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Problem Sheet 3

B3.3: Algebraic Curves

22 February, 2021

Question 1

(i) Show that given any 5 points in $\mathbb{CP}^2$, there is at least one conic passing through them. Show also that this conic is unique if no three of the points are collinear.

(ii) Let C be a quartic (degree 4) curve in $\mathbb{CP}^2$ with four singular points. Use the strong form of Bézout's theorem to show C must be reducible.

(iii) Show that $y^4 - 4xzy^2 - xz(x-z)^2 = 0$ defines a quartic with three singular points.

Proof. (i) *This is Question 3 of Sheet 2 of Projective Geometry.*

- First we consider the case where no three points are collinear.

  Let A, B, C, D, E be the given five points. By assumption A, B, C, D are in general position. By applying a projective transformation we may assume that $A = [1:0:0]$, $B = [0:1:0]$, $C = [0:0:1]$ and $D = [1:1:1]$. Suppose that $E = [\alpha_0 : \alpha_1 : \alpha_2]$. Let $\mathscr{C} : \sum_{i,j=0}^{2} \lambda_{i,j} x_i x_j = 0$ be a conic that contains the five points.

  $A, B, C \in \mathscr{C}$ implies that $\lambda_{0,0} = \lambda_{1,1} = \lambda_{2,2} = 0$. So $\mathscr{C}$ has the form

  $$\lambda_{0,1} x_0 x_1 + \lambda_{1,2} x_1 x_2 + \lambda_{2,1} x_2 x_0 = 0$$

  $D, E \in \mathscr{C}$ implies that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (1,1,1) = 0$, $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (\alpha_0, \alpha_1, \alpha_2) = 0$. Since $D \neq E$, $\langle (1,1,1) \rangle \neq \langle (\alpha_0, \alpha_1, \alpha_2) \rangle$. We deduce that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \in \langle (1,1,1), (\alpha_0, \alpha_1, \alpha_2) \rangle^{\perp}$, which is a 1-dimensional subspace. Hence the coefficients of the quadric is uniquely determined up to rescaling by a constant. The conic determined by the quadric is unique.

- Second, consider the case where A, B, C are collinear.

  If the five points are not collinear, let $ax + by + c = 0$ and $dx + ey + f = 0$ be the equations of the projective lines ABC and DE respectively. Then $(ax + by + c)(dx + ey + f) = 0$ defines a reducible conic passing through the five points.

  If the five points are collinear, then they define a projective line $ax + by + c = 0$. For any other line $dx + ey + f = 0$, $(ax + by + c)(dx + ey + f) = 0$ defines a reducible conic passing through the five points.

(ii) Suppose that C is irreducible. Let W, X, Y, Z be the four singular points of C. Choose a non-singular point $V \in C$. By (i) there exists a conic D passing through V, W, X, Y, Z. Since C is irreducible, C and D has no common components. By Bézout's Theorem,

$$\sum_{p \in C \cap D} I_p(C, D) = 8$$

By Proposition 13, we know that $I_p(C, D) \geqslant 2$ for $p \in \{W, X, Y, Z\}$. And $I_V(C, D) \geqslant 1$. Then we have

$$\sum_{p \in C \cap D} I_p(C, D) \geqslant 9$$

which is a contradiction. Hence C is reducible.

(iii) The polynomial $P(x, y, z) := y^4 - 4xzy^2 - xz(x-z)^2$ is homogeneous of degree 4. Hence its set of zeros is a quartic curve in $\mathbb{CP}^2$.

To find the singular points, we compute its partial derivatives.

$$\frac{\partial P}{\partial x} = -4zy^2 - z(x-z)^2 - 2xz(x-z), \qquad \frac{\partial P}{\partial y} = 4y^3 - 8xyz, \qquad \frac{\partial P}{\partial z} = -4xy^2 - x(x-z)^2 - 2xz(z-x)$$

By simple observation, we find that $P = \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ has at least three solutions: $[1:0:1]$, $[1:0:0]$ and $[0:0:1]$. So the curve has at least three singular points.

