Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3

B4.2: Functional Analysis II

26 February, 2020

Question 1

Let X and Y be real Hilbert spaces and $T \in \mathscr{B}(X,Y)$ be surjective. Show that there exists a unique bounded linear operator $R \in \mathscr{B}(Y,X)$ such that $TR = I_Y$ and $\|Ry\| \leqslant \|x\|$ for all $x \in X$ and $y \in Y$ satisfying $Tx = y$.

[*Hint: Follow the strategy of the proof of Theorem 2.3.4.*]

Proof. **Existence**: Consider the adjoint $T^* \in \mathscr{B}(X,Y)$. From Question 1 of Sheet 2 we know that

$$\ker T^* = \left(\overline{\operatorname{im} T}\right)^{\perp} = Y^{\perp} = \{0\}$$

Hence T^* is injective. By Theorem 2.3.4, $W := \operatorname{im} T^*$ is closed in X. Let $S \in \mathscr{B}(Y,W)$ be such that $Sy = T^*y$ for all $y \in Y$. Then S is bijective. The adjoint $S^* \in \mathscr{B}(W,Y)$ is also bijective. By inverse mapping theorem, we have $(S^*)^{-1} \in \mathscr{B}(Y,W)$. We claim that $R := (S^*)^{-1}$ is the operator we are looking for.

For $y, z \in Y$,

$$\left\langle TRy, z\right\rangle = \left\langle (S^*)^{-1}y, T^*z\right\rangle = \left\langle (S^{-1})^*y, Sz\right\rangle = \left\langle y, S^{-1}Sz\right\rangle = \left\langle y, z\right\rangle$$

Hence $TR = \mathrm{id}_Y$.

For $w, u \in W$,

$$\langle RTw, u\rangle = \left\langle w, T^*R^*u\right\rangle = \left\langle w, SS^{-1}u\right\rangle = \langle w, u\rangle$$

Hence $RT|_W = \mathrm{id}_W$. Since W is closed, by the projective theorem we have $X = W \oplus W^{\perp}$. Let $P_W \in \mathscr{B}(X,W)$ be the projection operator along the direct sum decomposition. Note that $\ker T = \left(\overline{\operatorname{im} T^*}\right)^{\perp} = W^{\perp}$. For $x \in X$,

$$Tx = TP_Wx + T(x - P_Wx) = TP_Wx$$

and hence

$$RTx = RTP_Wx = P_Wx$$

We deduce that $RT = P_W$. By Pythagoras' theorem, for $x \in X$,

$$\|x\|^2 = \|P_Wx\|^2 + \|x - P_Wx\|^2$$

For $x \in X$ and $y \in Y$ such that $Tx = y$,

$$\left\|Ry\right\| = \|RTx\| = \|P_Wx\| \leqslant \|x\|$$

Uniqueness: Suppose that $R' \in \mathscr{B}(Y,X)$ such that $TR' = \mathrm{id}_Y$ and $\left\|R'T\right\| \leqslant 1$.

For $y \in Y$, there exists $w \in W$ such that $Tw = y$. Since $TR' = \mathrm{id}_Y$, we have

$$T(R'y - w) = TR'y - Tw = y - y = 0 \implies R'y - w \in \ker T = W^{\perp}$$

On the other hand, $w \in W$. By Pythagoras' theorem,

$$\left\|R'y\right\|^2 = \left\|R'y - w\right\|^2 + \|w\|^2 \geqslant \|w\|^2$$

But by assumption $\left\|R'y\right\| \leqslant \|w\|$. Hence we must have $R'y = w$. We deduce that $\operatorname{im} R' \subseteq W$. Hence $\operatorname{im}(R - R') \subseteq W$. But $T(R - R') = 0$ implies that $\operatorname{im}(R - R') \subseteq \ker T = W^{\perp}$. We deduce that

$$\operatorname{im}(R - R') \subseteq W \cap W^{\perp} = \{0\} \implies R = R'$$

The bounded right inverse of T is unique. □

bounded alone is insufficient to obtain uniqueness, has to be $\|R'T\| \leq 1$.

