mathbb{A}^2$ isn't the identity, strictly speaking, but I know what you mean

Proof. This is already outlined in the lectures. Let $F : \mathbb{P}^2 \dashrightarrow \mathbb{A}^2$ be the rational map with identity on $U_0 \to \mathbb{A}^2$. Let $\alpha : \mathbb{A}^2 \to \mathbb{A}^1 \times \mathbb{A}^1$ be the canonical isomorphism. Let $i_1 : \mathbb{A}^1 \to \mathbb{P}^1$ be the inclusion map. Then

$$\phi := (i_1 \times i_1) \circ \alpha \circ F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

This isn't unique!

is a rational map. To show that it is a birational equivalence, the only non-trivial part is show that

$$i_1 \times i_1 : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \qquad (z_1, z_2) \mapsto ([1 : z_1], [1 : z_2])$$

Ah, OK, you've chosen $\iota_1(x) = [1:x]$

is a birational equivalence. ← It has inverse $([1:z_1],[1:z_2]) \mapsto (z_1,z_2)$ defined on $U_0 \times U_0$!

Let $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre embedding. Define $\pi : \mathbb{P}^3 \supseteq U_0 \to \mathbb{A}^1 \times \mathbb{A}^1$ by $[1 : z_{01} : z_{10} : z_{11}] \mapsto (z_{10}, z_{01})$. Then $\pi \circ \sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{A}^1 \ominus \mathbb{A}^1$ is a rational map. We can verify that

$\times$?

$$\pi \circ \sigma_{1,1} \circ (i_1 \times i_1)(z_1, z_2) = \pi([1 : z_2 : z_1 : z_1 z_2]) = (z_1, z_2)$$ ✓

So $\pi \circ \sigma_{1,1} \circ (i_1 \times i_1) = \mathrm{id}_{\mathbb{A}^1 \times \mathbb{A}^1}$. For the inverse direction, note that $(i_1 \times i_1) \circ \pi \circ \sigma_{1,1}$ is defined on $(U_0^1 \times U_0^1) \cap \sigma_{1,1}^{-1}(U_0^3)$, where $U_0^n = \{x_0 \neq 0\} \subseteq \mathbb{P}^n$. We have

$$(i_1 \times i_1) \circ \pi \circ \sigma_{1,1}([z_0 : z_1], [w_0 : w_1]) = (i_1 \times i_1) \circ \pi \left(\left[1 : \frac{w_1}{w_0} : \frac{z_1}{z_0} : \frac{z_1 w_1}{z_0 w_0} \right] \right) = (i_1 \times i_1) \left(\frac{z_1}{z_0}, \frac{w_1}{w_0} \right) = ([z_0 : z_1], [w_0 : w_1])$$

Hence $(i_1 \times i_1) \circ \pi \circ \sigma_{1,1} = \mathrm{id}_{\mathbb{P}^1 \times \mathbb{P}^1}$. Then $\pi \circ \sigma_{1,1}$ is an inverse to $i_1 \times i_1$. This concludes the proof. ✓ □

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Good, though you could have done this in one line!

Question 6

In these examples, attempt a resolution of singularities of the given variety by successively blowing up singular points.

(a) Find a resolution of singularities of the affine curve $C_1 = \mathbb{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$. Deduce that C_1 is rational, in other words birationally equivalent to $\mathbb{P}^1$.

Hint. Try blowing up the curve at the origin, and consider the map that projects to the exceptional divisor.

(b) Desingularise the affine curve $C_2 = \mathbb{V}(y^2 - x^4 - x^5) \subseteq \mathbb{A}^2$. Draw a picture of the series of blow-ups.

(c) Desingularise the affine surface $S_2 = \mathbb{V}(-xy + z^2) \subseteq \mathbb{A}^3$.