Next we shall show that the curve has at most three singular points. By (ii) it suffices to show that the curve is irreducible.

Suppose that P has a linear factor $ax + by + cz$. Since P contains a term y^4 but not xy^3 or zy^3, $a = c = 0$. But P is not divisible by y. We deduce that P does not have a linear factor.

Suppose that P is reducible. Then there exists quadratic polynomials $Q(x,y,z)$ and $R(x,y,z)$ such that

$$P(x,y,z) = y^4 - 4xzy^2 - xz(x-z)^2 = \left(y^2 + Q(x,y,z)\right)\left(y^2 + R(x,y,z)\right)$$

Then

$$Q + R = -4xz, \qquad QR = -xz(x-z)^2$$

By unique factorisation, we can list all the possible polynomials Q and R, and find that the equation has no solutions. We conclude that P is irreducible and hence the curve has exactly three singular points. □

Question 2

Let $P(x,y,z)$ be a homogeneous polynomial of degree d defining a nonsingular curve C.

(i) Write down Euler's relation for P, P_x, P_y, P_z. Deduce that the Hessian determinant satisfies:

$$z\mathcal{H}_P(x,y,z) = (d-1)\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_x & P_y & P_z \end{pmatrix}$$

(ii) Deduce further that:

$$z^2\mathcal{H}_P(x,y,z) = (d-1)^2\det\begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}$$

(iii) Deduce that if $P(x,y,1) = y - g(x)$ then $[a,b,1]$ is a flex of C iff $b = g(a)$ and $g''(a) = 0$.

Proof. (*Inspired by physicists' convention in General Relativity, we shall denote the partial derivative* $\dfrac{\partial P}{\partial x}$ *by* $P_{,x}$ *instead of* P_x.)

(i) The Euler relation is given by

$$xP_{,x} + yP_{,y} + zP_{,z} = dP$$

The first partial derivatives $P_{,i}$ are homogeneous of degree $(d-1)$. Hence the Euler relation is given by

$$P_{,i} = \sum_{j\in\{x,y,z\}} jP_{,ij} = (d-1)P$$

Starting from the Hessian determinant

$$\mathcal{H}_P(x,y,z) = \det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,xz} \\ P_{,yx} & P_{,yy} & P_{,yz} \\ P_{,zx} & P_{,zy} & P_{,zz} \end{pmatrix}$$

We perform some elementary row operations:

$$\begin{aligned}
z\mathcal{H}_P(x,y,z) &= \det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,xz} \\ P_{,yx} & P_{,yy} & P_{,yz} \\ zP_{,zx} & zP_{,zy} & zP_{,zz} \end{pmatrix} \\
&= \det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,xz} \\ P_{,yx} & P_{,yy} & P_{,yz} \\ xP_{,xx} + yP_{,yx} + zP_{,zx} & xP_{,xy} + yP_{,yy} + zP_{,zy} & xP_{,xz} + yP_{,yz} + zP_{,zz} \end{pmatrix} \\
&= \det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,xz} \\ P_{,yx} & P_{,yy} & P_{,yz} \\ (d-1)P_{,x} & (d-1)P_{,y} & (d-1)P_{,z} \end{pmatrix} \\
&= (d-1)\det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,xz} \\ P_{,yx} & P_{,yy} & P_{,yz} \\ P_{,x} & P_{,y} & P_{,z} \end{pmatrix}
\end{aligned}$$

