Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3

Radiations

B2: Symmetry & Relativity

Well done!

6 December, 2020

Question 1

Obtain the electric field of a uniformly moving charge, as follows: place the charge at the origin of the primed frame S' and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$.

Solution. Let $\mathbf{v}$ be the velocity of the moving charge and S' be the inertial frame moving with the charge. In the frame S' the charge is stationary. The fields of it is given by

$$\mathbf{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3}, \qquad \mathbf{B}' = 0$$

We write down the transformations of fields and coordinates:

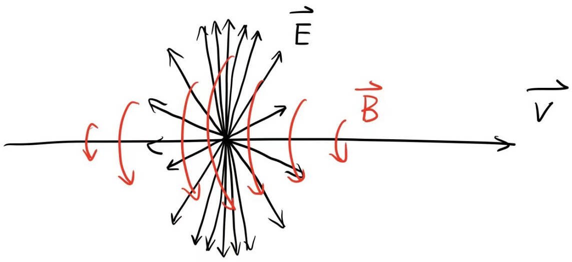
$$\mathbf{E}_{/\!/} = \mathbf{E}'_{/\!/} \qquad \mathbf{E}_{\perp} = \gamma(\mathbf{E}'_{\perp} - \mathbf{v}\wedge\mathbf{B}')$$
$$\mathbf{B}_{/\!/} = \mathbf{B}'_{/\!/} \qquad \mathbf{B}_{\perp} = \gamma(\mathbf{B}'_{\perp} + \mathbf{v}\wedge\mathbf{E}'/c^2)$$
$$\mathbf{r}'_{/\!/} = \gamma(\mathbf{r}_{/\!/} - \mathbf{v}t) \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}$$

Hence the electric field in frame S is given by

$$\mathbf{E}_{/\!/} = \mathbf{E}'_{/\!/} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'_{/\!/}}{x'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(\mathbf{r}_{/\!/} - \mathbf{v}t)}{(\gamma^2\,\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_{\perp}\|^2)^{3/2}}$$
$$\mathbf{E}_{\perp} = \gamma\mathbf{E}'_{\perp} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\mathbf{r}_{\perp}}{(\gamma^2\,\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_{\perp}\|^2)^{3/2}}$$
$$\mathbf{E} = \mathbf{E}_{/\!/} + \mathbf{E}_{\perp} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(\mathbf{r} - \mathbf{v}t)}{(\gamma^2\,\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_{\perp}\|^2)^{3/2}}$$

For the magnetic field, we have

$$\mathbf{B} = \mathbf{B}_{/\!/} + \mathbf{B}_{\perp} = \mathbf{B}'_{/\!/} + \gamma(\mathbf{B}'_{\perp} + \mathbf{v}\wedge\mathbf{E}'/c^2) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\gamma\mathbf{v}\wedge\mathbf{r}'}{r'^3} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\gamma\mathbf{v}\wedge\mathbf{r}}{(\gamma^2\,\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_{\perp}\|^2)^{3/2}} = \frac{\mathbf{v}\wedge\mathbf{E}}{c^2}$$ □



Question 2

A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ where q is the total charge. Find the fields due to a moving charged sphere by two methods, as follows.

[N.B. it will be useful to let the rest frame of the sphere be S' (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S' be in the standard configuration.]

(i) Field method: write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ and $r' \geqslant a$ where $r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.

(ii) Potential method: in the rest frame of the sphere the 3 -vector potential is zero, and the scalar potential is

$$\phi' = \frac{q}{8\pi\varepsilon_0 a}\left(3 - r'^2/a^2\right)$$

for $r' < a$, and

$$\phi' = \frac{q}{4\pi\varepsilon_0 r'}$$

for $r' \geqslant a$.

Form the 4-vector potential, transform it, and thus show that both ϕ and A are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle $\partial/\partial x$ and $\partial/\partial x'$, etc.]

