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Notes on
Category Theory

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Examinable Syllabus

C2.7 Category Theory

- Introduction: universal properties in linear and multilinear algebra.
- Categories, functors, natural transformations. Examples including categories of sets, groups, rings, vector spaces and modules, topological spaces. Groups, monoids and partially ordered sets as categories. Opposite categories and the principle of duality. Covariant, contravariant, faithful and full functors. Equivalences of categories.
- Adjoints: definition and examples including free and forgetful functors and abelianisations of groups. Adjunctions via units and counits, adjunctions via initial objects.
- Representables: definitions and examples including tensor products. The Yoneda lemma and applications.
- Limits and colimits, including products, equalizers, pullbacks and pushouts. Monics and epics. Interaction between functors and limits.
- Monads and comonads, algebras over a monad, Barr-Beck monadicity theorem (proof not examinable). The category of affine schemes as the opposite of the category of commutative rings.

C2.2 Homological Algebra

- Overview of category theory: adjoint functors, limits and colimits, Abelian categories.
- Chain complexes: complexes of R-modules and in an abelian category, operations on chain complexes, long exact sequences, chain homotopies, mapping cones and cylinders.
- Derived functors: delta functors, projective and injective resolutions, left and right derived functors, adjoint functors and exactness, balancing Tor and Ext.
- Tor and Ext: Tor and flatness, Ext and extensions, universal coefficients theorems, Kunneth formula, Koszul resolutions.
- Group homology and cohomology: definition, basic properties, cyclic groups, interpretation of H^1 and H^2, the Bar resolution.

Chapter 1

Basic Concepts

1.1 Categories and Morphisms

Let us go straight into the central definition.

Definition 1.1. Categories

A category C consists of the following data:

1. A class of **objects**, denoted by $\mathrm{Obj}(\mathsf{C})$;
2. A class of **morphisms**, denoted by $\mathrm{Mor}(\mathsf{C})$;
3. The maps $\iota : \mathrm{Mor}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{C})$ and $\tau : \mathrm{Mor}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{C})$, which assigns the **source** object and the **target** object to a morphism.

   For $f \in \mathrm{Mor}(\mathsf{C})$, if $X = \iota(f)$ and $Y = \tau(f)$, then we say that f is a morphism from X to Y, and write

   $$f : X \longrightarrow Y \qquad \text{or} \qquad X \xrightarrow{\ f\ } Y$$

   The class of morphisms from X to Y is denoted by $\mathrm{Hom}_{\mathsf{C}}(X, Y) := \iota^{-1}(X) \cap \tau^{-1}(Y)$.
4. An **identity** morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ for each object X.
5. For objects X, Y, Z, we have the **composition** of morphisms:

   $$\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, Z), \qquad (f, g) \mapsto f \circ g$$

   satisfying

   (a) Associativity: Suppose that X, Y, Z, W are objects. Let $f \in \mathrm{Hom}_{\mathsf{C}}(Z, W)$, $g \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$, $h \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$. Then

   $$f \circ (g \circ h) = (f \circ g) \circ h$$

   (b) For any morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we have

   $$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$$

Remark. Set-Theoretic Issues.

A problem related to the Zermelo-Fraenkel set theory is that, for the definition above, $\mathrm{Obj}(\mathsf{C})$ and $\mathrm{Mor}(\mathsf{C})$ may be proper classes. Our way to avoid the issue is to introduce the following concept:

Definition 1.2. Grothendieck Universe

A set $\mathscr{U}$ is called a Grothendieck universe, if it satisfies:

1. Transitivity: $\forall x(x \in \mathscr{U} \implies x \subseteq \mathscr{U})$;
2. Pairing: $\forall x \forall y(x, y \in \mathscr{U} \implies \{x, y\} \in \mathscr{U})$;
3. Power set: $\forall x(x \in \mathscr{U} \implies \mathscr{P}(x) \in \mathscr{U})$;
4. Union: $\forall x \forall f\left((x \in \mathscr{U} \wedge f : x \to \mathscr{U}) \implies \bigcup_{i \in x} f(i) \in \mathscr{U}\right)$;
5. Minimal inductive set: $\mathbb{N} \in \mathscr{U}$.

A set x is said to be $\mathscr{U}$-small, if $x \in \mathscr{U}$.

