Topology & Groups

Michaelmas 2016

Question Sheet 7

Questions with an asterisk $*$ beside them are optional.

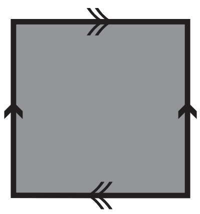
1. For each of the following subgroups of $\langle x, y\rangle = \pi_1(S^1 \vee S^1)$, construct a based covering map $p\colon (\tilde{X}, \tilde{b}) \to (S^1 \vee S^1, b)$ such that $p_*\pi_1(\tilde{X}, \tilde{b})$ is that subgroup:

   (i) $\langle x\rangle$

   (ii) $\{x^{n_1}y^{m_1}x^{n_2}y^{m_2}\dots y^{m_k} : \sum m_i \text{ is even}\}$.

   (iii) the kernel of the homomorphism $\langle x, y\rangle \to \mathbb{Z}\times\mathbb{Z}$ that sends x to $(1, 0)$, and y to $(0, 1)$.

2. Recall that the Klein bottle K is defined to be a square with the following side identifications.



   Construct a covering map from $\mathbb{R}^2$ to K and use it show that $\pi_1(K)$ is isomorphic to the group whose elements are pairs (m, n) of integers, with the non-abelian group operation given by

$$(m, n) \star (x, y) = (m + (-1)^n x, n + y).$$

3. Suppose that a group G has a left action on a path-connected space Y. This means that there is a homomorphism $\phi\colon G \to \mathrm{Homeo}(Y)$, where $\mathrm{Homeo}(Y)$ is the group of homeomorphisms of Y. This is known as a *covering space action* if each $y \in Y$ has an open neighbourhood U such that $\phi(g)(U) \cap U = \emptyset$ for each $g \in G - \{e\}$. Let Y/G denote the quotient space that identifies two points y_1 and y_2 in Y if and only if $\phi(g)(y_1) = y_2$ for some $g \in G$.

   (i) Prove that the quotient map $Y \to Y/G$ is a covering map.

   (ii) When Y is simply-connected, prove that $\pi_1(Y/G) \cong G$.

   (iii) Find a covering space action of $\mathbb{Z}\times\mathbb{Z}$ on $\mathbb{R}^2$, and use this to provide another proof that the fundamental group of the torus is $\mathbb{Z}\times\mathbb{Z}$.

4. Construct Cayley graphs for each of the following groups, with respect to the given generators:

   (i) $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^2, b^2 \rangle$, with respect to the generators a and b;

   (ii) $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^3, b^2 \rangle$, with respect to the generators a and b.

   [*You should find Question 2 on Problem Sheet 6 useful.*]

5. (i) Using covering spaces, prove that for each integer $n \geq 2$, F_n is a finite index subgroup of F_2, where F_n is the free group on n generators.

   (ii) Prove that if F_m is subgroup of F_n with index i, then $m = ni - i + 1$. Deduce that $m \geq n$.

   (iii) Prove that F_2 is a subgroup of F_3 (but necessarily its index is infinite). Construct a covering space of $S^1 \vee S^1 \vee S^1$ corresponding to this subgroup.

$*$ 6. Prove that there are 13 index 3 subgroups of F_2, of which 4 are normal.

7. Let G be the group $\langle x, y \mid x^3y^3 \rangle$. Let $\phi: G \to \mathbb{Z}/3\mathbb{Z}$ be the homomorphism that sends x and y to the generator of $\mathbb{Z}/3\mathbb{Z}$. Find a finite presentation for the kernel of this homomorphism. *[We know from Theorem VI.37 that a finite index subgroup of a finitely presented group is again finitely presented. To answer this question, you should use the proof of this theorem.]*

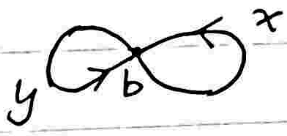
Overall: A+

Topology & Groups 7

Peize Liu

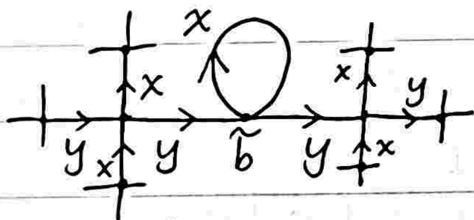
1. Let $X = S^1 \vee S^1$. $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = \langle x, y \mid \rangle$.