Solution. (i) By Gauss' Theorem we can write down the electric field of the uniformly charged sphere in the frame S' where the sphere is stationary:

$$\mathbf{E}' = \begin{cases} \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}'}{a^3}, & r' < a \\ \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}'}{r'^3}, & r' \geqslant a \end{cases}$$

The magnetic field in frame S' is $\mathbf{B}' = 0$. For $r' \geqslant a$, or equivalently $\gamma^2\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 \geqslant a^2$, the field is the same as a point charge. We can use the result of Question 1 and write down the fields in frame S:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(\mathbf{r}-\mathbf{v}t)}{(\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2)^{3/2}}, \qquad \mathbf{B} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\gamma\mathbf{v}\wedge\mathbf{r}}{(\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2)^{3/2}}$$

For $r' < a$, or equivalently $\gamma^2\|\mathbf{r}_{/\!/} - \mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 < a^2$, we have

$$\mathbf{E} = \mathbf{E}'_{/\!/} + \gamma\mathbf{E}'_\perp = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'_{/\!/} + \gamma\mathbf{r}'_\perp}{a^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(\mathbf{r}-\mathbf{v}t)}{a^3}$$
$$\mathbf{B} = \gamma\mathbf{v}\wedge\mathbf{E}'/c^2 = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\gamma\mathbf{v}\wedge\mathbf{r}}{a^3}$$

(ii) We are given the electromagnetic potential in the frame S':

$$\varphi' = \begin{cases} \dfrac{q}{8\pi\varepsilon_0 a}\left(3 - \dfrac{r'^2}{a^2}\right), & r' < a \\ \dfrac{q}{4\pi\varepsilon_0 r'}, & r' \geqslant a \end{cases}, \qquad \mathbf{A}' = \mathbf{0}$$

The Lorentz transformation from S' to S is given by

$$\varphi = \gamma(\varphi' + \mathbf{v}\cdot\mathbf{A}'), \qquad \mathbf{A}_{/\!/} = \gamma(\mathbf{A}'_{/\!/} + \mathbf{v}\varphi'/c^2), \qquad \mathbf{A}_\perp = \mathbf{A}'_\perp$$

Hence

$$\varphi = \gamma\varphi' = \begin{cases} \dfrac{\gamma q}{8\pi\varepsilon_0 a}\left(3 - \dfrac{\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2}{a^2}\right), & \gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 < a^2 \\ \dfrac{\gamma q}{4\pi\varepsilon_0}\dfrac{1}{\sqrt{\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2}}, & \gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 \geqslant a^2 \end{cases}$$

$$\mathbf{A} = \frac{\gamma\mathbf{v}}{c^2}\varphi' = \begin{cases} \dfrac{\gamma q\mathbf{v}}{8\pi\varepsilon_0 c^2 a}\left(3 - \dfrac{\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2}{a^2}\right), & \gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 < a^2 \\ \dfrac{\gamma q\mathbf{v}}{4\pi\varepsilon_0 c^2}\dfrac{1}{\sqrt{\gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2}}, & \gamma^2\|\mathbf{r}_{/\!/}-\mathbf{v}t\|^2 + \|\mathbf{r}_\perp\|^2 \geqslant a^2 \end{cases}$$

Then the fields are given by

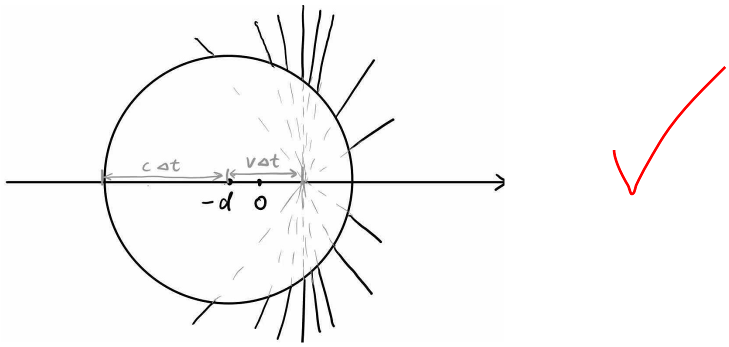
$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\wedge\mathbf{A}$$