We say that a category C is **locally $\mathscr{U}$-small**, if $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a $\mathscr{U}$-small set for any objects X, Y.
We say that a category C is **$\mathscr{U}$-small**, if $\mathrm{Mor}(\mathsf{C})$ is a $\mathscr{U}$-small set.

We adopt the following hypothesis, which is independent of ZFC:

Axiom 1.3. Tarski's Axiom

For each set x there exists a Grothendieck universe $\mathscr{U}$ such that $x \in \mathscr{U}$.

From now on we fix a Grothendieck universe $\mathscr{U}$, and assume that *all sets and categories involved in our discussion are $\mathscr{U}$-small.*

We give some constructions of categories.

Definition 1.4. Subcategories

Suppose that C is a category. We say that C' is a subcategory of C, if:

1. $\mathrm{Obj}(\mathsf{C}') \subseteq \mathrm{Obj}(\mathsf{C})$ and $\mathrm{Mor}(\mathsf{C}') \subseteq \mathrm{Mor}(\mathsf{C}')$;
2. For $X \in \mathrm{Obj}(\mathsf{C}')$, the identity morphisms of $\mathrm{Hom}_{\mathsf{C}}(X, X)$ and $\mathrm{Hom}_{\mathsf{C}'}(X, X)$ coincide;
3. The source map $\iota' : \mathrm{Mor}(\mathsf{C}') \to \mathrm{Obj}(\mathsf{C}')$, the target map $\tau' : \mathrm{Mor}(\mathsf{C}') \to \mathrm{Obj}(\mathsf{C}')$, and the compositions of morphisms are restricted from those of C.

If $\mathrm{Hom}_{\mathsf{C}'}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for any $X, Y \in \mathrm{Obj}(\mathsf{C}')$, then we say that C' is a **full subcategory** of C.

Definition 1.5. Opposite Categories

Suppose that C is a category. The opposite category C^{op} consists of the following data:

1. $\mathrm{Obj}(\mathsf{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathsf{C})$;
2. For any objects X, Y, $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y)$;
3. For morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(Y, Z)$ and $g \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y)$, the composition $f \circ^{\mathrm{op}} g$ in C^{op} is the reversed composition $g \circ f$ in C;
4. the identity morphisms of $\mathrm{Hom}_{\mathsf{C}}(X, X)$ and $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, X)$ coincide.

Remark. In other words, the opposite category C^{op} is C with all arrows reversed.

Next we give some examples of categories.

Example 1.6. Examples of Categories

1. Let $(S, \leqslant)$ be a **poset** (partially ordered set). S is a category, where the objects are elements of S, and for any $a, b \in S$, $\mathrm{Hom}(a, b)$ is a singleton if $a \leqslant b$ and is empty otherwise.

For any $n \in \mathbb{N}$, n is the totally order set $\{0, \ldots, n-1\}$, so we can define a corresponding category structure on n. We denote the resulting category by n.

2. The category of ($\mathscr{U}$-small) sets is denoted by Set. The morphisms are maps between sets.
3. The category of groups is denoted by Grp. The morphisms are group homomorphisms. Grp has a full subcategory Ab, whose objects are the Abelian groups.
4. The category of rings is denoted by Rng. The morphisms are ring homomorphisms. The category of rings with multiplicatively identity is denoted by Ring. The morphisms are unital ring homomorphisms.
5. The category of left-modules over the ring R is denoted by R-Mod. The morphisms are left R-module homomorphisms. Similarly, the category of right-modules over the ring R is denoted by Mod-R.
6. The category of vector spaces over a field k is denoted by k-Vect. It has a full subcategory k-$\mathsf{Vect}^{\mathrm{fd}}$, whose objects are finite-dimensional vector spaces.
7. The category of topological spaces is denoted by Top. The morphisms are continuous maps.
8. Given a set S, we can define the **discrete category** $\mathsf{Disc}(S)$, whose objects are elements are elements of S, and morphisms are the identity morphisms id_x for each $x \in S$.

The generalisation of injections and surjections in category theory are monomorphisms and epimorphisms.