Question 2

Let X be a Hilbert space and $T \in \mathscr{B}(X)$. Show that the graph $\Gamma(T)$ of T is a closed subspace of $X \times X$ and that

$$\Gamma(T)^{\perp} = \left\{\left(-T^*x, x\right) : x \in X\right\}$$

By considering the corresponding orthogonal decomposition of $(x,0)$, prove that $I+T^*T$ maps X onto X. Here the space $X\times X$ is endowed with the inner product

$$\langle (x_1,x_2),(y_1,y_2)\rangle_{X\times X} = \langle x_1,y_1\rangle_X + \langle x_2,y_2\rangle_X$$

Proof. $T\in\mathscr{B}(X)$. That $\Gamma(T)$ is closed in $X\times X$ follows directly from the closed graph theorem. ✓

Let $\Phi := \{(-T^*x,x):\ x\in X\}$. It is clear that Φ is a linear subspace of $X\times X$. ✓ For $x,y\in X$,

$$\langle (x,Tx),(-T^*y,y)\rangle = \langle x,-T^*y\rangle + \langle Tx,y\rangle = -\langle Tx,y\rangle + \langle Tx,y\rangle = 0$$

Hence $\Phi\subseteq\Gamma(T)^\perp$. ✓ On the other hand, for $(a,b)\in\Gamma(T)^\perp$, we have

$$\langle (a,b),(x,Tx)\rangle = \langle a,x\rangle + \langle b,Tx\rangle = \langle a+T^*b,x\rangle = 0$$

for all $x\in X$. ✓ Therefore $a+T^*b=0$, and $(a,b)=(-T^*b,b)\in\Phi$. ✓ we deduce that $\Gamma(T)^\perp\subseteq\Phi$. Hence

$$\Gamma(T)^\perp = \Phi = \{(-T^*x,x):\ x\in X\}$$ ✓

Since $\Gamma(T)$ is closed, by the projection theorem, $X\times X = \Gamma(T)\oplus\Gamma(T)^\perp$. ✓ Consider the decomposition of $(x,0)\in X\times X$ along the direct sum. There exists $y,z\in X$ such that

$$(x,0) = (y,Ty) + (-T^*z,z) \implies \begin{cases} x = y - T^*z \\ 0 = Ty + z \end{cases} \implies x = y - T^*(-Ty) = (\mathrm{id}+T^*T)y\in\mathrm{im}(\mathrm{id}+T^*T)$$ ✓

Hence $\mathrm{id}+T^*T$ is surjective. □

Question 3

Let X and Y be real Banach spaces and $T\in\mathscr{B}(X,Y)$. Assume that $Z=TX$ is a finite-codimensional subspace of Y and let $\{y_1+Z,\dots,y_m+Z\}$ be a basis for Y/Z. Define $\widehat{T}: X\oplus\mathbb{R}^m\to Y$ by

$$\widehat{T}(x,(v_1,\dots,v_m)) = T(x) + \sum_{j=1}^m v_j y_j$$

Show that $\widehat{T}$ is a surjective bounded linear operator. Hence, by applying the open mapping theorem, deduce that Z is closed.

Proof. It is clear that $\widehat{T}$ is linear by definition. ✓ For $y\in Y$, there exists $\sum_j v_j y_j\in Y$ such that $y\in\sum_j v_j y_j + Z$. ✓ Therefore, there exists $x\in X$ such that $y = T(x)+\sum_j v_j y_j$. Then we have

$$\widehat{T}(x,(v_1,\dots,v_m)) = T(x) + \sum_{j=1}^m v_j y_j = y$$

Hence $\widehat{T}$ is surjective. ✓

We equip $X\oplus\mathbb{R}^m$ with the norm

$$\|(x,(v_1,\dots,v_m))\| := \|x\|_X + \sum_{j=1}^m |v_j|$$

Let $M>\max\{\|T\|,\|y_1\|,\dots,\|y_m\|\}$. Then for $(x,(v_1,\dots,v_m))\in X\oplus\mathbb{R}^m$,

$$\left\|\widehat{T}(x,(v_1,\dots,v_m))\right\| \leqslant \|T(x)\| + \sum_{j=1}^m |v_j|\,\|y_j\| \leqslant \|T\|\,\|x\| + \sum_{j=1}^m |v_j|\,\|y_j\| \leqslant M\left(\|x\| + \sum_{j=1}^m |v_j|\right) = M\|(x,(v_1,\dots,v_m))\|$$ ✓

