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Problem Sheet 4

B1.2: Set Theory

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We use the first-order language $\mathcal{L} := \{\in, \subseteq; P, \bigcup, \mathcal{P}; \varnothing, \omega\}$, where $\in$ and $\subseteq$ are binary predicates, P is a binary function, $\bigcup$ and $\mathcal{P}$ are unary functions, and $\varnothing$ and ω are constants.

The equality symbol $\doteq$ is used in $\mathcal{L}$ which indicates that two terms have the same value under any model and assignment. The equality symbol $=$ is used in metalanguage which indicates that two strings are equal.

The ZF axioms we shall use in this sheet are listed below:

ZF1 *Extensionality:* $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x \doteq y)$;

ZF2 *Empty Set:* $\forall x \, \neg x \in \varnothing$;

ZF3 *Pairs:* $\forall x \forall y \forall z (z \in P(x, y) \leftrightarrow (x \doteq z \vee y \doteq z))$;

ZF4 *Unions:* $\forall x \forall y (y \in \bigcup x \leftrightarrow \exists z (y \in z \wedge z \in x))$;

ZF5 *Comprehension Scheme: Let* $\varphi \in \mathrm{Form}(\mathcal{L})$ *and* $z, w_1, \ldots, w_k \in \mathrm{Free}(\varphi)$. *Then* $\forall x \forall w_1 \cdots \forall w_k \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi))$;

ZF6 *Power Sets:* $\forall x \forall y (y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$;

ZF7 *Infinity:* $\exists x (\varnothing \in x \wedge \forall y (y \in x \to y^+ \in x))$, *where* y^+ *is defined to be* $\bigcup P(y, P(y, y))$;

ZF8 *Replacement Scheme: Let* $\varphi \in \mathrm{Form}(\mathcal{L})$ *and* $x, y, w_1, \ldots, w_k, A \in \mathrm{Free}(\varphi)$. *Then* $\forall A \forall w_1 \cdots \forall w_k (\forall x (x \in A \to \exists! y \, \varphi) \to \exists B \forall y (y \in B \leftrightarrow \exists x (x \in A \wedge \varphi)))$, *where* $\exists! y \, \varphi$ *is defined to be* $(\exists y \, \varphi \wedge \forall z \forall y ((\varphi \wedge \varphi[z/y]) \to y \doteq z))$.

ZF9 *Foundation:* $\forall x (\neg x \doteq \varnothing \to \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x)))$.

AC *Choice:* $\forall x (\neg x \doteq \varnothing \to \exists y \forall z (z \in x \to \exists! w \, w \in y \cap z))$.

The predicate $\subseteq$ is introduced for convenience. It satisfies $\forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \to z \in y))$.

The constant ω is the smallest inductive set, whose existence and uniqueness follows from the Axioms of Infinity, Comprehension Scheme, and Extensionality.

Question 1

(a) Let X be a well-ordered set, and $x \in X$. Show that either x is the greatest element in X or x has an immediate successor (that is an element $x^* \in X$ with $x < x^*$ such that there is no $y \in X$ with $x < y < x^*$).

(b) Let $X \subset \mathbb{R}$ such that the inherited order $<$ from $\mathbb{R}$ is a well-order on X. Prove that X must be countable.

[*Hint: consider the intervals* (x, x^*).]

Proof. (a) Suppose that x is not the greatest element in X. Then the set $\{y \in X : x < y\} \subseteq X$ is non-empty. Since X is well-ordered, it has a minimal element x^*. We claim that x^* is an immediate successor of x. If not, then there exists $z \in X$ such that $x < z < x^*$. Then $z \in \{y \in X : x < y\}$, contradicting the minimality of x^*.