Definition 1.7. Monomorphisms, Epimorphisms, Isomorphisms

Suppose that C is a category. Let $X, Y \in \mathrm{Obj}(\mathsf{C})$. We say that $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ is

1. a monomorphism, if for any $Z \in \mathrm{Obj}(\mathsf{C})$ and $g, h \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$, we have that $f \circ g = f \circ h \implies g = h$. (*Left cancellation*)
2. an epimorphism, if for any $Z \in \mathrm{Obj}(\mathsf{C})$ and $g, h \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$, we have that $g \circ f = h \circ f \implies g = h$. (*Right cancellation*)
3. an isomorphism, if there exists $g \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

If there exists an isomorphism $f : X \to Y$, then we say that X and Y are isomorphic and write $X \cong Y$.

Remark. In the categories Set and Grp, the monomorphisms and epimorphisms are exactly the injective and surjective maps. This is not the case in Ring. For example, the inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in Ring, but is certainly not a surjective map. This also warns us that

$$\text{monomorphism} + \text{epimorphism} \neq \text{isomorphism}$$

Remark. A category in which each morphism is an isomorphism is called a **groupoid**. A group is a category with a single object (the group elements are the morphisms).

1.2 Functors and Natural Transformations

Definition 1.8. Functors

Let C and C' be categories. A functor $F : \mathsf{C} \to \mathsf{C}'$ consists of the following data:

1. A map between objects $F : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{C}')$.
2. For $X, Y \in \mathrm{Obj}(\mathsf{C})$, a map between morphisms $F : \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}'}(FX, FY)$, satisfying
   (a) For $X \in \mathrm{Obj}(\mathsf{C})$, we have $F(\mathrm{id}_X) = \mathrm{id}_{FX}$;
   (b) For $X, Y, Z \in \mathrm{Obj}(\mathsf{C})$, $f \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we have $F(f \circ g) = F(f) \circ F(g)$.

We say that a functor $F : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{C}')$ is

1. **faithful**, if for any $X, Y \in \mathrm{Obj}(\mathsf{C})$, the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}'}(FX, FY)$ is injective;
2. **full**, if for any $X, Y \in \mathrm{Obj}(\mathsf{C})$, the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}'}(FX, FY)$ is surjective;
3. **fully faithful**, if for any $X, Y \in \mathrm{Obj}(\mathsf{C})$, the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}'}(FX, FY)$ is bijective.

Remark. In old literature, the functor $F : \mathsf{C} \to \mathsf{C}'$ is called a **covariant functor**. $G : \mathsf{C} \to \mathsf{C}'$ is called a **contravariant functor**, if is a functor $G : \mathsf{C}^{\mathrm{op}} \to \mathsf{C}'$.

Functors compose in the natural way: If $G : \mathsf{C} \to \mathsf{C}'$ and $F : \mathsf{C}' \to \mathsf{C}''$ are two functors, then the composite functor $F \circ G : \mathsf{C} \to \mathsf{C}''$ consists of the composite morphisms

$$\mathrm{Obj}(\mathsf{C}) \xrightarrow{\quad G \quad} \mathrm{Obj}(\mathsf{C}') \xrightarrow{\quad F \quad} \mathrm{Obj}(\mathsf{C}'')$$

$$\mathrm{Hom}_{\mathsf{C}}(X,Y) \xrightarrow{G} \mathrm{Hom}_{\mathsf{C}'}(GX,GY) \xrightarrow{F} \mathrm{Hom}_{\mathsf{C}''}(FGX,FGY)$$

Example 1.9. Examples of Functors

1. For a category C there is obviously an identity functor $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$.
2. A subcategory C' of C gives the inclusion functor $\iota : \mathsf{C}' \hookrightarrow \mathsf{C}$, which is always faithful. ι is full if C' is a full subcategory.
3. Let G be a group. We can always "forget" the group structure on G and regard it as a set. This defines a **forgetful functor** $\mathsf{Grp} \to \mathsf{Set}$. The functor is faithful and not full.
4. The Abelianisation $G \mapsto G/G'$, where the derived subgroup is defined by $G' := \{xyx^{-1}y^{-1} : x, y \in G\}$, gives a functor $\mathsf{Grp} \to \mathsf{Ab}$. The functor is neither faithful nor full.
5. For any based topological space (X, b) we can define its fundamental group $\pi_1(X, b)$. It gives a functor $\mathsf{Top}_\bullet \to \mathsf{Grp}$, where $\mathsf{Top}_\bullet$ is the category of based topological spaces. Similarly, the homology groups $X \mapsto H_i(X, \mathbb{Z})$ gives a family of functors $\mathsf{Top} \to \mathsf{Ab}$.
6. The category of all ($\mathscr{U}$-small) category is denoted by Cat. The morphisms are functors between categories.
7. Let S be a set. We can define a discrete category $\mathsf{Disc}(S)$ on S. This gives a fully faithful functor $\mathsf{Set} \to \mathsf{Cat}$.