(ii) Starting from the result of (i), We perform some elementary column operations:

$$\begin{aligned}
z^2\mathscr{H}_P(x,y,z) &= (d-1)\det\begin{pmatrix} P_{,xx} & P_{,xy} & zP_{,xz} \\ P_{,yx} & P_{,yy} & zP_{,yz} \\ P_{,x} & P_{,y} & zP_{,z} \end{pmatrix} \\
&= (d-1)\det\begin{pmatrix} P_{,xx} & P_{,xy} & xP_{,xx}+yP_{,xy}+zP_{,xz} \\ P_{,yx} & P_{,yy} & xP_{,yx}+yP_{,yy}+zP_{,yz} \\ P_{,x} & P_{,y} & xP_{,x}+yP_{,y}+zP_{,z} \end{pmatrix} \\
&= (d-1)\det\begin{pmatrix} P_{,xx} & P_{,xy} & (d-1)P_{,x} \\ P_{,yx} & P_{,yy} & (d-1)P_{,y} \\ P_{,x} & P_{,y} & dP \end{pmatrix} \\
&= (d-1)^2\det\begin{pmatrix} P_{,xx} & P_{,xy} & P_{,x} \\ P_{,yx} & P_{,yy} & P_{,y} \\ P_{,x} & P_{,y} & dP/(d-1) \end{pmatrix}
\end{aligned}$$

(iii) Since $P(x,y,1) = y - g(x)$, we have

$$P_{,xx}(x,y,1) = -g''(x), \qquad P_{,xy}(x,y,1) = 0, \qquad P_{,yy}(x,y,1) = 0, \qquad P_{,x}(x,y,1) = -g'(x), \qquad P_{,y}(x,y,1) = 1$$

Hence

$$\mathscr{H}_P(x,y,1) = \det\begin{pmatrix} -g''(x) & 0 & -g'(x) \\ 0 & 0 & 1 \\ -g'(x) & 1 & \dfrac{d}{d-1}(y-g(x)) \end{pmatrix} = g''(x)$$

Hence

$$\begin{aligned}
[a:b:1] \text{ is a flex of } C &\iff P(a,b,1) = 0 \wedge \mathscr{H}_P(a,b,1) = 0 \\
&\iff b = g(a) \wedge g''(a) = 0
\end{aligned}$$

The result suggests that the definition of inflection point using Hessian determinant coincides with the usual definition using second derivative. □

Question 3

Let C and D be nonsingular projective curves of degree n and m in $\mathbb{P}^2$. Show that if C is homeomorphic to D then either $n = m$ or $\{n,m\} = \{1,2\}$.

Proof. Projective curves in $\mathbb{CP}^2$ are topologically compact connected Riemann surfaces, which are orientable. By the classification of compact surfaces, a projective curve is uniquely determined by its genus g up to homeomorphism. If C and D are homeomorphic, by the degree-genus formula, we have

$$g_C = \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(m-1)(m-2) = g_D$$

Hence $(n-1)(n-2) = (m-1)(m-2)$. It follows that either $n = m$ or $\{n,m\} = \{1,2\}$. □

Question 4

Show that if C is the conic $y^2 = xz$ then the map

$$f : \mathbb{P}^1 \to C$$

given by

$$f : [s,t] \mapsto \left[s^2, st, t^2\right]$$

is a homeomorphism. Deduce without using the degree-genus formula that all nonsingular conics have genus zero.

Proof. Since the map $z \mapsto z^2$ is surjective onto $\mathbb{C}$, f is a surjective map. Suppose that $[s^2 : st : t^2] = [s'^2 : s't' : t'^2]$. Then there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $s = \pm\lambda s'$, $t = \pm\lambda t'$ and $st = \lambda^2 s' t'$. Hence $(s, t) = \pm\lambda(s', t')$ and $[s, t] = [s', t']$. We deduce that f is injective.

To show that f is a homeomorphism, it suffices to show that it is locally a homeomorphism. Without loss of generality we assume that $t \neq 1$. Then $f : [s/t : 1] \mapsto [s^2/t^2 : s/t : 1]$ is locally the map $\widetilde{f} : \mathbb{C} \to \mathbb{C}^2$ given by $z \mapsto (z^2, z)$. This is clearly a local homeomorphism because the derivative is non-vanishing and we can apply the inverse function theorem.

Next we shall show that all non-singular conic can be put into the form $y^2 = xz$ by a projective transformation.