(b) Suppose for contradiction that X is uncountable. Since X has at most one greatesr element, the set X^* of elements in X which has immediate successors is also uncountable. For each $x \in X^*$, let x^* be the immediate successor of x. By part (i) we know that the interval $(x, x^*) \subseteq \mathbb{R} \backslash X$. For $x, y \in X^*$ with $x \neq y$, the intervals (x, x^*) and (y, y^*) are disjoint. Hence $\mathbb{R} \backslash X$ has a subset which is the union of uncountably many disjoint open intervals. Since $\mathbb{R}$ is separable, for each $(x, x^*) \subseteq \mathbb{R} \backslash X$ there exists a rational number $q_x \in (x, x^*)$. The map $x \mapsto q_x$ is clearly an injection from X^* to $\mathbb{Q}$. But $\mathbb{Q}$ is countable, contradiction. □

Question 2

Let $<_A, <_B$ be strict total orders on sets A, B respectively. We define the *sum* $(A, <_A) + (B, <_B)$ and the *product* $(A, <_A) \times (B, <_B)$ of the orders as follows.

For the sum, we assume A, B are disjoint (which can always be arranged by replacing them by $A' = \{0\} \times A$, $B' = \{1\} \times B$ with the obvious orders on them). Then $(A, <_A) + (B, <_B)$ is the set $A \cup B$ with the order $<_+$ in which elements of A or B are ordered by $<_A, <_B$ respectively and all elements of A precede all elements of B.

The product $(A, <_A) \times (B, <_B)$ is $A \times B$ with the *reverse lexicographic order*, that is $(a, b) <_\times (a', b')$ iff $b < b'$, or $b = b'$ and $a < a'$.

(i) Draw illustrative pictures (coloured pens may be helpful) of the orders

$$\omega + 4, \quad 4 + \omega, \quad \omega + \omega, \quad \omega \cdot \omega$$

(ii) Prove that $<_+, <_\times$ are well-orders if $<_A, <_B$ are well-orders.

[*You may omit the (tedious) verification that they are strict orders and that they are total.*]

Proof. (i) $\omega + 4$:

$$0 < 1 < 2 < 3 < 4 < \cdots < \bar{0} < \bar{1} < \bar{2} < \bar{3}$$

$4 + \omega$:

$$0 < 1 < 2 < 3 < \bar{0} < \bar{1} < \bar{2} < \bar{3} < \bar{4} < \cdots$$

$\omega + \omega$:

$$0 < 1 < 2 < 3 < 4 < \cdots < \bar{0} < \bar{1} < \bar{2} < \bar{3} < \bar{4} < \cdots$$

$\omega \cdot \omega$:

$$\begin{array}{ccccccccc} & & \langle 0,0\rangle & < & \langle 1,0\rangle & < & \langle 2,0\rangle & < & \cdots \\ \cdots & < & \langle 0,1\rangle & < & \langle 1,1\rangle & < & \langle 2,1\rangle & < & \cdots \\ \cdots & < & \langle 0,2\rangle & < & \langle 1,2\rangle & < & \langle 2,2\rangle & < & \cdots \end{array}$$

(ii) Assume that we have proven that $<_+$ and $<_\times$ are strict total orders provided that $<_A$ and $<_B$ are strict total orders.

For non-empty $S \subseteq A \cup B$, if $A \cap S \not\doteq \varnothing$, then $A \cap S$ has an minimal element a_0 with respect to $\langle A, <_A\rangle$. It is clear that a_0 is the minimal element of S with respect to $\langle A \cup B, <_+\rangle$. If $A \cap S \doteq \varnothing$, then $B \cap S$ has an minimal element b with respect to $\langle B, <_B\rangle$. It is clear that b_0 is the minimal element of S with respect to $\langle A \cup B, <_+\rangle$. Hence $<_+$ is a well-order.

For non-empty $S \subseteq A \times B$, let $S_B := \{b \in B : \langle a, b\rangle \in S\}$. $S_B \subseteq B$ is non-empty. It has an minimal element $b_0 \in B$ with respect to $\langle B, <_B\rangle$. Let $S_A := \{a \in A : \langle a, b_0\rangle \in S\}$. $S_A \subseteq A$ is non-empty. It has an minimal element $a_0 \in A$ with respect to $\langle A, <_A\rangle$. It is clear that $\langle a_0, b_0\rangle$ is the minimal element of S with respect to $\langle A \times B, <_\times\rangle$. Hence $<_\times$ is a well-order. □

Question 3

Let α, β, γ be ordinals. Show that

(i) if $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.