Definition 1.10. Natural Transformations

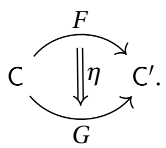
Let $F, G : \mathsf{C} \to \mathsf{C}'$ be two functors. A natural transformation $\eta : F \Rightarrow G$ is a family of morphisms $\eta_X \in \mathrm{Hom}_{\mathsf{C}'}(FX, GX)$, $X \in \mathsf{C}$, such that the diagram commutes

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

for any morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$.

We say that $\eta : F \Rightarrow G$ is a **natural isomorphism**, if every $\eta_X \in \mathrm{Hom}_{\mathsf{C}'}(FX, GX)$ is an isomorphism. We shall write $F \simeq G$ if there is a natural isomorphism between F and G.

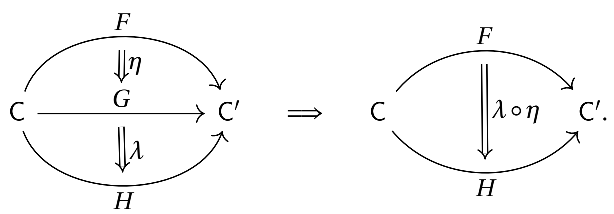
Natural transformations are often represented by the diagram



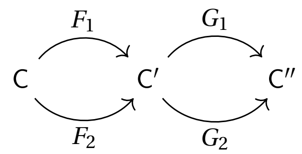
Remark. It is instructive to think natural transformations as *homotopies* between two functors.

Compositions of Natural Transformations.

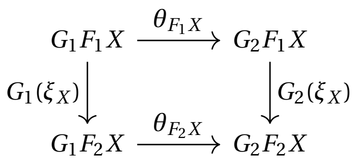
1. For three functors $F, G, H : \mathsf{C} \to \mathsf{C}'$ and natural transformations $\eta : F \Rightarrow G$ and $\lambda : G \Rightarrow H$, we can define the **vertical composition** $\lambda \circ \eta : F \Rightarrow H$ to be the the family $\{\lambda_X \circ \eta_X : X \in \mathrm{Obj}(\mathsf{C})\}$. It is represented by the diagram



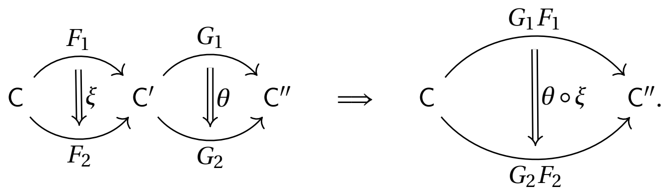
2. Consider the following diagram:



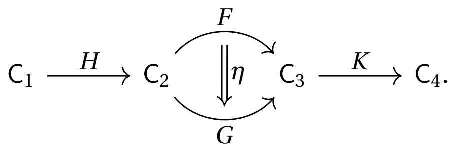
with natural transformations $\xi : F_1 \Rightarrow F_2$ and $\theta : G_1 \Rightarrow G_2$. For each $X \in \mathrm{Obj}(\mathsf{C})$ we have the commutative diagram



We can define the **horizontal composition** $\theta \circ \xi : G_1 \circ F_1 \Rightarrow G_2 \circ F_2$ such that, $(\theta \circ \xi)_X : G_2F_2X \to G_1F_1X$ is given by the diagonal composition of the above diagram for each $X \in \mathrm{Obj}(\mathsf{C})$. It is represented by the diagram



3. As a special case of horizontal composition, we can compose a natural transformation with functors. Consider the following diagram:



We can define $\eta \circ H : FH \Rightarrow FG$ to be the horizontal composition $\eta \circ \mathrm{id}_H$. And similarly define $K \circ \eta : KF \Rightarrow KG$ to be the horizontal composition $\mathrm{id}_K \circ \eta$. Here id_H and id_K are identity natural transformations on the functors H and K respectively.