Proof. Let $\pi : \mathrm{B}_0\,\mathbb{A}^2 \to \mathbb{A}^2$ be the projection map. Then

$$\mathrm{B}_0\,\mathbb{A}^2 = \{((x, y), [a : b]) : xb - ya = 0\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_0) \cup (\mathbb{A}^2 \times U_1)$$

(a) Let $f(x, y) = y^2 - x^2 - x^3$. Note that $\nabla f|_p = 0$ if and only if $p = 0$. Hence $\mathrm{Sing}(C_1) = \{0\}$. ✓

The blow-up of C_1 at the origin:

$$\mathrm{B}_0\,C_1 = (\mathrm{B}_0\,C_1)_a \cup (\mathrm{B}_0\,C_1)_b = (\mathrm{B}_0\,C_1 \cap (\mathbb{A}^2 \times U_0)) \cup (\mathrm{B}_0\,C_1 \cap (\mathbb{A}^2 \times U_1))$$

On the affine chart $\mathbb{A}^2 \times U_0$ we have the local coordinates $x, y, u := b/a$. Then

$$(\mathrm{B}_0\,C_1)_a = \overline{\{(x, y, u) : y = ux,\ y^2 = x^2 + x^3,\ (x, y) \neq (0, 0)\}} = \{(x, y, u) : y = ux,\ u^2 = 1 + x\} = \mathbb{V}(y - ux, u^2 - x - 1)$$ ✓

On the affine chart $\mathbb{A}^2 \times U_1$ we have the local coordinates $x, y, v := a/b$. Then

$$(\mathrm{B}_0\,C_1)_b = \overline{\{(x, y, v) : x = vy,\ y^2 = x^2 + x^3,\ (x, y) \neq (0, 0)\}} = \{(x, y, v) : x = vy,\ 1 = v^2 + v^3 y\} = \mathbb{V}(x - vy, v^2 + v^3 y - 1)$$ ✓

Both $(\mathrm{B}_0\, C_1)_a$ and $(\mathrm{B}_0\, C_1)_b$ are non-singular. So the blow-up $\mathrm{B}_0\, C_1$ is a non-singular quasi-projective variety.

Let $\sigma : \mathrm{B}_0\, C_1 \to E_0 := \pi^{-1}(\{0\})$ be the projection onto the exceptional divisor, which is a rational map. The inverse $\sigma^{-1} : E_0 \to \mathrm{B}_0\, C_1$ is given by $\sigma^{-1} : U_0 \to (\mathrm{B}_0\, C_1)_a,\ u \mapsto (1-u^2, u-u^3, u)$; and $\sigma^{-1} : U_1 \to (\mathrm{B}_0\, C_1)_b$,
$(u^2-1,\ u^3-u,\ u)$
$v \mapsto \left(\dfrac{1-v^2}{v^2}, \dfrac{1-v^2}{v^3}, v\right)$. Hence σ^{-1} is rational. We deduce that $\mathrm{B}_0\, C_1 \simeq \mathbb{P}^1$. Since $\mathrm{B}_0\, C_1 \simeq C_1$, we conclude that C_1 is rational.

$p = (-4/5, 0)$ also satisfies $\nabla f|_p = 0$!

(b) Let $f(x,y) = y^2 - x^4 - x^5$. Note that $\nabla f|_p = 0$ if and only if $p = 0$. Hence $\mathrm{Sing}(C_2) = \{0\}$. We compute the blow-up of C_2 at the origin. On the affine chart $\mathbb{A}^2 \times U_0$ we have the local coordinates $x, y, u := b/a$. Then

$$\begin{aligned}(\mathrm{B}_0\, C_2)_a &= \overline{\{(x,y,u) \colon\ y = ux,\ y^2 = x^4 + x^5,\ (x,y) \neq (0,0)\}} \\ &= \{(x,y,u) \colon\ y = ux,\ u^2 = (1+x)x^2\} = \mathbb{V}\big(y - ux, u^2 - x^2(x+1)\big)\end{aligned}$$

$\cong \mathbb{V}(u^2 - x^2(x+1)) \subseteq \mathbb{A}^2_{x,u}$

We note that $(\mathrm{B}_0\, C_2)_a$ is singular: the Jacobian

$$J = \begin{pmatrix} -u & 1 & -x \\ 3x^2 - 2x & 0 & 2u \end{pmatrix}$$

has rank 1 at $p = (0,0,0) \in \mathbb{A}^2 \times U_0 = \mathbb{A}^3_{x,y,u}$.

