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Problem Sheet 2

C2.2: Homological Algebra

Overall mark: α

2 November, 2021

Section A: Introductory

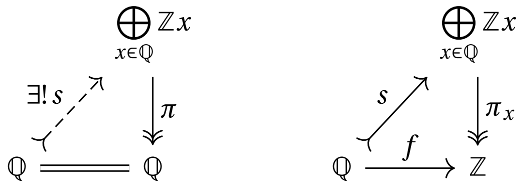
Question 1

Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Proof. Suppose that $\mathbb{Q}$ is a projective $\mathbb{Z}$-module. Let F be the free $\mathbb{Z}$-module generated by elements in $\mathbb{Q}$, that is,

$$F := \bigoplus_{x \in \mathbb{Q}} \mathbb{Z}x$$

Let $\pi : F \to \mathbb{Q}$ be the canonical projection. Since $\mathbb{Q}$ is projective, there exists $s : \mathbb{Q} \to F$ such that $\pi \circ s = \mathrm{id}_{\mathbb{Q}}$.



As s is injective, we can embed $\mathbb{Q}$ into F as a submodule. There exists $x \in \mathbb{Q}$ such that the projection $\pi_x : F \to \mathbb{Z}e_x \cong \mathbb{Z}$ satisfies that $f := \pi_x \circ s \neq 0$. Let $f(1) = m \in \mathbb{Z} \setminus \{0\}$. We have $1 = m \cdot f(1/m^2)$, where both $f(1/m^2) \in \mathbb{Z}$ and $m \in \mathbb{Z}$. This is a contradiction. $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. □

Section B: Core

Question 2

Write an injective resolution for $\mathbb{Z}$ as a $\mathbb{Z}$-module.

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Since $\mathbb{Z}$ is a principal ideal domain, every divisible $\mathbb{Z}$-module is injective. Hence $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective. Therefore the short exact sequence gives an injective resolution for $\mathbb{Z}$. □

Question 3

Write free resolutions for:

1. $\mathbb{Z}/2$ in $\mathbb{Z}$-Mod,
2. $\mathbb{Z}/2$ in $(\mathbb{Z}/2)[x]$-Mod,
3. $\mathbb{Z}/2$ in $\mathbb{Z}[x]$-Mod,
4. $\mathbb{Z}/2$ in $\mathbb{Z}[x]/\langle 2x \rangle$-Mod.

Proof.

1. We know that the following sequence of $\mathbb{Z}$-modules is exact:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2 \longrightarrow 0$$

where $\mathbb{Z}$ is a free $\mathbb{Z}$-module. The short exact sequence gives an injective resolution for $\mathbb{Z}/2$ as a $\mathbb{Z}$-module.

2. We know that the following sequence of $(\mathbb{Z}/2)[x]$-modules is exact:

$$0 \longrightarrow (\mathbb{Z}/2)[x] \xrightarrow{\cdot x} (\mathbb{Z}/2)[x] \xrightarrow{\mathrm{ev}} \mathbb{Z}/2 \longrightarrow 0$$

where $\pi : (\mathbb{Z}/2)[x] \to \mathbb{Z}/2$ is the evaluation homomorphism induced by $x \mapsto 0$. The short exact sequence gives an injective resolution for $\mathbb{Z}/2$ as a $(\mathbb{Z}/2)[x]$-module.

3. First we consider the projection $p\colon \mathbb{Z}[x] \to \mathbb{Z}/2$. It is easy to observe that $\mathbb{Z}/2 \cong \dfrac{\mathbb{Z}[x]}{\langle 2, x\rangle}$.

$$\mathbb{Z}[x] \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 0$$

Then $\ker p = \langle 2, x\rangle$ by first isomorphism theorem. We can construct a surjection $q\colon \mathbb{Z}[x] \oplus \mathbb{Z}[x] \to \ker p$ by $(f, g) \mapsto 2f(x) + xg(x)$.

$$\mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{\ q\ } \ker p \longrightarrow 0$$