By explicitly computing the partial derivatives (or muddling with the vector identities) we should obtain the same result as in (i). (*I don't know the advantage of using the potential method in the uniform motion. The calculation is significantly* *harder than the field method. Some of the calculations are already incorporated in the transformation rules of fields, which are derived from the Lorentz transformation of the 4-potential.)* □

Question 3

In a frame S a point charge first moves uniformly along the negative x-axis in the positive x direction, reaching the point $(-d,0,0)$ at $t=-\Delta t$, and then it is slowed down until it comes to rest at the origin at $t=0$. Sketch the lines of electric field in S at $t=0$, in the region $(x+d)^2+y^2+z^2>(c\Delta t)^2$.

Solution. The observers in frame S located in the region $(x+d)^2+y^2+z^2>(c\Delta t)^2$ does not know the deceleration of the charge, because the electric field propagates in the speed of light. They will simply observe a field which looks like a uniformly moving charge. □



Question 4

Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$A^\mu = \frac{q}{4\pi\varepsilon_0}\frac{U^\mu/c}{-R_\nu U^\nu}$$

where U^μ and R^μ are suitably chosen 4-vectors which you should define in your answer.

Proof. Let us derive the retarded potential $A(X)$ for general charge distribution $J(X_s)$. Starting from the d'Alembert's equation with Lorenz gauge:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\mu = -\mu_0 J^\mu$$

The general solution is obtained by

$$A^\mu(X) = \int -\mu_0 G(X;X_s)J^\mu(X_s)\,\mathrm{d}^4X_s$$

where $G(X,X_s)$ is the Green's function satisfying

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(X;X_s) = \delta^4(X-X_s)$$

The Green's function should only depends on the difference $X-X_s$. So $G=G(X-X_s)=G(t-t_s,\mathbf{x}-\mathbf{x}_s)$ Let $R:=X-X_s$, $\mathbf{r}:=\mathbf{x}-\mathbf{x}_s$ and $r:=\|\mathbf{x}-\mathbf{x}_s\|$.

We take inverse Fourier transforms:

$$G(t-t_s,\mathbf{r}) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega(t-t_s)}\,\widehat{G}(\omega,\mathbf{r})\,\mathrm{d}\omega$$

$$\delta^4(X-X_s) = \delta(t-t_s)\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega(t-t_s)}\,\mathrm{d}\omega$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(t-t_s,\mathbf{r}) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\mathrm{e}^{\mathrm{i}\omega(t-t_s)}\,\widehat{G}(\omega,\mathbf{r})\right)\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\left(\mathrm{e}^{\mathrm{i}\omega(t-t_s)}\,\widehat{G}(\omega,\mathbf{r})\right)\mathrm{d}\omega$$

Hence we have

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widehat{G}(\omega, \mathbf{r}) = \delta^3(\mathbf{r})$$

We only concern the spherical symmetric solutions. So $\widehat{G}(\omega, \mathbf{r}) = \widehat{G}(\omega, r)$. In the spherical coordinates, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \widehat{G}}{\partial r}\right) + \frac{\omega^2}{c^2}\widehat{G}(\omega, r) = \delta(r)$$

To eliminate the δ-function we multiply both sides by r:

$$\frac{\partial^2}{\partial r^2}(r\widehat{G}) + \frac{\omega^2}{c^2}r\widehat{G}(\omega, r) = r\delta(r) = 0$$

Hence the general solution is given by

$$\widehat{G}(\omega, r) = \frac{A\,\mathrm{e}^{\mathrm{i}\omega r/c} + B\,\mathrm{e}^{-\mathrm{i}\omega r/c}}{4\pi r}$$

