☺ Well done!

Peize Liu
St. Peter's College
University of Oxford

Amazing presentation

Problem Sheet 1

B3.2: Geometry of Surfaces

25 October, 2020

Question 1

Define the Euler characteristic of a surface with a subdivision. By choosing a suitable subdivision show that the Euler characteristic of a torus is zero.

An engineer constructs a vessel in the shape of a torus from a finite number of steel plates. Each plate is in the form of a not necessarily regular curvilinear polygon with n edges. The plates are welded together along the edges so that at each vertex n distinct plates are joined together, and no plate is welded to itself. What is the number n ? Justify your answer.

[*You may assume that the Euler characteristic is independent of the choice of subdivision of the surface.*]

Proof. Let X be a topological space and $f : S^{n-1} \to X$ be a continuous map. The space obtained by attaching an n-cell to X along f, $X \cup_f D^n$, is defined to be the quotient of the disjoint union $X \sqcup D^n$, such that each $x \in X$ is identified with $f^{-1}(\{x\})$.

a

Let X be a topological surface. A cellular decomposition of X is a chain $K_0 \subseteq K_1 \subseteq K_2 = X$, where K_0 is a finite set (the "0-cells") and each K_n is obtained by attaching finitely many n-cells to K_{n-1}. We define the Euler characteristic of X:

$$\chi(X) := \sum_{n=0}^{2} (-1)^n c_n$$

where c_n is the number of n-cells.

A torus $T^2 \cong S^1 \times S^1$ has the fundamental group $\langle a, b \mid aba^{-1}b^{-1} \rangle$ so it can be constructed with one 0-cell, two 1-cells and one 2-cell. It is visually shown in Figure 1.

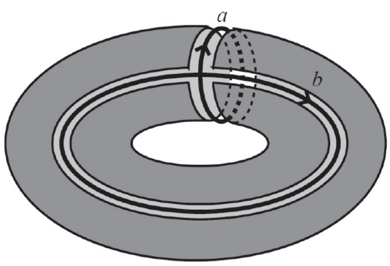


Figure 1: Cellular Decomposition of a Torus, borrowed from P13, Lecture notes for Topology & Groups

So the Euler characteristic of the torus is

$$\chi(T^2) = 1 - 2 + 1 = 0$$

For the second part of the question, suppose that the torus is constructed by welding m such plates. Each vertex on T^2 corresponds to n vertices of the plates; each edge on T^2 corresponds to 2 edges of the plates; and each face on T^2 corresponds to 1 face of the plates. Hence the torus is subdivided into $mn/n = m$ vertices, $mn/2$ edges and m faces. The Euler characteristic

$$\chi(T^2) = m - \frac{mn}{2} + m = 0$$

Hence the only possibility is that $n = 4$. $n = 4$ works if we consider the torus as the side identification of a square, as shown in Figure 2. $\square$

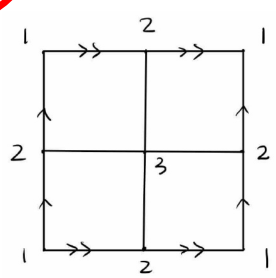


Figure 2:

Question 2. The Thomsen Graph / The Three Amenities Problem. E

Let H_1, H_2, H_3, G, W, E be six points on a sphere. Show that it is not possible to join each of H_1, H_2, H_3 to each of G, W, E by curves intersecting only at their end points (nine curves in all).

[*You may assume that such a configuration of curves would give a subdivision of the sphere.*]

By drawing a diagram show that such a construction is possible on the projective plane. Decide whether it is possible on the torus or the Klein bottle.

Proof. For simplicity, we called these points $A_1, A_2, A_3, B_1, B_2, B_3$ where each A_i is connected to each B_j by a curve on S^2. Suppose that these curves are non-intersecting away from the vertices. Such configuration defines a subdivision of the sphere. It has 6 vertices and 9 edges. For each face on S^2, it is surrounded by at least 4 edges $A_i - B_k - A_j - B_\ell$ ($i \neq j$ and $k \neq \ell$). Each edge connects 2 faces on S^2, so there are at most $\lfloor 9 \cdot 2/4 \rfloor = 4$ faces. Since $\chi(S^2) = 2$, S^2 has 5 faces, which is a contradiction. Hence no such configuration is possible.