Definition 1.11. Quasi-Inverse, Equivalence

Let C and C' be two categories. We say that the functor $F : \mathsf{C} \to \mathsf{C}'$ is an equivalence of categories C and C', if there exist functor $G : \mathsf{C}' \to \mathsf{C}$ such that $G \circ F \simeq \mathrm{id}_{\mathsf{C}}$ and $F \circ G \simeq \mathrm{id}_{\mathsf{C}'}$. G is called a quasi-inverse of F.

If we have $G \circ F = \mathrm{id}_{\mathsf{C}}$ and $F \circ G = \mathrm{id}_{\mathsf{C}'}$, then we say that G is the **inverse** of F. We say that C and C' are **isomorphic** categories.

Lemma 1.12. Uniqueness of Quasi-Inverse

Suppose that G_1 and G_2 are quasi-inverses of the functor $F : \mathsf{C} \to \mathsf{C}'$. Then there exists a natural isomorphism $\eta : G_1 \Rightarrow G_2$.

Proof. We have the horizontal composition of natural isomorphisms:

$$G = \mathrm{id}_{\mathsf{C}'} \circ G \simeq G' \circ F \circ G \simeq G' \circ \mathrm{id}_{\mathsf{C}} = G'$$ □

Definition 1.13. Essential Surjection

Let C and C' be categories. We say that a functor $F : \mathsf{C} \to \mathsf{C}'$ is essentially surjective, if for any $Y \in \mathrm{Obj}(\mathsf{C}')$ there exists $X \in \mathrm{Obj}(\mathsf{C})$ such that $FX \cong Y$.

Theorem 1.14

Let C and C' be categories. For a functor $F : \mathsf{C} \to \mathsf{C}'$, the following are equivalent:

1. F is an equivalence;

2. F is fully faithful and essentially surjective.

Proof.

$1 \Longrightarrow 2$: Let $G : \mathsf{C}' \to \mathsf{C}$ be a quasi-inverse of F. Let $\varphi : FG \Rightarrow \mathrm{id}_{\mathsf{C}'}$ and $\psi : GF \Rightarrow \mathrm{id}_{\mathsf{C}}$ be natural isomorphisms.

For $X \in \mathrm{Obj}(\mathsf{C}')$, we have $X = \varphi_X(FGX) \cong FGX$. Hence F is essentially surjective.

For $X, Y \in \mathsf{C}$. the composition

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathsf{C}}(X,Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathsf{C}'}(FX,FY) & \xrightarrow{G} & \mathrm{Hom}_{\mathsf{C}}(GFX,GFY) & \xrightarrow{\psi} & \mathrm{Hom}_{\mathsf{C}}(X,Y) \\ f & \longmapsto & F(f) & \longmapsto & GF(f) & \longmapsto & \psi_Y \circ GF(f) \circ \psi_X^{-1} \end{array}$$

is the identity map on $\mathrm{Hom}_{\mathsf{C}}(X,Y)$. Hence F is faithful. Interchanging F and G we find that G is also faithful. Hence F must be full.

$2 \Longrightarrow 1$: We shall construct a full subcategory D of C, which is called the **skeleton** of C as follows.

Using the axiom of choice we can choose a representative for each isomorphism class of objects in C. Let D be the full subcategory of C with these objects. Let $\iota : \mathsf{D} \to \mathsf{C}$ be the inclusion functor. We shall show that ι is an equivalence. For each $X \in \mathrm{Obj}(\mathsf{C})$, we can choose an isomorphism $\eta_X : X \to \kappa(X) \in \mathsf{D}$. Furthermore we choose $\eta_X = \mathrm{id}_X$ if $X \in \mathrm{Obj}(\mathsf{D})$. Now we have a map $\kappa : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{D})$. We extend it to a functor $\kappa : \mathsf{C} \to \mathsf{D}$ by defining

$$\kappa(f) := \eta_Y \circ f \circ \eta_X^{-1} \in \mathrm{Hom}_{\mathsf{D}}(\kappa(X), \kappa(Y))$$

for $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$. In this way, $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow \iota\kappa$ becomes a natural isomorphism. Since $\kappa\iota = \mathrm{id}_{\mathsf{D}}$, we deduce that κ is a quasi-inverse of ι. Hence D is equivalent to C.