Let $\widehat{G}^{(\pm)}(\omega, r) = \dfrac{1}{4\pi r}\mathrm{e}^{\pm\mathrm{i}\omega r/c}$. Substituting into the inversion formula:

$$G^{(\pm)}(t - t_s, r) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega(t-t_s)}\frac{\mathrm{e}^{\pm\mathrm{i}\omega r/c}}{4\pi r}\,\mathrm{d}\omega = \frac{1}{2\pi r}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega(t-t_s\pm r/c)}\,\mathrm{d}\omega = \frac{1}{4\pi r}\delta(t - t_s \pm r/c)$$

(*The whole method of Green's function is valid only in the sense of distributions.*)

$G^{(+)}$ is the retarded Green's function and $G^{(-)}$ is the advanced Green's function. For the situtation where there is no incoming wave, we only keep $G^{(+)}$ (and add a minus sign to meet the limiting case). So the 4-potential is given by

$$A^\mu(X) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(X_s)}{\|\mathbf{x}-\mathbf{x}_s\|}\delta\left(t - t_s + \frac{\|\mathbf{x}-\mathbf{x}_s\|}{c}\right)\mathrm{d}^4 X_s = \frac{\mu_0}{4\pi}\int \frac{J^\mu(X_s)_{\mathrm{ret}}}{\|\mathbf{x}-\mathbf{x}_s\|}\,\mathrm{d}^3\mathbf{x}_s$$

where $J^\mu(t_s, \mathbf{x}_s)_{\mathrm{ret}} = J^\mu\left(t - \dfrac{\|\mathbf{x}-\mathbf{x}_s\|}{c},\ \mathbf{x}_s\right)$.

Now we focus on moving charges and derive the Liénard-Wiechart potential. For a charge q with 3-velocity v, we have

$$\rho(\mathbf{x}_s) = q\delta^3(\mathbf{x}_s - \mathbf{x}'(t)), \qquad \mathbf{j}(\mathbf{x}_s) = q\mathbf{v}\delta^3(\mathbf{x}_s - \mathbf{x}'(t))$$

We change the parameter to proper time τ. The 4-current is given by

$$J^\mu(X_s) = q\int U^\mu(\tau)\delta^3(\mathbf{x}_s - \mathbf{x}'(\tau))\delta(t_s - t'(\tau))\,\mathrm{d}\tau = q\int U^\mu(\tau)\delta^4(X_s - X'(\tau))\,\mathrm{d}\tau$$

where $X'(\tau)$ is the worldline of the charge and $U(\tau)$ is the 4-velocity of the charge. Hence the 4-potential of the charge is given by

$$\begin{aligned}
A^\mu(X) &= \frac{\mu_0}{4\pi}\int \frac{1}{\|\mathbf{x}-\mathbf{x}_s\|}\delta\left(t - t_s + \frac{\|\mathbf{x}-\mathbf{x}_s\|}{c}\right)\left(q\int U^\mu(\tau)\delta^4(X_s - X'(\tau))\,\mathrm{d}\tau\right)\mathrm{d}^4 X_s \\
&= \frac{\mu_0}{4\pi}q\int U^\mu(\tau)\left(\int \frac{1}{\|\mathbf{x}-\mathbf{x}_s\|}\delta\left(t - t_s + \frac{\|\mathbf{x}-\mathbf{x}_s\|}{c}\right)\delta^4(X_s - X'(\tau))\,\mathrm{d}^4 X_s\right)\mathrm{d}\tau \\
&= \frac{\mu_0}{4\pi}q\int U^\mu(\tau)\frac{1}{\|\mathbf{x}-\mathbf{x}'\|}\delta\left(t - t' + \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right)\mathrm{d}\tau
\end{aligned}$$

To evaluate the integral, we can use a trick to deal with the integrand [1]. Note that

$$\begin{aligned}
\delta\left(\|X - X'\|^2\right) &= \delta\left(c^2(t-t')^2 - \|\mathbf{x}-\mathbf{x}'\|^2\right) = \delta\left(\left(c(t-t') + \|\mathbf{x}-\mathbf{x}'\|\right)\left(c(t-t') - \|\mathbf{x}-\mathbf{x}'\|\right)\right) \\
&= \frac{1}{2c\,\|\mathbf{x}-\mathbf{x}'\|}\left(\delta\left(t - t' + \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right) + \delta\left(t - t' - \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right)\right)
\end{aligned}$$

[1] Section 12.11 of *Classical Electrodynamics* by Jackson.