[*Hint: induction on γ, or use Theorem 14.7.*]

(ii) if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$, i.e. left cancellation holds.

(iii) right cancellation "$\alpha + \gamma = \beta + \gamma$ implies $\alpha = \beta$" fails, by giving a counterexample.

(iv) if γ is a limit ordinal then $\alpha + \gamma$ is a limit ordinal.

Proof. (i) We use transfinite induction on γ.

Base case: If $\gamma \doteq 0$, then the result holds vacuously.

Successor case: Suppose that $(\beta < \delta \to \alpha + \beta < \alpha + \delta)$ holds for ordinal δ. If $\delta < \beta$, then $\delta^+ \leqslant \beta$. $(\beta < \delta^+ \to \alpha + \beta < \alpha + \delta^+)$ holds vacuously. If $\delta \geqslant \beta$, then $\beta < \delta^+$. By definition of ordinal addition, $\alpha + \delta^+ \doteq (\alpha + \delta)^+ > \alpha + \delta$. Then $\alpha + \beta < \alpha + \delta^+$.

Limit case: Suppose that λ is a limit ordinal and $(\beta < \delta \to \alpha + \beta < \alpha + \delta)$ holds for all ordinals $\delta < \lambda$. If $\beta \geqslant \lambda$, then $(\beta < \lambda \to \alpha + \beta < \alpha + \lambda)$ holds vacuously. If $\beta < \lambda$, by definition of ordinal addition, $\alpha + \lambda \doteq \bigcup\{\alpha + \delta : \delta < \lambda\} > \alpha + \beta$.

(ii) If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$; if $\beta > \gamma$, then $\alpha + \beta > \alpha + \gamma$. Hence $\alpha + \beta \doteq \alpha + \gamma$ implies that $\beta \doteq \gamma$.

(iii) $1 + \omega \doteq \bigcup\{1 + n : n \in \omega\} \doteq \omega \doteq \bigcup\{2 + n : n \in \omega\} \doteq 2 + \omega$ but $1 \not\doteq 2$.

(iv) By definition, $\alpha + \gamma \doteq \bigcup\{\alpha + \beta : \beta < \gamma\}$. For ordinal $\delta < \alpha + \gamma$, there exists ordinal β such that $\delta \doteq \alpha + \beta$. $\delta^+ \doteq \alpha + \beta^+$. Since γ is a limit ordinal, $\beta < \gamma$ implies that $\beta^+ < \gamma$. Hence $\delta^+ < \alpha + \gamma$. By Theorem 12.10, $\alpha + \gamma$ is a limit ordinal. □

Question 4

For any two ordinals α, β, exactly one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ hold. Determine which of these holds when

(i) $\alpha = (\omega + 1) \cdot 2, \quad \beta = 2 \cdot (\omega + 1)$

(ii) $\alpha = (\omega + 1) \cdot \omega, \quad \beta = \omega \cdot (\omega + 1)$

Proof. (i) $\alpha = (\omega + 1) \cdot 2 \doteq (\omega + 1) + (\omega + 1)$.

$\beta = 2 \cdot (\omega + 1) \doteq 2 \cdot \omega + 2 \cdot 1 \doteq \bigcup\{2n : n \in \omega\} + 2 \doteq \omega + 2 \doteq (\omega + 1) + 1$.

Since $1 < \omega + 1$, we have $2 \cdot (\omega + 1) < (\omega + 1) \cdot 2$. That is, $\beta \in \alpha$.

(ii) $\alpha = (\omega + 1) \cdot \omega \doteq \bigcup\{(\omega + 1) \cdot n : n \in \omega\}$.