Since 2 and x are coprime in $\mathbb{Z}[x]$, for $(f, g) \in \ker q$, we have $f \in \langle x\rangle$ and $g \in \langle 2\rangle$. So $\ker q = \langle x\rangle \oplus \langle 2\rangle$. Finally it is easy to see that there is a bijection $r\colon \mathbb{Z}[x] \oplus \mathbb{Z}[x] \to \ker q$ given by $(f, g) \mapsto (xf, 2g)$.

$$0 \longrightarrow \mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{\ r\ } \ker q \longrightarrow 0$$

Patching the three exact sequences together, we obtain the free resolution of $\mathbb{Z}/2$ as a $\mathbb{Z}[x]$-module:

$$0 \longrightarrow \mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{\ r\ } \mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{\ q\ } \mathbb{Z}[x] \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 0$$

4. We will have the same argument with part (3), constructing the following exact sequence

$$\frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ r\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ q\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 0$$

except that $r\colon (f, g) \mapsto (xf, 2g)$ is no longer injective. As $2x = 0$ in $\dfrac{\mathbb{Z}[x]}{\langle 2x\rangle}$, we have $\ker r = \langle 2\rangle \oplus \langle x\rangle$.

Let $s\colon \dfrac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \dfrac{\mathbb{Z}[x]}{\langle 2x\rangle} \to \ker r$ given by $(f, g) \mapsto (2f, xg)$. Then $\ker s = \langle x\rangle \oplus \langle 2\rangle = \ker q$. We have

$$\frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ s\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ r\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ q\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 0$$

$$\frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ s\ } \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \oplus \frac{\mathbb{Z}[x]}{\langle 2x\rangle} \xrightarrow{\ r\ } \ker s \longrightarrow 0$$

By patching the two sequences we obtain an unbounded free resolution for $\mathbb{Z}/2$ as a $\mathbb{Z}[x]/\langle 2x\rangle$ module:

$$\cdots \xrightarrow{\ s\ } R^2 \xrightarrow{\ r\ } R^2 \xrightarrow{\ s\ } R^2 \xrightarrow{\ r\ } R^2 \xrightarrow{\ q\ } R \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 0$$

where $R := \mathbb{Z}[x]/\langle 2x\rangle$. □

Question 4

Let R be a commutative ring, $r \in R$, and $M \in R$-Mod. Define $R\left[r^{-1}\right] := \frac{R[x]}{rx-1} = \operatorname{coker}(R[x] \overset{rx-1}{\to} R[x])$ and $M\left[r^{-1}\right] = \operatorname{coker}(M[x] \overset{rx-1}{\to} M[x])$ where $M[x] = \left\{\sum_i m_i x^i\right\}$ is viewed naturally as an $R[x]$-module.

Show that $M \otimes_R R\left[r^{-1}\right] \simeq M\left[r^{-1}\right]$.

Proof. We have the short exact sequence of R-modules

$$0 \longrightarrow \langle rx - 1\rangle_{R[x]} \longrightarrow R[x] \longrightarrow R[r^{-1}] \longrightarrow 0$$

where $\langle rx-1\rangle_{R[x]}$ is the ideal of $R[x]$ generated by $(rx-1)$. Since the functor $M \otimes_R -$ is right exact, we have the following exact sequence

$$M \otimes_R \langle rx-1\rangle_{R[x]} \longrightarrow M \otimes_R R[x] \longrightarrow M \otimes_R R[r^{-1}] \longrightarrow 0$$

Similarly we have the short exact sequence of R-modules

$$0 \longrightarrow \langle rx-1\rangle_{M[x]} \longrightarrow M[x] \longrightarrow M[r^{-1}] \longrightarrow 0$$

where $\langle rx-1\rangle_{M[x]}$ is the submodule of $M[x]$ generated by $(rx-1)$. We can patch the two sequences together as follows

$$\begin{array}{ccccccc} \langle rx-1\rangle_{M[x]} & \longrightarrow & M[x] & \longrightarrow & M[r^{-1}] & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \| \\ M \otimes_R \langle rx-1\rangle_{R[x]} & \longrightarrow & M \otimes_R R[x] & \longrightarrow & M \otimes_R R[r^{-1}] & \longrightarrow & 0 \end{array}$$

The homomorphisms α, β, γ are given by

$$\alpha : \quad \sum_{i=1}^{n} m_i x^i (rx-1) \longmapsto \sum_{i=1}^{n} m \otimes_R x^i (rx-1)$$

$$\beta : \quad \sum_{i=1}^{n} m_i x^i \longmapsto \sum_{i=1}^{n} m \otimes_R x^i$$

$$\gamma : \quad \sum_{i=1}^{n} m_i r^{-i} \longmapsto \sum_{i=1}^{n} m_i \otimes_R r^{-i}$$

It is straight forward to check that γ is well-defined, everything is an R-module homomorphism, and the diagram above commutes.