For the physical retarded potential, we take $t > t'$. We can write

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|}\delta\left(t - t' + \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right) = 2c\mathbf{1}_{\{t>t'\}}\delta\left(\|X - X'\|^2\right)$$

On the other hand, as a function of the proper time τ, we have

$$\delta\left(\|X - X'\|^2\right) = \sum_i \delta(\tau-\tau_i)\left|\frac{\mathrm{d}\|X - X'(\tau)\|^2}{\mathrm{d}\tau}\right|^{-1}\Bigg|_{\|X-X'(\tau_i)\|^2=0} = \sum_i \frac{\delta(\tau-\tau_i)}{-2(X_\nu - X'_\nu)U^\nu}\Bigg|_{\|X-X'(\tau_i)\|^2=0}$$

Note that $\|X - X'(\tau_i)\| = 0$ has two solutions, one earlier and one later than t. The physical solution is the earlier point τ_0. Hence

$$\mathbf{1}_{\{t>t'\}}\delta\left(\|X - X'\|^2\right) = \frac{\delta(\tau-\tau_0)}{-2R_\nu U^\nu(\tau_0)}$$

where $R := X - X'$. Finally we obtain the Liénard-Wiechart potential for moving charges

$$A^\mu(X) = \frac{\mu_0 q}{4\pi}\int U^\mu(\tau)\frac{2c\delta(\tau-\tau_0)}{-2R_\nu U^\nu(\tau_0)}\mathrm{d}\tau = \frac{q}{4\pi\varepsilon_0 c}\frac{U^\mu(\tau_0)}{-R_\nu U^\nu(\tau_0)}$$ □

this is a correct derivation but it's not what the problem asked. See tutorial

Question 5

The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\kappa^3}\left(\frac{\hat{\mathbf{n}} - \mathbf{v}/c}{\gamma^2 r^2} + \frac{\hat{\mathbf{n}}\wedge[(\hat{\mathbf{n}} - \mathbf{v}/c)\wedge\mathbf{a}]}{c^2 r}\right)$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$ and $\kappa = 1 - v_r/c = 1 - \hat{\mathbf{n}}\cdot\mathbf{v}/c$, and

$$\mathbf{B} = \hat{\mathbf{n}}\wedge\mathbf{E}/c$$

where $\mathbf{r}$ is the vector from the source point to the field point, and $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified? A charged particle moves along the x axis with constant proper acceleration ("hyperbolic motion"), its worldline being given by

$$x^2 - t^2 = \alpha^2$$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows:

(i) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$$x_s = \alpha + \frac{y^2}{2\alpha}$$

(ii) Show that the velocity and acceleration at the source event are

$$v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}$$

$$a_s = \frac{\alpha^2}{x_s^3}$$

(iii) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of x_s v_s, and a_s. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\hat{\mathbf{n}}, \mathbf{v}, \mathbf{a}$, and $\hat{\mathbf{n}}\wedge(\hat{\mathbf{n}}\wedge\mathbf{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.

(iv) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis such that their separation is constant, comment on the forces they exert on one another.

Solution. For this question only we use the natural unit $c = 1$.