By finite induction, we can prove that $(\omega + 1) \cdot n \doteq \underbrace{(\omega + 1) + \cdots + (\omega + 1)}_{n \text{ times}} \doteq \omega + \underbrace{(1 + \omega) + \cdots + (1 + \omega)}_{(n-1) \text{ times}} + 1 \doteq \omega + \underbrace{\omega + \cdots + \omega}_{(n-1) \text{ times}} + 1 \doteq \omega \cdot n + 1$.

Note that $\omega \cdot n + 1 < \omega \cdot n + \omega \doteq \omega \cdot (n+1)$. Then $\alpha \doteq \bigcup\{(\omega+1) \cdot n : n \in \omega\} \doteq \bigcup\{\omega \cdot (n+1) : n \in \omega\} \doteq \omega \cdot \omega \doteq \omega^2$.

$\beta = \omega \cdot (\omega + 1) \doteq \omega \cdot \omega + \omega \cdot 1 \doteq \omega^2 + \omega$.

Since $0 < \omega$, $\omega^2 < \omega^2 + \omega$. Hence $\alpha \in \beta$. □

Question 5

Ordinal exponentiation $\beta \mapsto \alpha^\beta$ for any ordinal $\alpha > 0$ was defined in lectures. Prove that if α, β are countable, with $\alpha > 0$, then α^β is countable.

[*Observe the difference with cardinal exponentiation on this point.*]

Assume AC. It can be done without much more work.

Proof. We precede the proof by 3 steps.

Step 1: We use transfinite induction on β to show that if α and β are countable then $\alpha + \beta$ is countable.

Base case: $\alpha + 0 = \alpha$ is countable.

Successor case: Suppose that $\alpha + \beta$ is countable. Then $\alpha + \beta^+ = (\alpha + \beta)^+ = (\alpha + \beta) \cup \{\alpha + \beta\}$ is countable.

Limit case: Suppose that λ is a limit ordinal. $\alpha + \beta$ is countable for all ordinals $\beta < \lambda$. Then $\alpha + \lambda = \bigcup\{\alpha + \beta : \beta < \lambda\}$. Since λ is countable, $\alpha + \lambda$ is a countable union of countable sets. By AC $\alpha + \lambda$ is countable.

Step 2: We use transfinite induction on β to show that if α and β are countable then $\alpha \cdot \beta$ is countable.

Base case: $\alpha \cdot 0 = 0$ is countable.

Successor case: Suppose that $\alpha \cdot \beta$ is countable. Then $\alpha \cdot \beta^+ = \alpha \cdot \beta + \alpha$. By Step 1, $\alpha \cdot \beta^+$ is countable.

Limit case: Suppose that λ is a limit ordinal. $\alpha \cdot \beta$ is countable for all ordinals $\beta < \lambda$. Then $\alpha \cdot \lambda = \bigcup\{\alpha \cdot \beta : \beta < \lambda\}$. Since λ is countable, $\alpha \cdot \lambda$ is a countable union of countable sets. By AC $\alpha \cdot \lambda$ is countable.

Step 3: We use transfinite induction on β to show that if α and β are countable then α^β is countable.

Base case: $\alpha^0 = 1$ is countable.

Successor case: Suppose that α^β is countable. Then $\alpha^{\beta^+} = \alpha^\beta \cdot \alpha$. By Step 2, α^{β^+} is countable.

Limit case: Suppose that λ is a limit ordinal. α^β is countable for all ordinals $\beta < \lambda$. Then $\alpha^\lambda = \bigcup\{\alpha^\beta : \beta < \lambda\}$. Since λ is countable, α^λ is a countable union of countable sets. By AC α^λ is countable. □

Question 6

Let P be a non-empty partially strictly ordered set and assume no element of P is maximal (i.e. for every $x \in P$ there exists $y \in P$ with $x < y$). Use AC to show there exists a function $f : \omega \to P$ with $f(n) < f(n^+)$ for all $n \in \omega$.