By applying a suitable projective transformation we may assume that the conic C passes through $[1:0:0]$, and the tangent line of C at $[1:0:0]$ is $z = 0$. Suppose that C is defined by the polynomial

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$$

Then $a = 0$ and $d = 0$. Since C is non-singular, it is irreducible, and hence $bf \neq 0$. The conic equation becomes

$$\left(\sqrt{b}y\right)^2 = -z(fx + ey + cz)$$

The projective transformation $[x : y : z] \mapsto [fx + ey + cz : \sqrt{b}y : -z]$ takes C to the conic $y^2 = xz$.

We deduce that all non-singular conics are homeomorphic to the projective line $\mathbb{CP}^1$, which is homeomorphic to S^2, and has genus 0. □

Question 5

Let $f : X \to Y$ be a (nonconstant) holomorphic map of compact connected Riemann surfaces, where X is the Riemann sphere. Show that Y is homeomorphic to X.

Proof. This is Question 2 of Sheet 2 of Geometry of Surfaces.

Recall the Riemann-Hurwitz formula from Geometry of Surfaces:

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{x \in X} (v_f(x) - 1)$$

Since Y is a compact connected Riemann surface, it is orientable. So by classification theorem of compact surfaces, Y is homeomorphic to some connected sum of tori. $\chi(Y) = 2 - 2g$ for some $g \in \mathbb{N}$.

Suppose that $n \geqslant 1$. Then

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{x \in X} (v_f(x) - 1) \leqslant \deg f \cdot \chi(Y) \leqslant 0$$

But we know that X is the Riemann sphere, so $\chi(X) = 2$. This is a contradiction. Hence $g = 0$ and $Y \cong S^2$. We deduce that Y is homeomorphic to X. □

Question 6

(i) Let U be a connected open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic. Show that if $a \in U$, then for sufficiently small real positive r, we have:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$

(ii) Deduce that if $|f|$ has a local maximum at $a \in U$, then $|f|$ is constant on some neighbourhood of a.

(iii) Deduce that if $|f|$ has a local maximum at $a \in U$, then f is constant on U.

(iv) Now suppose S is a compact connected Riemann surface and $f : S \mapsto \mathbb{C}$ is a holomorphic function. Show that f is constant. (You may assume the Identity Theorem for Riemann surfaces, that is, if two holomorphic maps on a Riemann surface agree then on a nonempty open set then they agree everywhere).

Proof. (i) *This is Question 1 of Sheet 6 of Complex Analysis.*

For sufficiently small $r > 0$, $\gamma^*(a,r) \subseteq U$. Since f is holomorphic in U, by Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i}\oint_{\gamma(a,r)} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+r e^{i\theta})}{a+r e^{i\theta}-a} i r e^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f\left(a+r e^{i\theta}\right) d\theta$$

(ii) Since $|f|$ attains local maximum at $a \in U$, there exists $R > 0$ $|f(a)| \geqslant |f(z)|$ for all $z \in B(a,R)$. For $r \in (0,R]$, by part (i) we have

$$|f(a)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f\left(a+r e^{i\theta}\right) d\theta\right| \leqslant \sup_{\theta\in[0,2\pi]}\left|f\left(a+r e^{i\theta}\right)\right|$$

But $|f(a)|$ is a local maximum. Hence

$$|f(a)| = \sup_{\theta\in[0,2\pi]}\left|f\left(a+r e^{i\theta}\right)\right|$$

and

$$\int_0^{2\pi}\left(f\left(a+r e^{i\theta}\right) - \sup_{\theta\in[0,2\pi]}\left|f\left(a+r e^{i\theta}\right)\right|\right) d\theta = 0$$

Since f is continuous, f is constant on $\gamma^*(a,r)$. Since r is arbitrary, we deduce that f is constant on $B(a,R)$.

(iii) $B(a,R) \subseteq U$ has a limit point in U. Since U is connected, by the identity theorem, f is constant on U.

(iv) Since S is compact and f is continuous, $f(S)$ is bounded in $\mathbb{C}$. $|f|$ attains maximum at some $a \in \mathbb{C}$. By (ii) f is constant in an open neighbourhood of a. By the (stronger) q4identity theorem, f is constant. □