It is trivial that β is injective. Every element in $M \otimes_R \langle rx-1\rangle_{R[x]}$ has the form $m \otimes_R \sum a_i x^i (rx-1)$, where $m \in M$ and $a_i \in R$. Then we have

$$\alpha\left(\sum_{i=1}^{n} m a_i x^i (rx-1)\right) = \sum_{i=1}^{n} a_i \alpha(m x^i (rx-1)) = m \otimes_R \sum_{i=1}^{n} a_i x^i (rx-1)$$

Hence α is surjective.

Now by the **Four Lemma** (a half of the five lemma), we conclude that γ is an isomorphism. So $M[r^{-1}] \cong M \otimes_R R[r^{-1}]$ as R-modules. ✓ □

Question 5

Prove the general Frobenius reciprocity formula (Tensor-Hom adjunction):

$\mathrm{Hom}_S(A, \mathrm{Hom}_R(B,C)) \cong \mathrm{Hom}_R(A \otimes_S B, C)$. where A is a right S-module, B is an (S,R)-bimodule, and C is a right R-module.

α *Proof.* This is a very straightforward verification.

Let $\sigma \in \mathrm{Hom}_S(A, \mathrm{Hom}_R(B,C))$. σ defines a map $\tilde{\sigma} : A \times B \to C$ by $\tilde{\sigma}(a,b) := \sigma(a)(b)$. We note the $\tilde{\sigma}$ defines a balanced product:

$$\begin{aligned} \tilde{\sigma}(a+a',b) &= \sigma(a+a')(b) = (\sigma(a)+\sigma(a'))(b) = \sigma(a)(b)+\sigma(a')(b) = \tilde{\sigma}(a,b)+\tilde{\sigma}(a',b) \\ \tilde{\sigma}(a,b+b') &= \sigma(a)(b+b') = \sigma(a)(b)+\sigma(a)(b') = \tilde{\sigma}(a,b)+\tilde{\sigma}(a,b') \\ \tilde{\sigma}(as,b) &= \sigma(as)(b) = (\sigma(a)s)(b) = \sigma(a)(sb) = \tilde{\sigma}(a,sb) \end{aligned}$$

By the universal property of $A \otimes_S B$, there exists a unique right R-module homomorphism $\varphi : A \otimes_S B \to C$ such that $\sigma(a)(b) = \varphi(a \otimes_S b)$. We claim that $\sigma \mapsto \varphi$ defines an isomorphism of *Abelian groups*

$$F\colon \operatorname{Hom}_{\mathsf{Mod}\text{-}S}(A, \operatorname{Hom}_{\mathsf{Mod}\text{-}R}(B,C)) \to \operatorname{Hom}_{\mathsf{Mod}\text{-}R}(A \otimes_S B, C)$$

The assignment $\sigma \mapsto \varphi$ trivially preserves addition. If $F(\sigma) = 0$, then $\sigma(a)(b) = 0$ for all $a \in A$ and $b \in B$. Hence $\sigma(a) = 0$ for all $a \in A$. Hence $\sigma = 0$. This implies that F is injective.