We label the paths as follows:



As $\tilde{X}$ is locally homeomorphic to X, $\tilde{X}$ should be a graph which looks like:  locally at each vertex. We claim that any graph with degree 4 at each vertex is a covering space of X.

(i) $(\tilde{X}, \tilde{b})$ is given as follows:

$\tilde{X} :=$ 

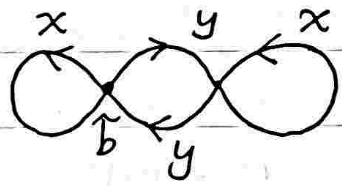
Clearly $\tilde{X} \simeq S^1 \Rightarrow \pi_1(\tilde{X}) = \mathbb{Z} \Rightarrow p_* \pi_1(\tilde{X}, \tilde{b}) = \langle x \rangle$.

(ii) Note that

$\pi_1(X) = \{x^{n_1} y^{m_1} \cdots x^{n_k} {}^{m_k} : \sum m_i \text{ even}\} \sqcup \{x^{n_1} y^{m_1} \cdots x^{n_k} y^{m_k} : \sum m_i \text{ odd}\}$

The subgroup has index 2. By Proposition 6.16, we know that $p^{-1}(b)$ has cardinality 2.

$(\tilde{X}, \tilde{b})$ is given as follows:

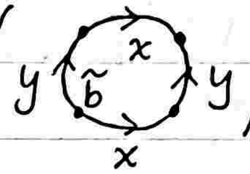
$\tilde{X} :=$ 

Clearly any loop based at $\tilde{b}$ traverses y evenly many times.

$\pi_1(\tilde{X}, \tilde{b}) = \{x^{n_1} y^{m_1} \cdots x^{n_k} m^{m_k} : \sum m_i \text{ even}\}$.

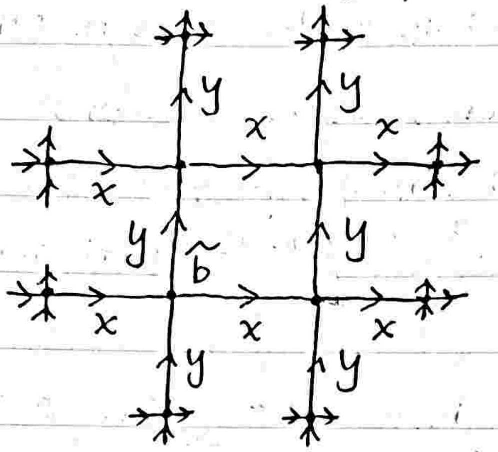
(iii) $\mathbb{Z} \times \mathbb{Z} = \langle x, y \mid xyx^{-1}y^{-1} \rangle$

Hence the kernel would be the subgroup $\langle xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}$

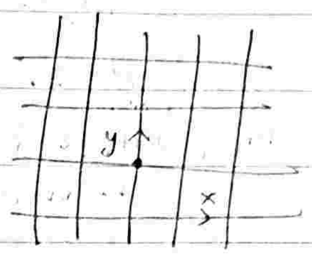
Given that $\pi_1\Big($  $\Big) \cong \mathbb{Z}$

We construct the covering space $(\tilde{X}, \tilde{b})$ as follows:

$\tilde{X} :=$



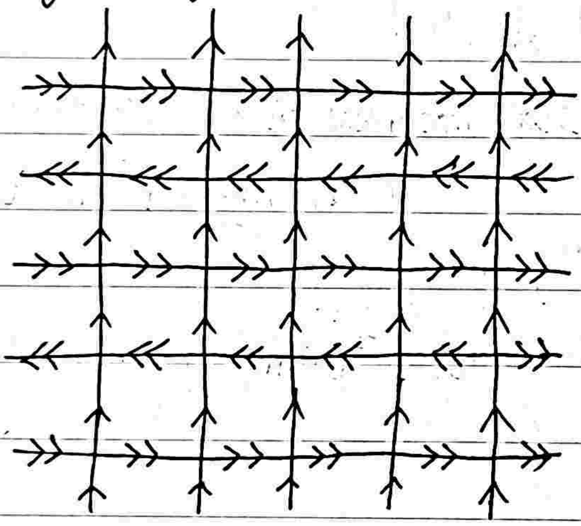
$\tilde{X} :=$



Great

H+

2. The covering map from $\mathbb{R}^2$ to K can be visualized by:



More specifically, let $\sim$ be the equivalence relation given by: $(x,y) \sim (x+1, y)$, $(x,y) \sim (-x, y+1)$ $(\forall x, y \in \mathbb{R})$

Then the quotient space $K := \mathbb{R}^2/\sim$ is the Klein bottle. The projection $p : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ is the covering map.

