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Problem Sheet 1

B1.1: Logic

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Question 1 β

(a) Which of the following are formulae of $\mathcal{L}$? Give reasons.

(i) $(p_3 \to p_1)$

(ii) $p_1 \to p_2 \to p_3$

(iii) $(\neg p_5 \wedge \neg p_6) = \neg p_{11}$

(iv) $(p \leftrightarrow \neg p)$

(v) $((p_1 \vee \neg p_1) \to (\neg p_2))$

(b) Prove carefully that for any formula ϕ, the number of left parentheses occuring in ϕ is equal to the number of right parentheses occuring in ϕ.

Proof. We note the following fact: ψ begins with (if and only if it contains a binary symbol, if and only if it has length greater than 2.

Not actually a proof :P

(a) (i) This is a formula, formed by applying Rule III to p_3 and p_1. ✓

(ii) This is not a formula. Any formula with length greater than 1 must begin with either ¬ or (. ✓

(iii) This is not a formula. = is not in the alphabet of $\mathcal{L}$. ✓

(iv) This is a formula. By Rule I, p is a formula. By Rule II, $\neg p$ is a formula. By Rule III, $(p \leftrightarrow \neg p)$ is a formula.

Strictly speaking, p without subscripts is not a propositional variable — but that's a minor issue

(v) This is not a formula. It can be expressed as $(\phi \to \psi)$ where $\phi = (p_1 \wedge \neg p_1)$ and $\psi = (\neg p_2)$ are formulae. But $(\neg p_2)$ is not a formula, because it begins with (and does not contain any binary symbols.

This argument invokes the unique readability theorem. It might be easier to just quote Q2(a)

(b) We use induction on the number of left parentheses of ϕ:

If ϕ has no left parentheses, it must be either p or $\neg p$ for some propositional variable p. Hence ϕ has no right parentheses. The result holds.

φ may also be ¬¬p, etc. You could consider both negation and binary operators in one induction by considering length of formula instead — that's the more standard method

Suppose that the result holds for formulae with no more than n left parentheses. Assume that ϕ now have $n+1$ left parentheses. Since $n+1 > 0$, there exists formulae ψ_1 and ψ_2 such that $\phi = (\psi_1 \star \psi_2)$ for some binary symbol $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$. Both ψ_1 and ψ_2 have at most n left parentheses. By induction hypothesis, both ψ_1 and ψ_2 have equal number of left and right parentheses. It follows that ϕ has equal number of left and right parentheses. □

By dealing with negation and binary operators separately, you left out the cases ¬(φ * ψ), etc. again

Question 2 α-

(a) Prove that the length of a formula with exactly n occurences of the negation symbol and m occurences of binary connectives is $4m + n + 1$. Check this for the formulae in Question 1.(a).

Did you forgot this part? :P

(b) List all formulae of $\mathcal{L}$ of length $\leqslant 6$.

Proof. (a) We use induction on the length of the formula ϕ.

Base case: If the length of ϕ is 1, then $\phi = p$ is a propositional variable. In this case $m = n = 0$. Length $\ell = 1 = 4m + n + 1$.

Induction case: Suppose that the result holds for all formulae of length less than ℓ. Assume that ϕ has length ℓ. ϕ is one of the following:

(i) p, a propositional variable;

(ii) $\neg\psi$, where ψ is a formula;

Given that the length > 1 in the inductive case, you can simply list cases (ii), (iii) only

(iii) $(\psi_1 \star \psi_2)$, where ψ_1 and ψ_2 are formulae, and $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$.

For (i), this is the base case. For (ii), ψ is a formula with m binary symbols and $n-1$ negation symbols. It has length $\ell - 1$. By induction hypothesis, $\ell - 1 = 4m + (n-1) + 1$. Hence $\ell = 4m + n + 1$. For (iii), ψ_1 and ψ_2 are formulae of length less than ℓ. Suppose that ψ_1 has m_1 binary symbols, n_1 negation symbols and length ℓ_1, and ψ_2 has m_2 binary symbols, n_2 negation symbols, and length ℓ_2. Then

$$\ell_1 = 4m_1 + n_1 + 1, \quad \ell_2 = 4m_2 + n_2 + 1, \quad m = m_1 + m_2 + 1, \quad n = n_1 + n_2, \quad \ell = \ell_1 + \ell_2 + 3.$$