Similarly we construct the skeleton D' of C'. Let $\iota' : \mathsf{D}' \to \mathsf{C}'$ be the inclusion functor and $\kappa' : \mathsf{C}' \to \mathsf{D}'$ be its quasi-inverse. It is easy to verify that $G := \kappa' \circ F \circ \iota : \mathsf{D} \to \mathsf{D}'$ is still fully faithful and essentially surjective.

For $Y \in \mathrm{Obj}(\mathsf{D}')$, there exists $X \in \mathrm{Obj}(\mathsf{D})$ such that $GX \cong Y$, by essential surjectivity of G. But this implies that $GX = Y$ by our construction. Hence $G : \mathrm{Obj}(\mathsf{D}) \to \mathrm{Obj}(\mathsf{D}')$ is surjective. Suppose that $X, Y \in \mathrm{Obj}(\mathsf{D})$ such that $GX = GY$. Since G is fully faithful, we have $X \cong Y$. Then $X = Y$ again by our construction. Hence $G : \mathrm{Obj}(\mathsf{D}) \to \mathrm{Obj}(\mathsf{D}')$ is injective. We deduce that $G : \mathsf{D} \to \mathsf{D}'$ is invertible. Let $H := \iota \circ G^{-1} \circ \kappa' : \mathsf{C}' \to \mathsf{C}$. We have

$$F \circ H = F \circ \iota \circ G^{-1} \circ \kappa' \simeq \iota' \circ \kappa' \circ F \circ \iota \circ G^{-1} \circ \kappa' = \iota' \circ G \circ G^{-1} \circ \kappa' = \iota' \circ \kappa' \simeq \mathrm{id}_{\mathsf{C}'}$$
$$H \circ F = \iota \circ G^{-1} \circ \kappa' \circ F \simeq \iota \circ G^{-1} \circ \kappa' \circ \iota \circ \kappa = \iota \circ G^{-1} \circ G \circ \kappa = \iota \circ \kappa \simeq \mathrm{id}_{\mathsf{C}}$$

Hence H is a quasi-inverse of F. F is an equivalence. □

Example 1.15. Some Elementary Linear Algebra

We fix a field k. Recall that $k\text{-}\mathsf{Vect}^{\mathrm{fd}}$ is the category of finite-dimensional vector spaces over k. We define the category Mat as follows: The object set $\mathrm{Obj}(\mathsf{Mat}) = \mathbb{N}$. For $m, n \in \mathbb{N}$, we define $\mathrm{Hom}_{\mathsf{Mat}}(m,n) := M_{m\times n}(k)$, the set of all $m \times n$ matrices over k. The composition of morphisms is the usual matrix multiplication. By convention, $M_{0\times n}(k) = M_{m\times 0}(k) = \{0\}$.

We define a functor $F : \mathsf{Mat} \to k\text{-}\mathsf{Vect}^{\mathrm{fd}}$ as follows. For $n \in \mathrm{Obj}(\mathsf{Mat})$, $F(n) := k^n = M_{n\times 1}(k)$. For $A \in \mathrm{Hom}_{\mathsf{Mat}}(m,n)$, the linear map $F(A) : k^m \to k^n$ is the matrix multiplication $v \mapsto Av$.

It is easy to verify that F is a fully faithful and essentially surjective functor (just linear algebra!). So the categories Mat and $k\text{-}\mathsf{Vect}^{\mathrm{fd}}$ are equivalent.

Definition 1.16. Functor Categories

Given two ($\mathscr{U}$-small) categories C and C', the functor category $\mathsf{Fun}(\mathsf{C},\mathsf{C}')$ consists of the following data:

- The objects of $\mathsf{Fun}(\mathsf{C},\mathsf{C}')$ are the morphisms from C to C';
- The morphisms from $F : \mathsf{C} \to \mathsf{C}'$ to $G : \mathsf{C} \to \mathsf{C}'$ are the natural transformations $F \Rightarrow G$. (This is why we call natural transformations *morphisms of functors*.)
- The composition of morphisms is given by the vertical composition of natural transformations.

Lemma 1.17

Let C and C' be categories. There is a natural isomorphism $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{C}'^{\mathrm{op}}) \simeq \mathsf{Fun}(\mathsf{C}, \mathsf{C}')^{\mathrm{op}}$, in which every natural transformation $\eta : F \Rightarrow G$ is mapped to $\eta^{\mathrm{op}} : G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$.