Hence $\widehat{T}$ is bounded.

It is clear that X is closed in $X\oplus\mathbb{R}^m$. If $\widehat{T}(x,(v_1,\dots,v_m))\in\mathrm{im}\,T=Z$, then $\sum_j v_j y_j + Z = Z$ and $(v_1,\dots,v_m)=0$. ✓ Hence $T((X\oplus\mathbb{R}^m)\setminus X) = Y\setminus Z$. Finally, by open mapping theorem, $Y\setminus Z$ is open in Y. Hence Z is closed in Y. □

Strictly speaking you should write $(X\oplus\mathbb{R}^m)\setminus(X\oplus\{0\})$

Question 4

Let X be a Banach space.

(a) Show that if a sequence (x_n) in X converges weakly, then its weak limit is unique.

(b) Suppose that $x_n \to x$ in X and $\ell_n \to \ell$ in X^*. Show that $\ell_n(x_n) \to \ell(x)$.

(c) Suppose in addition that X is a Hilbert space. Show that if $x_n \rightharpoonup x$ in X and if $\|x_n\| \to \|x\|$, then $x_n \to x$.

(d) Prove (c) when the assumption that X is a Hilbert space is replaced by the assumption that X is uniformly convex: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and if $\|x - y\| \geqslant \varepsilon$ then $\|x + y\| \leqslant 2(1-\delta)$.

[*Hint: Consider the sequence* $\frac{1}{2}(x_n + x)$ *and use Theorem 3.2.2.*]

Proof. (a) Suppose that x and x' are weak limits of (x_n). Then by definition, for any $\ell \in X^*$, $\ell(x_n) \to \ell(x)$ and $\ell(x_n) \to \ell(x')$ as $n \to \infty$. By uniqueness of limit in $\mathbb{R}$, $\ell(x) = \ell(x')$. In particular, $\ell(x - x') = 0$ for all $\ell \in X^*$. Suppose that $x \neq x'$. Then there exists a linear functional $f \in \operatorname{span}\{x - x'\}^*$ such that $f(x - x') = \|x - x'\|$. By Hahn-Banach Theorem, there exists an extension F of f on X^*. So $F(x - x') = \|x - x'\| \neq 0$. This is a contradiction. We deduce that the weak limit is unique.

(b) Since (x_n) is convergent in X, it is bounded in X. So there exists $M > 0$ such that $\|x_n\| \leqslant M$ for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$.

$$\lim_{n\to\infty} x_n = x \implies \exists N_1 \in \mathbb{N}\ \forall n > N_1\ \|x_n - x\| < \varepsilon;$$
$$\lim_{n\to\infty} \ell_n = \ell \implies \exists N_2 \in \mathbb{N}\ \forall n > N_2\ \|\ell_n - \ell\| < \varepsilon.$$

Then for $n > \max\{N_1, N_2\}$,

$$|\ell_n(x_n) - \ell(x)| \leqslant |\ell_n(x_n) - \ell(x_n)| + |\ell(x_n) - \ell(x)| \leqslant \|\ell_n - \ell\|_* \|x_n\| + \|\ell\|_* \|x_n - x\| < \varepsilon(M + \|\ell\|_*)$$

We deduce that $\ell_n(x_n) \to \ell(x)$ in $\mathbb{R}$ as $n \to \infty$.