We deduce that $\ell = 4m + n + 1$. ✓

(b) The equation $4m + n + 1 \leqslant 6$ $(m, n \in \mathbb{N})$ has the following solutions:

$$(m, n) = (0,0),\ (0,1),\ (0,2),\ (0,3),\ (0,4),\ (0,5),\ (1,0),\ (1,1).$$

The possible formulae are:

$$p_1,\ \neg p_1,\ \neg\neg p_1,\ \neg\neg\neg p_1,\ \neg\neg\neg\neg p_1,\ \neg\neg\neg\neg\neg p_1,$$
$$(p_1 \wedge p_2),\ (p_1 \vee p_2),\ (p_1 \to p_2),\ (p_1 \leftrightarrow p_2),$$
$$(\neg p_1 \wedge p_2),\ (\neg p_1 \vee p_2),\ (\neg p_1 \to p_2),\ (\neg p_1 \leftrightarrow p_2),$$
$$(p_1 \wedge \neg p_2),\ (p_1 \vee \neg p_2),\ (p_1 \to \neg p_2),\ (p_1 \leftrightarrow \neg p_2),$$

where p_1 and p_2 can be replaced by arbitrary propositional variables. What about ¬(p1 * p2) ? □

Question 3 β

Can a proper initial segment of a formula ever be a formula again? How about final proper segments?

Proof. A string of formula cannot have a proper initial substring that is also a formula. By Question 2, a formula must have equal number of left and right parentheses.

We use induction on the length of the formula to prove: a proper initial substring of a formula contains more left parentheses than right parentheses.

This is not true! Consider "¬" as initial segment of "¬ψ". It is true only if you use weak inequality "no less"

Base case: Suppose that ϕ is a formula of length 1. It does not contain proper initial substrings.

Note: Some accept empty string (but not a formula). That's also okay

Induction case: Suppose that the result holds for formulae of length less than n. Assume that ϕ is a formula of length n. ϕ is one of the following:

(i) p, a propositional variable;

(ii) $\neg\psi$, where ψ is a formula;

(iii) $(\psi_1 \star \psi_2)$, where ψ_1 and ψ_2 are formulae, and $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$.

A proper way to attempt this question:
You inductive hypothesis should be both
1) a proper initial segment of a formula contains no less left parenthesis than right parenthesis; and
2) a proper initial segment is not a formula

For (i), this is the base case. For (ii), a proper initial substring of ϕ is either $\neg$ or $\neg\chi$, where χ is a proper initial substring of ψ. By induction hypothesis, χ is not a formula. Hence $\neg\chi$ is not a formula. For (iii), a proper initial substring of ϕ is one of the following:

(a) $($;

(b) $(\chi$, where χ is a proper initial substring of ψ_1;

(c) $(\psi_1$;

(d) $(\psi_1\star$;

(e) $(\psi_1 \star \chi$, where χ is a proper initial substring of ψ_2;

(f) $(\psi_1 \star \psi_2$.

Your proof still generally applies:
- for case (ii), both (1) and (2) come trivially from inductive hypothesis;
- for case (iii), you only need to invoke the weak inequality from inductive hypothesis (because from (a) - (f) you always have one extra left parenthesis outside, so it suffices even if χ has as many left parentheses as right) — Here the whole proper initial segment does have more left parentheses than right, but it's restricted to case (iii) only

Note that by induction hypothesis, χ has more left parentheses than right parentheses. ψ_1 and ψ_2 has equal number of left and right parentheses by Question 2.(a). Hence in all situations χ has more left parentheses than right parentheses.

A string of formula can have a proper final substring that is also a formula. For example $\neg p$ is formula, and p is also a formula. ✓ □

Question 4 α

Prove the Unique Readability Theorem.

Proof. Let ϕ be a formula. By definition, ϕ is one of the following:

(i) p, a propositional variable;

(ii) $\neg\psi$, where ψ is a formula;

(iii) $(\psi_1 \star \psi_2)$, where ψ_1 and ψ_2 are formulae, and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

If we look at the first symbol, we observe that these three readings are mutually exclusive. ✓

The first case is trivial. For the second case, if $\phi = \neg\psi = \neg\chi$, then $\psi = \chi$. For the third case, suppose that $\phi = (\psi_1 \star \psi_2) = (\chi_1 * \chi_2)$. Then $\psi_1 \star \psi_2 = \chi_1 * \chi_2$ as strings. If $\star \neq *$, it follows that either ψ_1 is a proper initial substring of χ_1 or χ_1 is a proper initial substring of ψ_1. But χ_1 and ψ_1 are both formulae. This is impossible by Question 3. Hence $\star = *$ and $\psi_1 = \chi_1$. It follows that $\psi_2 = \chi_2$. The theorem is hence proven. $\square$

The condition that the two binary symbols are not "equal" is not good — It might be that the two binary symbols are the same, but appear at different locations, then you still do not have unique readability. You should rather directly assume that ψ1 and χ1 are not the same, then invoke Q3.