Proof. Define $\varphi : P \to \mathcal{P}(P)$ by $\varphi(x) := \{y \in P : y > x\}$. Since P has no maximal element, $\varphi(x) \doteq\!\!\!\!/\ \varnothing$ for all $x \in P$. By Axiom of Choice, there exists a function $g : \mathcal{P}(P)\backslash\{\varnothing\} \to P$ such that $g(A) \in A$ for $A \in \mathcal{P}(P)\backslash\{\varnothing\}$. Consider $h := g \circ \varphi : P \to P$. Fix $a_0 \in P$. By Recursion theorem, there exists a function $f : \omega \to P$ such that $f(0) \doteq a_0$ and $f(n^+) \doteq h(f(n))$ for $n \in \omega$. Note that $h \circ f(n) \doteq g \circ \varphi \circ f(n) \doteq g(\{y \in P : y > f(n)\}) \in \{y \in P : y > f(n)\}$. Hence $f(n^+) > f(n)$ for all $n \in \omega$. □

Question 7

Let R be a commutative ring with identity $1 \neq 0$.

(i) Prove that the union of a chain of proper ideals is a proper ideal.

(ii) Use Zorn's Lemma to prove that R has a maximal ideal.

Proof. See my algebra notes. □

Question 8

Let $\mathcal{A}$ be a non-empty set of non-empty sets and $X = \bigcup \mathcal{A}$. Assume Zorn's Lemma and prove the existence of a function $F : \mathcal{A} \to X$ with $F(A) \in A$ for each $A \in \mathcal{A}$.

[*Hint: apply ZL to the partially ordered set P of partial maps from $\mathcal{A}$ to X, regarded as a subset of $\mathcal{P}(\mathcal{A} \times X)$, ordered by inclusion.*]

Deduce that Zorn's Lemma implies the Axiom of Choice.

Proof. Let $\mathscr{S}$ be the set of maps $f : \mathcal{B} \to \bigcup \mathcal{A}$ where $\mathcal{B} \subseteq \mathcal{A}$ and $f(A) \in A$ for each $A \in \mathcal{B}$. Let $\mathscr{S}$ be partially ordered by set inclusion.

$\mathscr{S}$ is non-empty. Take $A_0 \in \bigcup \mathcal{A}$ and $a_0 \in A_0$. Define $f : \{A_0\} \to \bigcup \mathcal{A}$ by $f(A_0) = a_0$. Then $f \in \mathscr{S}$. For each chain $\mathscr{T} \subseteq \mathscr{S}$, we claim that $\bigcup \mathscr{T} \in \mathscr{S}$. It is clear that $\bigcup \mathscr{T} \in \mathcal{P}(\mathcal{A} \times \bigcup \mathcal{A})$. For $\langle A, a\rangle, \langle A, b\rangle \in \bigcup \mathscr{T}$, there exists $f_1, f_2 \in \mathscr{T}$ such that $\langle A, a\rangle \in f_1$ and $\langle A, b\rangle \in f_2$. Since $\mathscr{T}$ is a chain, either $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. Without loss of generality we assume the former. Then $\langle A, a\rangle, \langle A, b\rangle \in f_2$. Since f_2 is a map, then $a \doteq b$. Hence $\bigcup \mathscr{T}$ is a map. For $\langle A, a\rangle \in \bigcup \mathscr{T}$, $\langle A, a\rangle \in f$ for some $f \in \mathscr{T}$. Then $a \in A$ since f is a choice function. We deduce that $\bigcup \mathscr{T}$ is a choice function.

By Zorn's Lemma, $\mathscr{S}$ has a maximal element F. We claim that F is a map from $\mathcal{A}$ to $\bigcup \mathcal{A}$. If not, then take $B \in \mathcal{A}\backslash \operatorname{dom}(F)$ and $b \in B$. Define $\tilde{F} : \operatorname{dom}(F) \cup B \to \bigcup \mathcal{A}$ by

$$\tilde{F}(A) \doteq \begin{cases} F(A), & A \in \operatorname{dom}(F); \\ b, & A \doteq B. \end{cases}$$

Then $\tilde{F} \in \mathscr{S}$ and $F \subsetneq \tilde{F}$, contradicting the maximality of F.

Wew conclude that Zorn's Lemma implies the Axiom of Choice. □