(c) Since X is a Hilbert space,

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\operatorname{Re}\langle x, x_n\rangle$$

The map $y \mapsto \langle x, y\rangle$ is a bounded linear functional. Since $x_n \rightharpoonup x$, we have $\langle x, x_n\rangle \to \langle x, x\rangle = \|x\|^2$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \|x_n - x\|^2 = \lim_{n\to\infty} \|x_n\|^2 + \|x\|^2 - 2\operatorname{Re}\left(\lim_{n\to\infty} \langle x, x_n\rangle\right) = \|x\|^2 + \|x\|^2 - 2\operatorname{Re}\|x\|^2 = 0$$

Hence $x_n \to x$ in X.

(d) Suppose that $x = 0$. Then $\|x_n\| \to 0$ directly implies that $x_n \to 0$ in X by definition. Now suppose that $\|x\| \neq 0$. Since $\|x_n\| \to \|x\|$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|x_n\| \geqslant \dfrac{1}{2}\|x\|$. By discarding the first N terms in the sequence, we assume that $\|x_n\| \neq 0$ for all $n \in \mathbb{N}$. Let $y_n := x_n / \|x_n\|$ and $y := x / \|x\|$. For $\ell \in X^*$,

$$\lim_{n\to\infty} \ell(y_n) = \lim_{n\to\infty} \frac{\ell(x_n)}{\|x_n\|} = \frac{\lim_{n\to\infty} \ell(x_n)}{\lim_{n\to\infty} \|x_n\|} = \frac{\ell(x)}{\|x\|} = \ell(y)$$

Hence $y_n \rightharpoonup y$ in X. Let $z_n := \dfrac{1}{2}(y_n + y)$. Then $z_n \rightharpoonup y$ in X. By Theorem 3.2.2,

$$1 = \|y\| \leqslant \liminf_{n\to\infty} \|z_n\| = \frac{1}{2} \liminf_{n\to\infty} \|y_n + y\|$$

Suppose that $y_n \not\to y$ in X. Then there exists $\varepsilon > 0$ such that $\limsup_{n\to\infty} \|y_n - y\| \geqslant \varepsilon$. By definition of uniform convexity, there exists $\delta > 0$ such that $\|y_n + y\| \leqslant 2(1-\delta)$ whenever $\|y_n - y\| \geqslant \varepsilon$. Hence

$$\liminf_{n\to\infty} \|y_n + y\| \leqslant 2(1-\delta)$$

This contradicts the equation above. We deduce that $y_n \to y$ in X. Finally, since $\|x_n\| \to \|x\|$ in $\mathbb{R}$ and $x_n = y_n \|x_n\|$, by algebra of limits we conclude that $x_n \to x$ in X. □

Question 5

All sequence spaces in this question are real.

(a) Let $1 < p < \infty$. Show that a sequence $(x_n) \subseteq \ell^p$ converges weakly to x if and only if it is bounded and $x_n(j) \to x(j)$ for every j.

[*Hint: Use weak sequential compactness or the inequality*

$$\left|\sum_j \alpha(j)\beta(j)\right| \leqslant \left\{\sum_j |\alpha(j)|^p\right\}^{1/p} \left\{\sum_j |\beta(j)|^q\right\}^{1/q}. \quad]$$

(b) Show that a sequence in ℓ^1 is weakly convergent if and only if it is strongly convergent.

[*Hint: For given* $\varepsilon > 0$, *construct inductively increasing sequences* n_k *and* m_k *such that* $\sum_{j \leqslant m_{k-1}} |x_{n_k}(j)| < \varepsilon/8$ *and* $\sum_{j > m_k} |x_{n_k}(j)| < \varepsilon/8$. *Then test the weak convergence against* $b \in \ell^\infty$ *given by* $b(j) = \operatorname{sign}(x_{n_k}(j))$ *for* $m_{k-1} < j \leqslant m_k$.]

(c) Let $1 \leqslant p < \infty$ and let $e_n \in \ell^p$ denote the sequence $(\delta_{nj})_{j=1}^\infty$ where δ is the Kronecker delta. Does (e_n) converge weakly or strongly in ℓ^p? If it converges (weakly or strongly), identify its limit.