The constructions for real projective plane, 2-torus, and the Klein bottle are shown in the following figure:

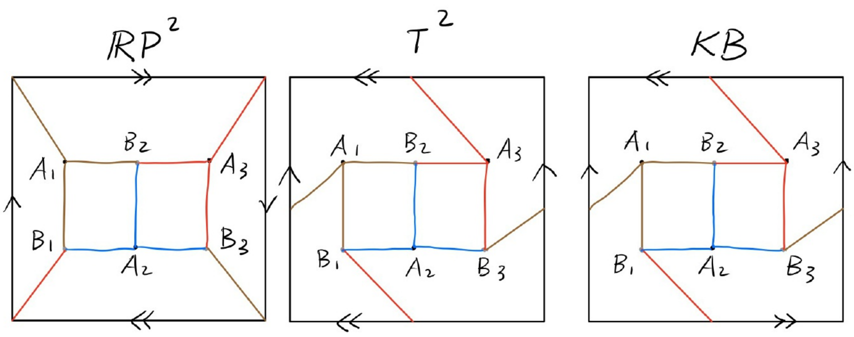


Figure 3: The Thomsen Graph on $\mathbb{R}P^2$, T^2 and KB

□

Question 3 B

Use the formula for the Euler characteristic to show that there are no more than five Platonic solids.

(*A Platonic solid is a convex polyhedron with congruent faces consisting of regular polygons and the same number of faces meet at each vertex.*)

What are the possibilities for subdividing a torus into polygons each with n sides, and such that k edges meet at each vertex?

Proof. A convex polyhedron is homeomorphic to S^2 so it has Euler characteristic 2. Suppose that the polyhedron is made of m n-sided polygons such that k edges meet at each vertex. Then the polyhedron has mn/k vertices, $mn/2$ edges and m faces. We have

$$\frac{mn}{k} - \frac{mn}{2} + m = 2$$

which implies that

$$\frac{1}{k} + \frac{1}{n} = \frac{1}{2} + \frac{2}{mn} > \frac{1}{2}$$

where $n, k \geqslant 3$, The only positive integer solutions of (n,k) are: $(3,3), (3,4), (4,3), (5,3), (3,5)$. So there are at most five distinct Platonic solids.

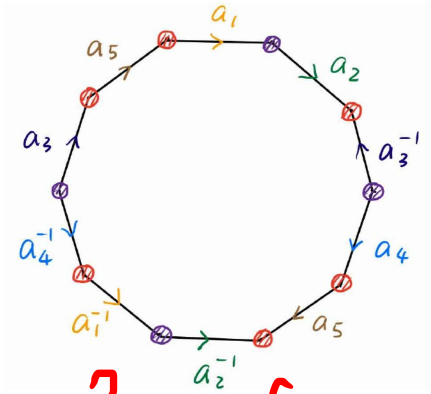
Since a torus has Euler characteristic 0, we deduce that $\frac{1}{k} + \frac{1}{n} = \frac{1}{2}$. The only possibility is $n = k = 4$, which is consistent with the result in Question 1. Careful!! There are 2 more possibilities. □

Question 4

(i) Calculate the Euler characteristic of the surface given in planar form by $a_1 a_2 a_3^{-1} a_4 a_5 a_2^{-1} a_1^{-1} a_4^{-1} a_3 a_5$. Show that the surface contains a Möbius band.

(ii) By looking for $xyx^{-1}y^{-1}$ terms (or using the classification of surfaces) show that the surface described by $b_1a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$ is homeomorphic to $T\#T$.

Proof. (i) The fundamental polygon is shown in the figure:



Be careful about the identifications!

We observe that the surface obtained has ~~2~~ 3 vertices, ~~10~~ 5 edges and 1 face. So the Euler characteristic is $2-10+1=-7$.