1.3 Universal Properties

Definition 1.18. Initial Objects, Final Objects, Null Objects

Let C be a category. An object $X \in \mathrm{Obj}(\mathsf{C})$ is said to be

1. initial, if $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a singleton for all $Y \in \mathrm{Obj}(\mathsf{C})$;
2. final, if $\mathrm{Hom}_{\mathsf{C}}(Y, X)$ is a singleton for all $Y \in \mathrm{Obj}(\mathsf{C})$;
3. null, if it is both initial and final.

Lemma 1.19

The initial and final objects are unique up to isomorphism.

Proof. Suppose that X and X' are initial in C. Then there exist unique morphisms $f : X \to X'$ and $g : X' \to X$. Hence $f \circ g : X' \to X'$ and $g \circ f : X \to X$ are also unique. Then $f \circ g = \mathrm{id}_{X'}$ and $g \circ f = \mathrm{id}_X$. We have $X \cong X'$. The proof for final objects are identical. □

Example 1.20. Examples of Initial and Final Objects

1. The poset category $(\mathbb{Z}, \leqslant)$ has no initial nor final objects.
2. In the category Set, the initial object is the empty set $\varnothing$, and the final objects are singletons $\{*\}$.
3. In the category Grp, $\{e\}$ is a null object.
4. In the category Ring, $\mathbb{Z}$ is an initial object.

The initial and final objects are the natural tools to express the universal properties which are already familar to readers through learning algebra. To make things rigorous, we need the following definition:

Definition 1.21. Comma Categories

Consider the functors $S : \mathsf{A} \to \mathsf{C}$ and $T : \mathsf{B} \to \mathsf{C}$. The comma category (S/T) consists of the following data:

- The objects are the triplets (A, B, f), where $A \in \mathrm{Obj}(\mathsf{A})$, $B \in \mathrm{Obj}(\mathsf{B})$, and $f \in \mathrm{Hom}_{\mathsf{C}}(SA, TB)$;
- The morphisms from (A, B, f) to (A', B', f') are pairs (g, h), where $g \in \mathrm{Hom}_{\mathsf{A}}(A, A')$ and $h \in \mathrm{Hom}_{\mathsf{B}}(B, B')$, such that the following diagram commutes:

$$\begin{array}{ccc} SA & \xrightarrow{S(g)} & SA' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ TB & \xrightarrow{T(h)} & TB' \end{array}$$

- The composition of morphisms is given by $(g_1, h_1) \circ (g_2, h_2) := (g_1 \circ g_2, h_1 \circ h_2)$. The identity morphism on (A, B, f) is $(\mathrm{id}_A, \mathrm{id}_B)$.

Example 1.22. Universal Property of Free Modules

Let R be a commutative ring with identity. The **free** R**-module** generated by the set X is the R-module $FX = \bigoplus_{x \in X} Rx$. There is a canonical inclusion map $\iota : X \hookrightarrow FX$. The free module FX can be characterised by the following universal property:

For any R-module N and set-map $f : X \to N$, there exists a unique R-module homomorphism $\varphi : FX \to N$ such that $f = \varphi \circ \iota$.

We construct a corresponding comma category as follows.

The free construction gives a functor $F : \mathsf{Set} \to R\text{-}\mathsf{Mod}$. Let $G : R\text{-}\mathsf{Mod} \to \mathsf{Set}$ be the forgetful functor. Then the canonical inclusion is a morphism $\iota : X \to GFX$.

The category 1 has a unique morphism. Any functor $j_X : 1 \to R\text{-}\mathsf{Mod}$ is equivalent to picking a object $X \in \mathrm{Obj}(R\text{-}\mathsf{Mod})$.

Consider the comma category $(X/G) := (j_X/G)$, corresponding to the functors $j_X : 1 \to \mathsf{Set}$ and $G : R\text{-}\mathsf{Mod} \to \mathsf{Set}$. The objects in (X/F) are of the form $(*, N, f)$, where $N \in \mathrm{Obj}(R\text{-}\mathsf{Mod})$ and $f \in \mathrm{Hom}_{\mathsf{Set}}(X, GN)$. The morphism from $(*, N, f)$ to $(*, N', f')$ is a R-module homomorphism $\psi : N \to N'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & & \searrow^{f'} & \\ f(GN) & & \xrightarrow{\ \psi\ } & & f'(GN') \end{array}$$

By the above definition, $(*, FX, \iota)$ is an initial object of (X/G). So by Lemma 1.19, the free module FX is unique up to an R-module homomorphism.