Proof. (a) " $\Longrightarrow$ " Suppose that (x_n) converges weakly in ℓ^p to x. By Theorem 3.2.1, (x_n) is bounded in ℓ^p. For each $j \in \mathbb{N}$, consider the bounded linear functional $\ell_j : \ell^p \to \mathbb{R}$, $y \mapsto y(j)$. Then by the definition of weak convergence,

$$\lim_{n\to\infty} x_n(j) = \lim_{n\to\infty} \ell_j(x_n) = \lim_{n\to\infty} \ell_j(x) = x(j)$$

" $\Longleftarrow$ " Suppose that $(x_n) \subseteq \ell^p$ is pointwise convergent to $x \in \ell^p$ and is bounded in ℓ^p. There exists $K > 0$ such that

$$\|x_n\|_p := \left(\sum_{j\in\mathbb{N}} |x_n(j)|^p\right)^{1/p} < K$$

for any $n \in \mathbb{N}$. But this implies that (x_n) is uniformly pointwise bounded. That is, $|x_n(j)| < K$ for all $n, j \in \mathbb{N}$.

Note that there is an isometric isomorphism $(\ell^p)^* \cong \ell^q$ where $q^{-1} + p^{-1} = 1$. Since (x_n) is bounded, by Hölder's inequality and bounded convergence theorem,

$$\lim_{n\to\infty} \left|\sum_{j\in\mathbb{N}} y(j)(x_n(j) - x(j))\right| \leqslant \lim_{n\to\infty} \|y\|_q \|x_n - x\|_p = \|y\|_q \left\|\lim_{n\to\infty} x_n - x\right\|_p = 0$$

for any $y \in \ell^q$. Hence $x_n \rightharpoonup x$ in ℓ^p.

(b) It is clear that the strong convergence implies the weak convergence. Suppose that $(x_n) \subseteq \ell^1$ converges weakly but not strongly to $x \in \ell^1$. Let $y_n := x_n - x$. Then there exists a subsequence (z_n) of (y_n) such that

$$\|z_n\|_1 = \sum_{j\in\mathbb{N}} |z_n(j)| \geqslant \varepsilon$$

for some $\varepsilon > 0$.

We shall construct the increasing sequences (n_k) and (m_k) in the following inductive way:

- Let $m_0 = 0$ and $n_0 = 0$.
- Suppose that we have constructed m_{k-1} and n_{k-1}. We claim that there exists $n_k \geqslant n_{k-1}$ such that

$$\sum_{j \leqslant m_{k-1}} |z_{n_k}(j)| < \frac{\varepsilon}{8}$$

From part (a) we know that $z_n(j) \to 0$ as $n \to \infty$ for each $n \in \mathbb{N}$. Then

$$\forall\, i \in \{0, \ldots, k-1\}\ \exists N_i \in \mathbb{N}\ \forall\, n > N_i\ |z_n(i)| < \frac{\varepsilon}{8(m_{k-1}+1)}$$

Let $n_k := \max\{n_{k-1}, N_0, \dots, N_{k-1}\}$. Then

$$\sum_{j \leq m_{k-1}} |z_{n_k}(j)| < \sum_{j=0}^{m_{k-1}} \frac{\varepsilon}{8(m_{k-1}+1)} = \frac{\varepsilon}{8}$$

- We claim that there exists $m_k \geq m_{k-1}$ such that

  $$\sum_{j \geq m_k} |z_{n_k}(j)| < \frac{\varepsilon}{8}$$

  Since $(z_{n_k}) \in \ell^1$,

  $$\|z_{n_k}\|_1 = \sum_{j \in \mathbb{N}} |z_{n_k}(j)| < \infty$$

  Choose sufficiently large m_k such that

  $$\sum_{j=0}^{m_k} |z_{n_k}(j)| > \|z_{n_k}\|_1 - \frac{\varepsilon}{8}$$

  and the result follows.