The surface contains a Möbius band because it contains two copies of the letter a_5. If we connect the two curves a_5 by two lines across the face, we will obtain a Möbius band.

(ii) Let X denotes this surface. Then X is obtained by gluing a 10-sided polygon with the boundary word $b_1a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$. Cyclically permutating this word we obtain $a_2^{-1}a_1^{-1}b_1^{-1}a_1b_1a_2b_3a_3^{-1}b_3^{-1}a_3$. We observe that $X = X_1\#X_2$, where X_1 is obtained by gluing a hexagon with the boundary word $a_2^{-1}a_1^{-1}b_1^{-1}a_1b_1a_2$, and X_2 is obtained by gluing a square with boundary word $b_3a_3^{-1}b_3^{-1}a_3$. For X_1, first we cyclically permute the boundary word: $a_2a_2^{-1}a_1^{-1}b_1^{-1}a_1b_1$. Next we contract $a_2a_2^{-1}$ and obtain $a_1^{-1}b_1^{-1}a_1b_1$. Now it is clear that both X_1 and X_2 are homeomorphic to T^2. We conclude that $X\cong T^2\#T^2$. ☐

Question 5 a

Suppose Z is a compact, connected surface with $\chi(Z)=n$. Compute the number of isomorphism classes of ordered pairs (X,Y) of compact, connected surfaces X,Y with $X\#Y\cong Z$, in the cases when

(i) Z is orientable;

(ii) Z is not orientable.

Proof. We know that $\chi(Z)=\chi(X\#Y)=\chi(X)+\chi(Y)-2$ and that Z is orientable if and only if both X and Y are orientable.

(i) Suppose that Z is orientable. Then both X and Y are orientable. By classification of compact surfaces, $X\cong\Sigma_{g_1}$ and $Y\cong\Sigma_{g_2}$ for some $g_1,g_2\in\mathbb{N}$. Therefore $\chi(X)=2-2g_1$ and $\chi(Y)=2-2g_2$. We have

$$n=\chi(Z)=2-2g_1+2-2g_2-2=2-2(g_1+g_2)$$

Hence $g_1+g_2=1-n/2$. In particular n is even and is not greater than 2. The possibilities are $(g_1,g_2)=(0,1-n/2),(1,-n/2),\ldots,(1-n/2,0)$. There are $2-n/2$ isomorphism classes.

(ii) Suppose that Z is not orientable.

- If neither X nor Y is orientable, then by classification of compact surfaces, X and Y are some connected sum of $\mathbb{R}P^2$, and we have $\chi(X)=2-h_1$ and $\chi(Y)=2-h_2$ for some $h_1,h_2\in\mathbb{Z}_+$. Then $n=2-(h_1+h_2)$ and hence $h_1+h_2=2-n$. The possibilities are $(h_1,h_2)=(1,1-n),(2,-n),\ldots,(1-n,1)$. There are $1-n$ isomorphism classes. $(n\leqslant 0)$

- Assume that exactly one of X and Y is orientable. First we consider that X is orientable and Y is not orientable. Therefore $\chi(X)=2-2g$ and $\chi(Y)=2-h$ for $g\in\mathbb{N}$ and $h\in\mathbb{Z}_+$.

$$n=2-2g+2-h+2=2-(2g+h)$$

So $2g+h=2-n$

" - "

If n is odd, then the possibilities are $(g,h) = (0,2-n),(1,-n),\ldots,((1-n)/2,1)$. There are $(1-n)/2$ isomorphism classes. If n is even, then hen the possibilities are $(g,h) = (0,2-n),(1,-n),\ldots,(-n/2,2)$. There are $-n/2$ isomorphism classes. We can exchange X and Y and the result is similar.

In summary: If n is even, then $n \leqslant 0$ and there are $1-n+2\cdot(-n)/2 = 1-2n$ isomorphism classes. If n is odd, then $n \leqslant 1$ and there are $1-n+2\cdot(1-n)/2 = 2-2n$ isomorphism classes. □

You have counted wrong!