1.4 Adjoints

Definition 1.23. Products and Coproducts of Categories

Let I be a $\mathscr{U}$-small set. Let $\{\mathsf{C}_i : i \in I\}$ be a family of categories.

1. The product category $\prod_{i \in I} \mathsf{C}_i$ consists of the following data:
   - The object set $\mathrm{Obj}\left(\prod_{i \in I} \mathsf{C}_i\right) := \prod_{i \in I} \mathrm{Obj}(\mathsf{C}_i)$. The objects are of the form $(X_i)_{i \in I}$.
   - For $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$, the morphisms $\mathrm{Hom}_{\prod_{i \in I} \mathsf{C}_i}((X_i)_{i \in I}, (Y_i)_{i \in I}) := \prod_{i \in I} \mathrm{Hom}_{\mathsf{C}_i}(X_i, Y_i)$.
   - The composition of morphisms is defined to be component-wise.
2. The coproduct category $\coprod_{i \in I} \mathsf{C}_i$ consists of the following data:
   - The object set $\mathrm{Obj}\left(\coprod_{i \in I} \mathsf{C}_i\right) := \coprod_{i \in I} \mathrm{Obj}(\mathsf{C}_i)$. The objects are of the form $(X_i)_{i \in I}$.
   - The morphisms
   $$\mathrm{Hom}_{\coprod_{i \in I} \mathsf{C}_i}(X_j, X_k) := \begin{cases} \mathrm{Hom}_{\mathsf{C}_j}(X_j . X_k) & j = k \\ \varnothing & j \neq k \end{cases}$$
   where each $X_j \in \mathrm{Obj}(\mathsf{C}_j)$.

Example 1.24. Hom Functor

Given a category C, the assignment $(X, Y) \mapsto \mathrm{Hom}_{\mathsf{C}}(X, Y)$ defines a binary functor

$$\mathrm{Hom}_{\mathsf{C}} : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$$

Indeed, a pair of morphisms $f : X' \to X$ and $g : Y \to Y'$ in C induces $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X', Y')$ via $\varphi \mapsto g \circ \varphi \circ f$.

Definition 1.25. Adjunction

(F, G, φ) is called an adjunction, if $F : \mathsf{C}_1 \to \mathsf{C}_2$ and $G : \mathsf{C}_2 \to \mathsf{C}_1$ are a pair of functors, and φ is a natrual isomorphism of functors:

$$\varphi : \mathrm{Hom}_{\mathsf{C}_2}(F(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}_1}(-, G(-))$$

In practice φ is often omitted. We say that (F, G) is an adjoint pair, F is the left adjoint of G, and G is the right adjoint of F.

1.5 Representable Functors

1.6 Limits and Colimits

Chapter 2

Homological Algebra

2.1 Abelian Categories

2.2 Chain Complexes and Exact Sequences

2.3 Homology and Cohomology

2.4 Derived Functors

Notations

- Cat: The category of ($\mathscr{U}$-small) categories.
- Set: The category of sets.
- Grp: The category of groups.
- Ab: The category of Abelian groups.
- Rng: The category of rings.
- Ring: The category of rings with multiplicative identity.
- R-Mod: The category of left modules over the ring R.
- Mod-R: The category of right modules over the ring R.
- (R,S)-Mod: The category of (R,S)-bimodules, where R and S are rings.
- k-Vect: The category of vector spaces over the field k.
- k-$\mathsf{Vect}^{\mathrm{fd}}$: The category of finite-dimensional vector spaces over the field k.
- Mat: The category of column vectors over a field k, with morphisms represented by matrices.
- R-Alg: The category of associative algebras over the ring R.
- Top: The category of topological spaces.
- $\mathsf{Top}_\bullet$: The category of based topological spaces.
- $\mathsf{Fun}(\mathsf{C},\mathsf{C}')$: The category of functors from C to C'.
- $\mathbb{N}$: The set of non-negetive integers.
- $\mathbb{Z}_+$: The set of positive integers.
- $\mathbb{Z}$: The set of integers.
- $\mathbb{Q}$: The set of rational numbers.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{C}$: The set of complex numbers.

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