For $\varphi \in \operatorname{Hom}_{\mathsf{Mod}\text{-}R}(A \otimes_S B, C)$, let $\sigma_a\colon b \mapsto \varphi(a \otimes_S b)$. Then we have

$$\sigma_a(br + b'r') = \varphi(a \otimes_S (br + b'r')) = \varphi((a \otimes_S b)r + (a \otimes_S b')r') = \varphi(a \otimes_S b)r + \varphi(a \otimes_S b')r' = \sigma_a(b)r + \sigma_a(b')r'$$

Hence $\sigma_a \in \operatorname{Hom}_{\mathsf{Mod}\text{-}R}(B,C)$. Let $\sigma\colon a \mapsto \sigma_a$. Then we have

$$\sigma(as + a's')(b) = \varphi((as + a's') \otimes_S b) = \varphi(a \otimes_S sb) + \varphi(a' \otimes_S s'b) = \sigma(a)(sb) + \sigma(a')(s'b) = (\sigma(a)s + \sigma(a')s')(b)$$

Hence $\sigma \in \operatorname{Hom}_{\mathsf{Mod}\text{-}S}(A, \operatorname{Hom}_{\mathsf{Mod}\text{-}R}(B,C))$. We deduce that F is surjective. □

Section C: Optional

Question 6

Show that every R-submodule of a free R-module M is free when R is a PID.

Proof. Let N be a R-submodule of M. Let X be a basis of M. We consider the set $\mathscr{S}$ of triplets (Y, Z, b), where

- $Z \subseteq Y \subseteq X$;
- $N_Y := N \cap \bigoplus_{y \in Y} Ry$ is free;
- $b\colon Z \to N$ is a map such that $\operatorname{im} b$ is a basis of N_Y.

Equip $\mathscr{S}$ with the partial order

$$(Y, Z, b) \leqslant_{\mathscr{S}} (Y', Z', b') \iff (Y \subseteq Y') \wedge (Z \subseteq Z') \wedge (b'|_Z = b)$$

$\mathscr{S}$ is non-empty, as $(\varnothing, \varnothing, \varnothing) \in \mathscr{S}$. Let $\{(Y_i, Z_i, b_i)\}_{i \in I}$ be a chain in $\mathscr{S}$. Let $Y := \bigcup_i Y_i$, $Z := \bigcup_i Z_i$ and $b = \bigcup_i b_i$. We claim that $(Y, Z, b) \in \mathscr{S}$. Indeed $Z \subseteq Y$. The union $\operatorname{im} b = \bigcup_i \operatorname{im} b_i$ is clearly linearly independent and spans N_Y. Hence N_Y is free.

Now by Zorn's Lemma, $\mathscr{S}$ has a maximal element, which will be denoted again by (Y, Z, b). Hopefully it does not cause any ambiguity in the subsequent discussions.

We claim that $Y = X$. Suppose for contradiction that it is not. Then we take $x \in X \setminus Y$. Consider the ideal

$$I := \left\{ a \in R\colon \left(ax + \bigoplus_{y \in Y} Ry \right) \cap N \neq \varnothing \right\}$$

If $I = \{0\}$, then $N_{Y \cup \{x\}} = N \cap \left(\bigoplus_{y \in Y} Ry \oplus Rx \right) = N_Y$. We have $(Y, Z, b) <_{\mathscr{S}} (Y \cap \{x\}, Z, b)$. This is a contradiction.

Suppose that $I \neq \{0\}$. Since R is a PID, $I = \langle c \rangle$ for some $c \in R$. Pick

$$m = cx + \sum_i a_i y_i \in \left(cx + \sum_{y \in Y} Ry \right) \cap N$$

We claim that $N_{Y \cup \{x\}} = N_Y \oplus Rm$. For $n \in N_{Y \cup \{x\}}$, $n = \sum_j b_j y'_j + rx$ for some $b, b_i \in R$ and $y'_j \in Y$. Then by definition

$r \in \langle c \rangle$. Let $r = sc$ for some $s \in R$. Hence

$$n = \sum_j b_j y'_j + scx = sm + \left(\sum_j b_j y'_j - \sum_i a_i y_i \right) \in N_Y + Rm$$

It is clear that $N_Y \cap Rm = \{0\}$ as $Y \cup \{x\}$ is linearly independent. This proves our claim. Now we let $Z' = Z \cup \{x\}$, $Y' = Y \cup \{x\}$, and $b' : Z' \to N$ which satisfies $b'|_Z = b$ and $b'(x) = m$. We have $(Y, Z, b) <_{\mathscr{S}} (Y', Z', b')$. This is a contradiction.

In conclusion, we have $X = Y$. Hence $N_Y = N \cap \bigoplus_{y \in X} Rx = N$ is free. □