On the affine chart $\mathbb{A}^2 \times U_1$ we have the local coordinates $x, y, v := a/b$. Then

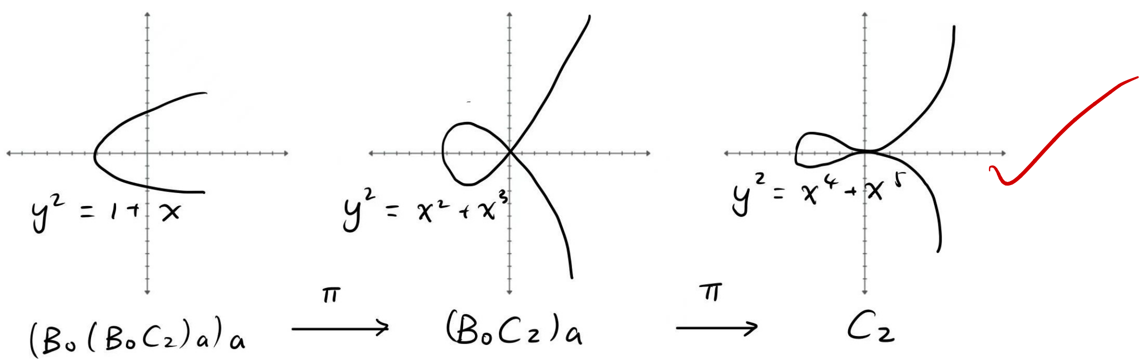
$$\begin{aligned}(\mathrm{B}_0\, C_2)_b &= \overline{\{(x,y,v) \colon\ x = vy,\ y^2 = x^4 + x^5,\ (x,y) \neq (0,0)\}} \\ &= \{(x,y,v) \colon\ x = vy,\ 1 = y^2(v^4 + v^5 y)\} = \mathbb{V}\big(x - vy, y^2 v^4(1+vy) - 1\big)\end{aligned}$$

Note that $(\mathrm{B}_0\, C_2)_b \cap E_0 = \varnothing$. This suggests that we only need to consider the desingularisation of $(\mathrm{B}_0\, C_2)_a$ (*why?*). See below

To desingularise $\mathrm{B}_0\, C_2$, we embed $(\mathrm{B}_0\, C_2)_a$ into $\mathbb{A}^2$ via the the following isomorphism (restricting to $(\mathrm{B}_0\, C_2)_a$)

$$\varphi \colon \mathbb{A}^3 \to \mathbb{A}^2, \qquad (x,y,u) \mapsto (x,u)$$

Then we identify $(\mathrm{B}_0\, C_2)_a$ as $\mathbb{V}\big(y^2 - x^2 - x^3\big) \subseteq \mathbb{A}^2_{x,y}$. We consider the blow-up of $C_2^{(1)} := (\mathrm{B}_0\, C_2)_a$ at the origin. The rest of the calculation is the same as in part (a). We obtain a non-singular quasi-projective curve $\mathrm{B}_0\, C_2^{(1)} = (\mathrm{B}_0\, C_2^{(1)})_a \cup (\mathrm{B}_0\, C_2^{(1)})_b$.



(c) Let $f(x,y,z) = -xy + z^2$. Then $\nabla f|_p = 0$ if and only if $p = (0,0,0)$. We blow up S_2 at the origin.

$$\begin{aligned}\mathrm{B}_0\, S_2 &= \overline{\{((x,y,z),[a:b:c]) \in \mathbb{A}^3 \times \mathbb{P}^2 \colon (x,y,z) = \lambda(a,b,c),\ \lambda \in k \setminus \{0\},\ xy = z^2\}} \\ &= \{((x,y,z),[a:b:c]) \in \mathbb{A}^3 \times \mathbb{P}^2 \colon (x,y,z) = \lambda(a,b,c),\ \lambda \in k,\ xy = z^2,\ ab = c^2\}\end{aligned}$$

The intersection with the exceptional divisor is a projective defined by the same equation, i.e. $E = \mathbb{V}(-ab + c^2) \subseteq$

To answer your question:

Reason 1: We only care about the birational class of the variety when desingularising

Reason 2: Let $\pi : Bl_0 \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be the projection. Then

$$\pi^{-1}(C_2 \setminus 0) = \pi^{-1}(C_2 \setminus 0) \cap \{a \neq 0\}$$

so

$$Bl_0 C_2 = \text{closure}(\pi^{-1}(C_2 \setminus 0) \cap \{a \neq 0\})$$

$$= (\pi^{-1}(C_2 \setminus 0) \cap \{a \neq 0\}) \cup \underbrace{((0,0),[1:0])}_{\text{exceptional divisor of } Bl_0 C_2}$$

$$= \{((x,y),[1:u]) : y = ux,\ u^2 = x^2 + x^3\}$$

$$\cong C_1$$

$\mathbb{P}^2$.