We identify $(\ell^1)^*$ with ℓ^∞. Consider $b \in \ell^\infty$ such that $b(j) = \operatorname{sgn}(z_{n_k})$ for $m_{k-1} < j \leq m_k$.

For sufficiently large k,

$$\langle b, z_{n_k} \rangle = \sum_{j \in \mathbb{N}} b(j) z_{n_k}(j) = \left(\sum_{j=0}^{m_{k-1}} + \sum_{j=m_k+1}^{\infty} \right) b(j) z_{n_k}(j) + \sum_{j=m_{k-1}+1}^{m_k} |z_{n_k}(j)|$$

where

$$\left| \left(\sum_{j=0}^{m_{k-1}} + \sum_{j=m_k+1}^{\infty} \right) b(j) z_{n_k}(j) \right| \leq \sum_{j=0}^{m_{k-1}} |z_{n_k}(j)| + \sum_{j=m_k+1}^{\infty} |z_{n_k}(j)| < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$$

In addition,

$$\varepsilon < \|x\| = \left(\sum_{j=0}^{m_{k-1}} + \sum_{j=m_k+1}^{\infty} \right) |z_{n_k}(j)| + \sum_{j=m_{k-1}+1}^{m_k} |z_{n_k}(j)| < \frac{\varepsilon}{4} + \sum_{j=m_{k-1}+1}^{m_k} |z_{n_k}(j)|$$

Therefore

$$\langle b, z_{n_k} \rangle > \frac{\varepsilon}{4} + \left(\varepsilon - \frac{\varepsilon}{4} \right) = \varepsilon$$

contradicting that $x_n \rightharpoonup 0$ in ℓ^1. We conclude that $x_n \to 0$ in ℓ^1.

(c) We claim that (e_n) converges weakly (but not strongly) to 0 in ℓ^p.

For any $n, m \in \mathbb{N}$ with $n \neq m$,

$$\|e_n - e_m\|_p = (1^p + (-1)^p)^{1/p} = 2^{1/p}$$

Hence (e_n) is not a Cauchy sequence in ℓ^p. It does not converge in norm.

Again we identify $(\ell^p)^*$ with ℓ^q where $p^{-1} + q^{-1} = 1$. For $y \in \ell^q$,

$$\sum_{j \in \mathbb{N}} |e_n(j) y(j)| = \sum_{j \in \mathbb{N}} |\delta_{nj} y(j)| = |y(n)| \to 0$$

as $n \to \infty$. Hence $e_n \rightharpoonup 0$ in ℓ^p. $\square$

Question 6

A sequence (ℓ_n) in the dual space X^* of a Banach space X is said to be weak* convergent to $\ell \in X^*$ if $\ell_n(x) \to \ell(x)$ for all $x \in X$.

(a) Show that weak* convergent sequences are bounded.

(b) Show that if X is separable, then the unit ball of X^* is weak* sequentially compact, i.e. every sequence (ℓ_n) in X^* with $\|\ell_n\|_* \leqslant 1$ has a weak* convergent subsequence.

[*Hint: Let* (x_n) *be a dense subset of* X. *Mimic the proof of the weak sequential compactness of the unit ball to construct a subsequence* (ℓ_{n_k}) *such that* $\ell_{n_k}(x_m)$ *is convergent for every* m.]

Proof. (a) Suppose that $\ell_n \rightharpoonup \ell$ in X^*. Then for any $x \in X$, $|(\ell_n - \ell)x| \to 0$ as $n \to \infty$. In particular the sequence $|(\ell_n - \ell)x|$ is bounded in $\mathbb{R}$ for each $x \in X$. By the uniform boundedness principle, there exists $K > 0$ such that $\|\ell_n - \ell\|_* < K$ for all $n \in \mathbb{N}$. By triangular inequality,

$$\sup_{n\in\mathbb{N}} \|\ell_n\|_* \leqslant \sup_{n\in\mathbb{N}} \|\ell_n - \ell\|_* + \|\ell\|_* \leqslant M + \|\ell\|_*$$

Hence the sequence (ℓ_n) is bounded in X^*.