(i) The field propagates in the speed of light. We have

$$(x_s - \alpha)^2 + y^2 = t^2$$

In addition, the motion of the source satisfies

$$x_s^2 - t^2 = \alpha^2$$

Subtracting the two equations we obtain

$$2\alpha^2 - 2x_s\alpha + y^2 = 0 \implies x_s = \alpha + \frac{y^2}{2\alpha}$$

(ii) Differentiate the equation of motion with repsect to t to obtain the velocity. The time $t = -\sqrt{x_s^2 - \alpha^2}$ because the casual relation.

$$x_s^2 - t^2 = \alpha^2 \implies 2x_s v_s = 2t = -2\sqrt{x_s^2 - \alpha^2} \implies v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}$$

The acceleration is given by

$$a_s = \frac{\mathrm{d}v_s}{\mathrm{d}t} = v_s \frac{\mathrm{d}v_s}{\mathrm{d}x_s} = v_s \left(\frac{1}{\sqrt{x_s^2 - \alpha^2}} + \frac{v_s}{x_s} \right) = \frac{1}{x_s} - \frac{x_s^2 - \alpha^2}{x_s^3} = \frac{\alpha^2}{x_s^3}$$

(iii) For $\alpha = 1$ and $y = 2$, we have $x_s = 3$, $v_s = -\frac{2\sqrt{2}}{3}$ and $a_s = \frac{1}{27}$. Then

$$\mathbf{r} = (-2,2,0), \quad \hat{\mathbf{n}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \mathbf{v} = \left(-\frac{2\sqrt{2}}{3}, 0, 0\right), \quad \mathbf{a} = \left(\frac{1}{27}, 0, 0\right), \quad \hat{\mathbf{n}} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a}) = \left(-\frac{1}{54}, \frac{1}{54}, 0\right), \quad \kappa = \frac{1}{3}, \quad \gamma = 3$$

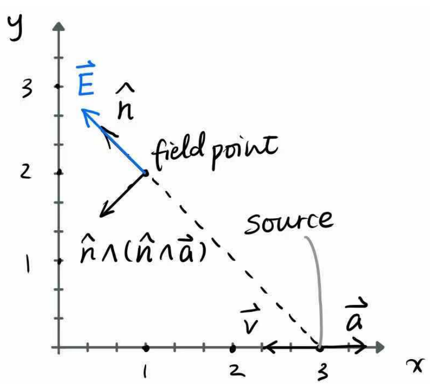
The electric field at $(0,1,2,0)$ is given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 \kappa^3 r} \left(\frac{\hat{\mathbf{n}} - \mathbf{v}}{\gamma^2 r} + \hat{\mathbf{n}} \wedge ((\hat{\mathbf{n}} - \mathbf{v}) \wedge \mathbf{a}) \right)$$

where

$$\frac{\hat{\mathbf{n}} - \mathbf{v}}{\gamma^2 r} + \hat{\mathbf{n}} \wedge ((\hat{\mathbf{n}} - \mathbf{v}) \wedge \mathbf{a}) = \left(-\frac{1}{108}, \frac{1}{108}, 0\right)$$

So the direction of $\mathbf{E}$ is $\frac{1}{\sqrt{2}}(-1,1,0)$.



(iv) The two charges repel each other through Coulomb force. Apart from the force along the rod, there is another component of the electric field which points towards the direction of velocity. Then the two charges as a whole exert a force on itself which is against the deceleration. □

Question 6

The far field due to an elementary wire segment dz carrying oscillating current I is given by

$$dE = \frac{I\sin\theta}{2\varepsilon_0 cr}\frac{dz}{\lambda}\cos(kr-\omega t)$$

Compare and contrast the case of a short antenna and the *half-wave dipole antenna*. Roughly estimate E in the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennae?

Solution. A thin antenna is modelled by an oscillating sinusoidal current given by $I(z,t) = I_0 \sin\left(\frac{kL}{2} - k|z|\right)\mathrm{e}^{-\mathrm{i}\omega t}$, so that the current at the end points $z = \pm\frac{L}{2}$ is identically zero. The time dependence of I is absorbed in the far field expression of the electric field.