On the affine chart $\mathbb{A}^3 \times U_0$ with local coordinates $x, y, z, \alpha := b/a, \beta := c/a$,

$$(\mathrm{B}_0\, S_2)_a = \mathbb{V}\big(y - \alpha x,\ z - \beta x,\ \beta^2 - \alpha\big)$$

This is a non-singular affine surface. By symmetry this is also true on the affine chart $\mathbb{A}^3 \times U_1$.

On the affine chart $\mathbb{A}^3 \times U_2$ with local coordinates $x, y, z, \alpha := a/c, \beta := b/c$,

$$(\mathrm{B}_0\, S_2)_c = \mathbb{V}\big(x - \alpha z,\ y - \beta z,\ \alpha\beta - 1\big)$$

This is also a non-singular affine surface. Hence $\mathrm{B}_0\, S_2$ is a non-singular quasi-projective variety. ✓ □

Great!

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Section C: Optional

Question 7

(a) Find an inductive procedure to desingularise the affine surface

$$S_n = \mathbb{V}\big(-xy + z^n\big) \subseteq \mathbb{A}^3$$

for a positive integer $n > 2$.

(b) By successively blowing up singular points, desingularise the surface

$$T_1 = \big\{x^2 + y^3 + z^3 = 0\big\} \subseteq \mathbb{A}^3$$

(c) Investigate whether it is possible to desingularise the affine surface

$$T_2 = \big\{x^2 - y^2 z = 0\big\} \subseteq \mathbb{A}^3$$

by a procedure with begins by blowing up the origin $(0,0,0) \in T_2$. How is this example different from all the earlier examples of singular affine surfaces?

Question 8

Assume that the base field $k = \mathbb{C}$. Show that the projective surface

$$X = \big\{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\big\} \subseteq \mathbb{P}^3$$

contains at least 27 lines.

Hint. Move two of the summands to the other side, and factorise both sides of the resulting equality.

Comment. It is possible to show that there exists a union of six disjoint lines

$$E = L_1 \cup \ldots \cup L_6 \subseteq X$$

and a birational morphism $\pi : X \to \mathbb{P}^2$ such that π is the blowup of six points in $\mathbb{P}^2$ and $E \subseteq \mathbb{P}^2$ is the exceptional locus of π. For details, see the last chapter of Reid: Undergraduate Algebraic Geometry.

Proof. This problem is quite combinatorial. Let $\sigma \in S_4$ be a permutation on $\{0,1,2,3\}$. The surface can be factorised as follows:

$$\begin{aligned}
& x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 \\
\implies & x_{\sigma(0)}^3 + x_{\sigma(1)}^3 = -(x_{\sigma(2)}^3 + x_{\sigma(3)}^3) \\
\implies & (x_{\sigma(0)} - x_{\sigma(1)})(x_{\sigma(0)} - \omega_+ x_{\sigma(1)})(x_{\sigma(0)} - \omega_- x_{\sigma(1)}) = -(x_{\sigma(2)} - x_{\sigma(3)})(x_{\sigma(2)} - \omega_+ x_{\sigma(3)})(x_{\sigma(2)} - \omega_- x_{\sigma(3)})
\end{aligned}$$

where $\omega_\pm = \dfrac{1 \pm \sqrt{5}}{2}$. For convenience we write

$$C^0_{a,b} := x_a - x_b, \qquad C^\pm_{a,b} = x_a - \omega_\pm x_b$$

for the hyperplanes in $\mathbb{P}^3$. For a fixed $\sigma \in S_4$, $C^\alpha_{\sigma(0),\sigma(1)} = C^\beta_{\sigma(2),\sigma(3)} = 0$ defines a line on the surface X, where $\alpha, \beta \in \{0, \pm\}$. This provides $3 \times 3 = 9$ distinct lines on X. It is easy to count that S_4 has 3 orbits. This gives at least $3 \times 9 = 27$ distinct lines on X. □

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Nice!

Challenge: prove (directly) that these are all of the lines