Let $b = (0,0) \in K \Rightarrow p^{-1}(b) = \mathbb{Z}^2 \subset \mathbb{R}^2$.

By Proposition 6.19, there is a bijective correspondence between $p^{-1}(b)$ and $\pi_1(K, b)/p_*\pi_1(\mathbb{R}^2, b) \cong \pi_1(K, b)$

($\mathbb{R}^2$ is contractible. $\pi_1(\mathbb{R}^2) = \{e\}$).

We identify the elements in $\pi_1(K, b)$ with the elements in $p^{-1}(b)$. To compute the group structure, we use Procedure 6.21: For $(m,n), (x,y) \in p^{-1}(b)$, they correspond to the loops $\tilde{l}_1$ from $(0,0)$ to (m,n) and $\tilde{l}_2$ from $(0,0)$ to (x,y). $p \circ \tilde{l}_2$ is a loop in K. We lift it to a path starting from (m,n):

If n is even, then (m,n) looks like $(0,0)$ locally. Then ~~$(m,n) * (x,y) = (m+x, n+$~~

$p \circ \tilde{l}_2$ lifts to a path from (m,n) to $(m+x, n+y)$, which is the translation of $\tilde{l}_2$.

If n is odd, then the orientation of the path is reversed in x-direction. $p \circ \tilde{l}_2$ lifts to a path from (m,n) to $(m-x, n+y)$.

In conclusion, the induced group operation is given by:

$$(m,n) * (x,y) = (m + (-1)^n x, n+y).$$

A Y

3.(i) This is immediate by definition:

For each $y \in Y$ there exists an open neighbourhood U of y such that $g_1(U) \cap g_2(U) = \phi$ for $g_1 \neq g_2$.

The projection $p: Y \twoheadrightarrow Y/G$ identifies the disjoint union of (mutually homeomorphic) open sets: $\{g(U) \subset Y : g \in G\}$.

For each $g_i(U)$, the restriction $p|_{g_i(U)} : g_i(U) \to U$ is a homeomorphism by definition.

Hence $p: Y \to Y/G$ is a covering map. + path-connected

(ii) If Y is simply-connected, then $\pi_1(Y) = \{e\}$, which is normal when embedded in $\pi_1(Y/G)$.

For each $g \in G$: g induces a homeomorphism which is also a cover transformation. Then G is the group of covering transformation of Y! Why? We claim that

$$G \cong \pi_1(Y/G)/p_* \pi_1(Y) \cong \pi_1(Y/G).$$

Let $f : \pi_1(Y/G) \to G$ defined as follows:

For $[l] \in \pi_1(Y/G, b)$, let $\tilde{l}$ be the lift of l in Y from $\tilde{b}$ to b_1. Let $g = f([l]) \in G$ be the homeomorphism such that $g(\tilde{b}) = b_1$. It is clear that f is a group homomorphism. f is surjective as Y is simply-connected.

By First Isomorphism Theorem:

$$G = \operatorname{Im} f \cong \pi_1(Y/G)/\operatorname{Ker} f.$$

For $[l] \in \operatorname{Ker} f$, $g = f([l]) = id \Rightarrow \tilde{b} = g(\tilde{b}) = b_1$

$\Rightarrow$ l lifts to a ~~pat~~ loop based at $\tilde{b}$ in Y, which is homotopic to a constant path. Hence $\operatorname{Ker} f = \{e\}$.

$\Rightarrow G \cong \pi_1(Y/G)$.

(iii) Let $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by:

$$(m,n) \cdot (x,y) = (x+m, y+n) \qquad ((m,n) \in \mathbb{Z} \times \mathbb{Z}, (x,y) \in \mathbb{R}^2)$$

Explain. ?

As $\mathbb{Z} \times \mathbb{Z}$ is discrete, it is clear that this is a covering space action. Moreover, $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$ is the torus (obvious by side identification). By (ii):

$$\pi_1(T^2) = \pi_1(\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})) \cong \mathbb{Z} \times \mathbb{Z}.$$

as $\mathbb{R}^2$ is simply-connected.