For the short antenna approximation, we assume that $kL \ll 1$. We use a Taylor expansion on the current:

$$I(z) \sim I_0\left(1 - \frac{2|z|}{L}\right)$$

The far field electric field is given by

$$E = \frac{\sin\theta\cos(kr-\omega t)}{2\varepsilon_0 cr\lambda}\int I(z)\,\mathrm{d}z = \frac{\sin\theta\cos(kr-\omega t)}{\varepsilon_0 cr\lambda}\int_0^{L/2} I_0\left(1-\frac{2z}{L}\right)\mathrm{d}z = \frac{I_0 L\sin\theta\cos(kr-\omega t)}{4\varepsilon_0 cr\lambda}$$

For the half-wave dipole antenna, we have $L = \lambda/2$. Then

$$I(z) = I_0\sin\left(k\left(\frac{\lambda}{4} - |z|\right)\right) = I_0\cos kz$$

The far field electric field is given by

$$E = \frac{\sin\theta\cos(kr-\omega t)}{2\varepsilon_0 cr\lambda}\int I(z)\,\mathrm{d}z = \frac{\sin\theta\cos(kr-\omega t)}{2\varepsilon_0 cr\lambda}\int_{-\lambda/4}^{\lambda/4} I_0\cos kz\,\mathrm{d}z = \frac{I_0\sin\theta\cos(kr-\omega t)}{2\pi\varepsilon_0 cr}$$

For longer antennae, we should bring back the phase factor $\mathrm{e}^{\mathrm{i}kz\cos\theta}$ in the integral, which we have neglected for the previous cases. The far field electric field is given by

$$E = \frac{I_0\cos(kr-\omega t)}{\varepsilon_0 cr}\int_0^{L/2}\sin\left(\frac{kL}{2} - kz\right)\cos(kz\cos\theta)\,\mathrm{d}z = \frac{I_0\cos(kr-\omega t)}{\varepsilon_0 cr}\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta}$$

Since $L > \lambda_2$, the superposition of electric field is significantly out of phase. We should observe a more complicated θ-distribution of the electric field. □

Question 7

Show that the space-space part of the energy-momentum tensor

$$T^{\mu\nu} = \varepsilon_0 c^2\left(-F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}\right)$$

is

$$\sigma^{ij} = \frac{1}{2}\varepsilon_0\left(E^k E_k + c^2 B^k B_k\right)\delta^{ij} - \varepsilon_0\left(E^i E^j + c^2 B^i B^j\right)$$

(Greek indices run over space and time, and Latin indices over space only.)

Use the stress-energy tensor $T^{\mu\nu}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 tesla. *Mu-metal* is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it "confines" the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

Solution. The electromagnetic field tensor of type $(2,0)$, $(1,1)$ and $(0,2)$ are given respectively by

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (F^{\mu}_{\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

From Question 9 in Sheet 3 we have shown that

$$F^{\mu\nu}F_{\mu\nu} = -\frac{2E^2}{c^2} + 2B^2$$

Next we compute $F^{\mu\lambda}F^{\nu}_{\lambda}$:

$$\begin{aligned}\left(F^{\mu\lambda}F^{\nu}_{\lambda}\right) &= \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{E}^{\mathsf{T}}/c \\ -\mathbf{E}/c & \mathscr{B} \end{pmatrix}\begin{pmatrix} 0 & -\mathbf{E}^{\mathsf{T}}/c \\ -\mathbf{E}/c & \mathscr{B} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{E}^{\mathsf{T}}\mathscr{B}/c \\ -\mathscr{B}\mathbf{E}/c & \mathbf{E}\mathbf{E}^{\mathsf{T}}/c^2 + \mathscr{B}^2 \end{pmatrix}\end{aligned}$$

where $\mathscr{B} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix} = (B_{ij})$ so that $B_{ij} = \varepsilon_{ijk}B^k$. Then

$$(\mathscr{B}^2)^i_j = \mathscr{B}^i_k\mathscr{B}^k_j = B_{ik}B_{kj} = \varepsilon_{ik\ell}\varepsilon_{kjm}B^\ell B^m = \left(\delta_{im}\delta_{j\ell} - \delta_{ij}\delta_{\ell m}\right)B^\ell B^m = B^iB^j - \delta_{ij}B^2$$