(b) Let $\{x_m\}_{m\in\mathbb{N}}$ be a countable dense subset of X. Without loss of generality let $x_0 = 0$ and assume that $x_m \neq 0$ for all $m > 0$. Let $y_m := x_m / \|x_m\|$ for all $m > 0$. Then $\{y_m\}_{m\in\mathbb{Z}_+}$ is a countable dense subset of the unit sphere $S \subseteq X$. We claim that there exists a subsequence (ℓ_{n_k}) of (ℓ_n) such that $\ell_{n_k}(y_m) \to \ell(y_m)$ as $k \to \infty$ for all $m > 0$.

We inductively construct a map $S : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as follows:

- Let $S(k, 0) = k$.
- Suppose that we have constructed $S(k, j-1)$ for each $k \in \mathbb{N}$. Note that the sequence $\{\ell_{S(k,j-1)}(y_j)\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$, because

$$\left|\ell_{S(k,j-1)}(y_j)\right| \leqslant \left\|\ell_{S(k,j-1)}\right\|_* \left\|y_j\right\| \leqslant 1$$

  Since $[-1, 1]$ is sequentially compact in $\mathbb{R}$, by Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{\ell_{S(k,j)}(y_j)\}_{k\in\mathbb{N}}$ of $\{\ell_{S(k,j-1)}(y_j)\}_{k\in\mathbb{N}}$. That is, $\ell_{S(k,j)}(y_j) \to a_j$ as $k \to \infty$ for some $a_j \in [-1, 1]$.

Now, consider the subsequence $\{\ell_{S(j,j)}\}_{j\in\mathbb{N}}$ of $\{\ell_n\}_{n\in\mathbb{N}}$. For each fixed m, $S(j, j)$ is a subsequence of $S(j, m)$ for $j \geqslant m$. Therefore $\ell_{S(j,j)}(y_m) \to a_m$ as $j \to \infty$ for each $m > 0$.

Next, consider an arbitrary $x \in X \setminus \{0\}$. Fix $\varepsilon > 0$. Since $\{y_m\}$ is dense in S, there exists y_m such that $\left\| y_m - \frac{x}{\|x\|} \right\| < \frac{\varepsilon}{3\|x\|}$. Since $\{\ell_{S(j,j)}(y_m)\}_{j\in\mathbb{N}}$ is a Cauchy sequence, there exists $M \in \mathbb{N}$ such that $\left|(\ell_{S(j,j)} - \ell_{S(i,i)})(y_m)\right| < \frac{\varepsilon}{3\|x\|}$ for all $i, j > M$. Then

$$\begin{aligned}
\left|\ell_{S(i,i)}(x) - \ell_{S(j,j)}(x)\right| &= \|x\| \left|\left(\ell_{S(i,i)} - \ell_{S(j,j)}\right)\left(\frac{x}{\|x\|}\right)\right| \\
&\leqslant \|x\| \left|\left(\ell_{S(i,i)} - \ell_{S(j,j)}\right)\left(\frac{x}{\|x\|} - y_m\right)\right| + \|x\| \left|\left(\ell_{S(i,i)} - \ell_{S(j,j)}\right)(y_m)\right| \\
&< \|x\| \left\|\ell_{S(i,i)} - \ell_{S(j,j)}\right\|_* \left\|\frac{x}{\|x\|} - y_m\right\| + \varepsilon \\
&< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon
\end{aligned}$$

Hence $\{\ell_{S(j,j)}(x)\}_{j\in\mathbb{N}}$ is a Cauchy sequence.

Finally, we define $\ell : X \to \mathbb{R}$ by

$$\ell(x) := \lim_{j\to\infty} \ell_{S(j,j)}(x)$$

It is clear that ℓ is bounded and linear. Hence $\ell \in X^*$. We conclude that $\ell_{S(j,j)} \rightharpoonup \ell$ as $j \to \infty$ in X^*. The closed unit ball in X^* is weak sequentially compact. □