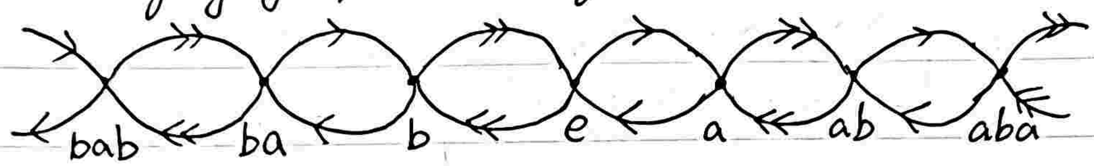
H

4.(i) $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a,b \mid a^2, b^2 \rangle = \langle a \mid a^2 \rangle * \langle b \mid b^2 \rangle$

By Q2 in Sheet 6, any $g \in \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is represented by $g = a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}$. Since $a^2 = b^2 = e$, the elements in $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ are of the form:

$e, a, ab, aba, abab \cdots$ or $b, ba, bab, baba, \cdots$

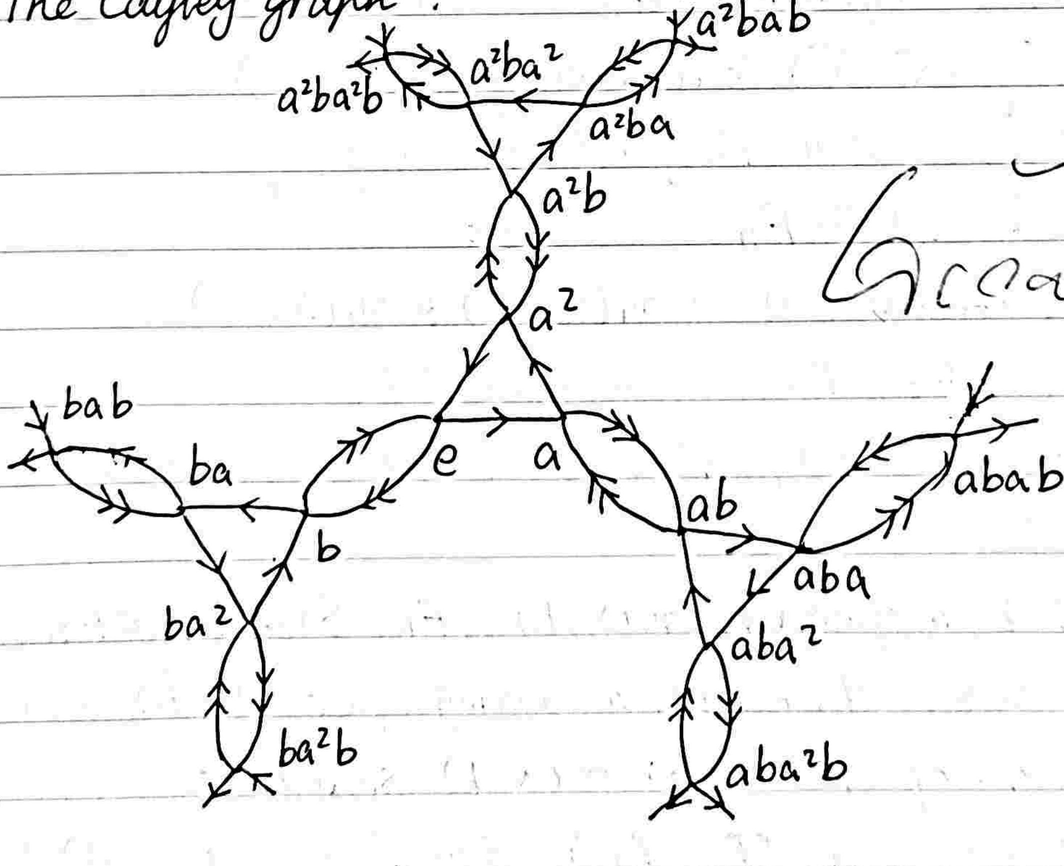
The Cayley graph is as follows:



(ii) Similar to (i), for any $g \in \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$:

$g = a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}$, $n_i \in \{0,1,2\}$, $m_i \in \{0,1\}$.