Hence $F^{i\nu}F^{j}_{\nu} = E^iE^j/c^2 + B^iB^j - \delta^{ij}B^2$. Substituting back to the energy-momentum tensor:

$$\begin{aligned}\sigma^{ij} &= \varepsilon_0 c^2\left(-F^{i\nu}F^j_\nu - \frac{1}{4}\delta^{ij}F_{\kappa\lambda}F^{\kappa\lambda}\right) = \varepsilon_0 c^2\left(-E^iE^j/c^2 - B^iB^j + \delta^{ij}B^2 + \delta^{ij}\left(\frac{E^2}{2c^2} - \frac{B^2}{2}\right)\right) \\ &= \frac{1}{2}\varepsilon_0\left(E^kE_k + c^2B^kB_k\right)\delta^{ij} - \varepsilon_0\left(E^iE^j + c^2B^iB^j\right)\end{aligned}$$

For a long straight solenoid, the fields are given by

$$\mathbf{E} = 0, \qquad \mathbf{B} = B\mathbf{e}_z$$

Hence

$$\sigma^{ij} = \varepsilon_0 c^2\left(\frac{1}{2}B^2\delta^{ij} - B^iB^j\right) \implies \left(\sigma^{ij}\right) = \frac{1}{2}\varepsilon_0 c^2 B^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Suppose that the solenoid occupies $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, 0 \leqslant z \leqslant L\}$. The μ-metal is placed at $z = L$. On $z = L^-$, the stress tensor component σ^{zz} is negative. On $z = L^+$, $\sigma^{zz} = 0$. The force exerted on the μ-metal is given by

$$F = \iint_{\{(x,y,z):\ x^2+y^2 \leqslant r^2,\ z=L\}} \sigma^{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j = -\frac{1}{2}\varepsilon_0 c^2 B^2 \cdot \pi r^2 = -1125\ \mathrm{N}$$

The minus sign indicates that the force is attractive. □

Question 8

Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction.

Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u, and frequency ν in the two frames satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$$

(Optional: can you prove this for any relative motion of the frame? [Hint: write $T^{\mu\nu}$ in terms of K^μ])

A confused student proposes that these quantities should transform like ν'/ν rather than ν'^2/ν^2, on the grounds that energy-momentum $N^\mu = (uc, \mathbf{N})$ is a 4-vector and so should transform in the same way as the wave vector. What is wrong with this argument?

Solution. Consider the plane wave travelling in the z-direction:

$$\mathbf{E} = E_0 \cos(kx - \omega t)\mathbf{e}_x, \qquad \mathbf{B} = B_0 \cos(kx - \omega t)\mathbf{e}_y, \qquad \mathbf{k} = k\mathbf{e}_z$$

The energy density is given by

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \frac{E_0^2}{\mu_0 c^2}\cos^2(kx - \omega t)$$

The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B} = \frac{E_0^2}{\mu_0 c^2}\cos^2(kx - \omega t)\mathbf{e}_z$$

The momentum flow has the (z, z)-component only:

$$\sigma^{zz} = \frac{E_0^2}{\mu_0 c^2}\cos^2(kx - \omega t)$$

Hence the energy-momentum tensor is given by

$$\left(T^{\mu\nu}\right) = \frac{E_0^2}{\mu_0 c^2}\cos^2(kx - \omega t)\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

In general the tensor is expressed as

$$T^{\mu\nu} = \frac{E_0^2}{\mu_0 c^2 \omega^2}\cos^2(K^\lambda X_\lambda)K^\mu K^\nu$$

from which we find that E_0^2/ω^2 is invariant under Lorentz transformations.

Note that $u, P, g \propto E_0^2$ and $\nu \propto \omega^2$. We deduce that in two different inertial frames the values of the quantities satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$$

The "energy-momentum" $N = (uc, \mathbf{N})$ is not a tangent vector field in the Minkowski spacetime, so it does not transform like a 4-vector under Lorentz transformations. In fact N is a part of the energy-momentum tensor, which is a tensor field. □