The Cayley graph:



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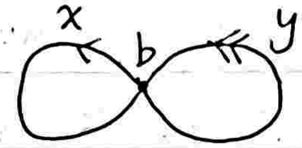
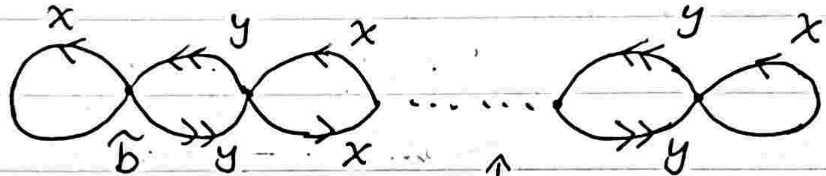
HT

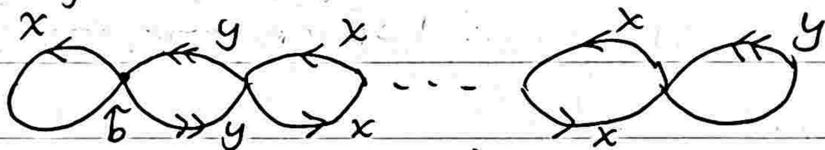
5.(i) Let $X = S^1 \vee S^1$, $\pi_1(X,b) = F_2 = \langle x, y \mid \rangle$, as shown below:

$X =$ 

We have discussed the possible covering spaces of $S^1 \vee S^1$. For each integer $n \geq 2$, let $(\tilde{X}_n, \tilde{b})$ be the following covering space of X:

$\tilde{X}_n :=$ 

(if n is odd) $\quad$ $(n-1)$ vertices

or 

(if n is even) $\quad$ $(n-1)$ vertices.

Clearly $(\tilde{X}_n, \tilde{b})$ is a covering space of X.

By Theorem 4.11, the fundamental group:

$\pi_1(\tilde{X}_n, \tilde{b}) \cong F_n$.

By Corollary 6.18 $p_* \pi_1(\tilde{X}_n, \tilde{b}) \leq \pi_1(X_n, b)$

$\Rightarrow F_n \leq F_2$. F_n is a subgroup of F_2.

By Proposition 6.19, the index $[F_2 : F_n] = \text{card } p^{-1}(b) = n-1$ is finite.

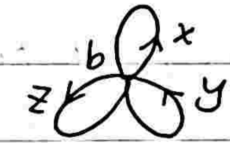
(ii) Let X be a space with $\pi_1(X, b) \cong F_n$. Since $F_m \leq F_n$, by Theorem 6.32 there exists a covering space $(\tilde{X}, \tilde{b})$ and a covering map $p : (\tilde{X}, \tilde{b}) \to (X, b)$ such that $\pi_1(\tilde{X}, \tilde{b}) \cong F_m$. $[F_n : F_m] = i$ implies that $p^{-1}(b)$ has cardinality i.

More specfically, we choose $X = \bigvee_{i=1}^{n} S^1$. $\tilde{X}$ is locally homeomorphic to X. Hence $\tilde{X}$ is a graph with i vertices and the degree of each vertex is $2n$. By handshaking lemma, $\tilde{X}$ has ni edges. Since $\tilde{X}$ is connected, the maximal tree of $\tilde{X}$ contains $(i-1)$ edges. Finally by Theorem 4.11, we have $\pi_1(\tilde{X}) \cong F_{ni-(i-1)}$

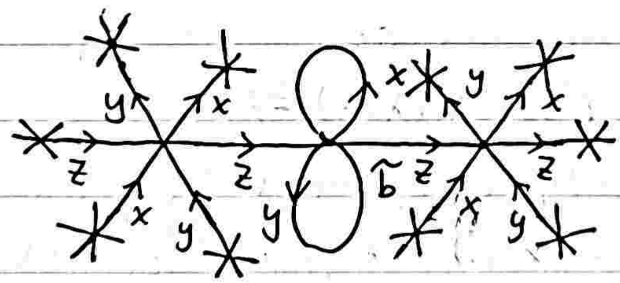
$\Rightarrow m = ni - i + 1$

Since $i \geq 1$, $m = i(n-1)+1 \geq n-1+1 = n$.

(iii) Let $F_3 = \langle x, y, z \mid \rangle$. Then $F_2 \cong \langle x, y\rangle \leq F_3$.
$\langle x, y\rangle$ is a subgroup of F_3 with infinite index. There are no subgroup of F_3 isomorphic to F_2 that has finite index, as proven in (ii). For $X = S^1 \vee S^1 \vee S^1$:

$X =$ 

we construct the covering space $(\tilde{X}, \tilde{b})$ as follows:
(similar to Q1 (i))

$\tilde{X} :=$ 

Great

A+

7. Let $H = \operatorname{Ker}\varphi$. Let (K, b) be the ~~simplicial~~ CW-complex with $\pi_1(K, b) \cong G$ (using the construction given in Corollary 5.34). By Theorem 6.32, there exists a covering space $(\tilde{K}, \tilde{b})$ and a covering map $p: (\tilde{K}, \tilde{b}) \to (K, b)$ such that $\pi_1(\tilde{K}, \tilde{b}) \cong H$.

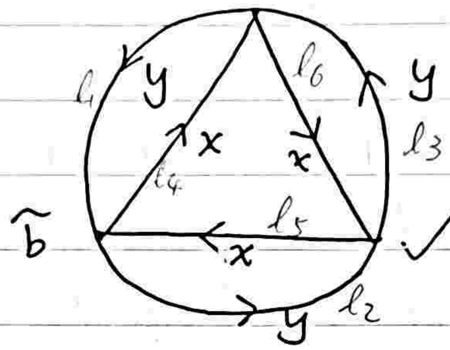
Notice that $p^{-1}(b) = [G:H] = |\mathrm{Im}\,\varphi| = |\mathbb{Z}/3\mathbb{Z}| = 3$.

Then $\tilde{K}$ has 3 0-cells.

We assume that $\varphi(x) = 1$, $\varphi(y) = 2$.

The 0-cells and 1-cells of $\tilde{K}$ are as follows:

One 2-cell in K is lifted to 3 2-cells in $\tilde{K}$.



~~$\tilde{K}$ has one 2-cell, which~~

The maximal tree is spanned by $l_1 \cdot l_2$. B

The relation $x^3y^3 = e$ in G corresponds to the 2-cell that runs over $l_5 \cdot l_4 \cdot l_6 \cdot l_3^3$. There's more than one!

Hence H is given by the presentation: ?

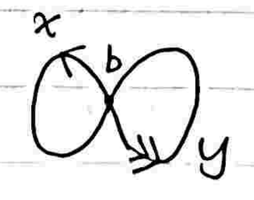
$H = \langle a, b, c, d \mid abcd^3 \rangle$ what are a, b, c, d?

$H = \langle a, b, c, d \mid dabc,\ dbca,\ dcab \rangle \cong \langle a, b, c \mid \cdots\cdots \rangle$

6. Let $X = S^1 \vee S^1$ so that $\pi_1(X, b) \cong F_2$.

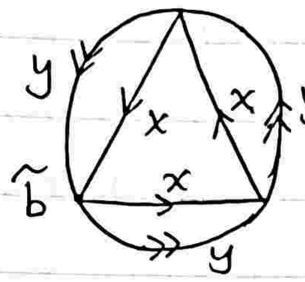
Let $H \leq F_2$ be a subgroup of index 3. If the covering space $p : (\tilde{X}, \tilde{b}) \to (X, b)$ satisfies that $\pi_1(\tilde{X}, \tilde{b}) = H$, then $|p^{-1}(b)| = 3$. We transform the question into the classification of directed graphs (path-connected) with 3 vertices and degree 4 at each vertex.

We write (X, b):



We list all the possibilities of $(\tilde{x}, \tilde{b})$ below :

①



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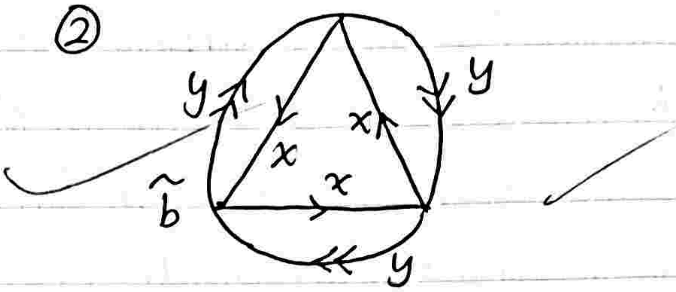

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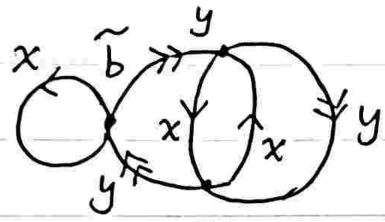

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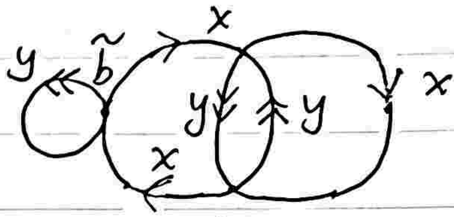

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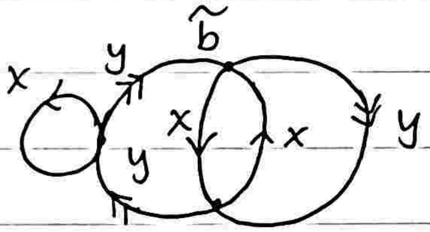

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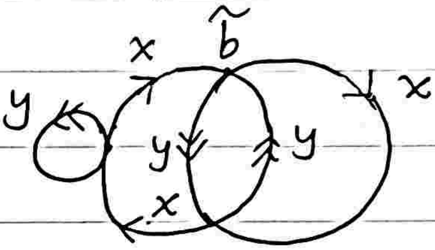

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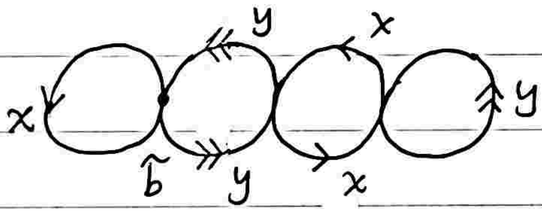

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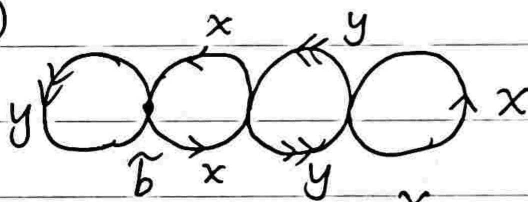

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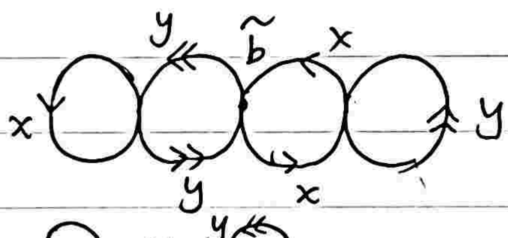

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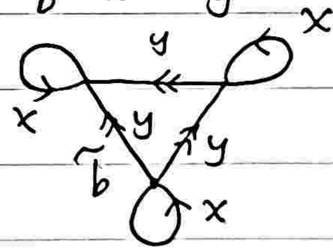

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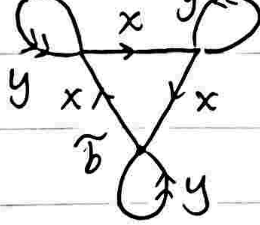

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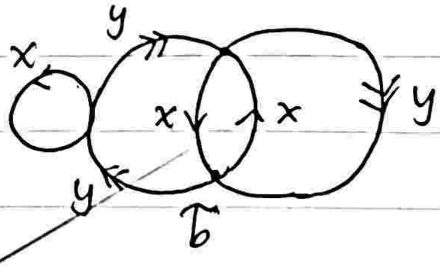

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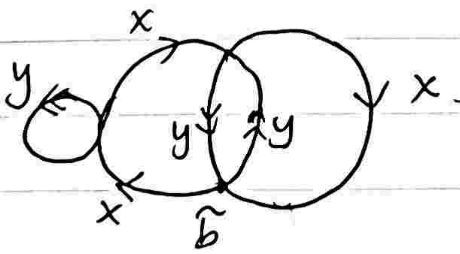

We claim that any other graph satisfying the conditions

is isomorphic to one of ① ~ ⑬.
Hence F_2 has 13 distinct subgroups of index 3.
Moreover, we see that in the cases ① ② ⑩ ⑪, the subgroup H is normal in $\pi_1(X, b)$, as the vertices are differed by a covering transformation (which is clear by symmetry). Very Well Done!

A^+