Topology & Groups

Michaelmas 2016

Question Sheet 3

Questions with an asterisk $*$ beside them are optional.

You may assume throughout this sheet that $\pi_1(S^1) \cong \mathbb{Z}$, and that a generator for $\pi_1(S^1)$ is represented by the loop $t \mapsto e^{2\pi i t}$.

1. Show that for a space X, the following three conditions are equivalent:

   (i) Every map $S^1 \to X$ is homotopic to a constant map.

   (ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

   Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

2. Let X and Y be spaces with basepoints x_0 and y_0. Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

   Deduce that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

3. A *retraction* of a space X onto a subspace A is a map $r\colon X \to A$ such that $ri = \mathrm{id}_A$, where $i\colon A \to X$ is the inclusion map.

(i) Prove that there is no retraction map $r\colon D^2 \to S^1$.

(ii) Our aim here is to show that any map $f\colon D^2 \to D^2$ has a fixed point. Suppose that, on the contrary, f has no fixed point; in other words $f(x) \neq x$ for all $x \in D^2$. Use the pairs $(x, f(x))$ to construct a retraction $D^2 \to S^1$. Thus, we deduce that any map $D^2 \to D^2$ must have a fixed point.

   This fact has many applications outside of topology. For example, it can be used to show that certain differential equations always have a solution.

4. For $n > 2$, prove that no two of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^n$ are homeomorphic.

$*$ 5. Show that there is no retraction of a Möbius band onto its boundary.

Topology & Groups 3

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1. (i) $\Rightarrow$ (ii): Suppose $f: S^1 \to X$ is continuous. Then $f \simeq c_{x_0}$ for some $x_0 \in X$ via a homotopy $H: S^1 \times [0,1] \to X$.

Let $\pi: S^1 \times [0,1] \to D^2$ defined by:

$$\pi(x,t) = (1-t)x.$$

Then $\pi(x,0) = \mathrm{id}_{S^1}$, $\pi(x,1) = 0 \in D^2$

Let $g: D^2 \to X$ defined by: $\boxed{\text{cannot use the gluing lemma to a disk with center removed!}}$

$$g \circ \pi(x,t) = H(x,t)$$ $\boxed{\text{just use constant map!}}$ say what this means

g is well-defined, as π is surjective and everywhere injective except for $t=1$, in which $g(0) = H(x,1) = 0$ is also well-defined.

why?

Then g is continuous, and $g|_{S^1} = H(x,0) = f$ as required.

(ii) $\Rightarrow$ (iii): For $x_0 \in X$ and a based loop $f: [0,1] \to X$ such that $f(0) = f(1) = x_0$, it induces a continuous map $\tilde{f}: S^1 \to X$ such that $f = \tilde{f} \circ g$, where $g: [0,1] \to S^1$ is given by $g(t) = e^{2\pi i t}$.

By (ii), $\tilde{f}$ extends to a continuous map $F: D^2 \to X$.

Since D^2 is contractible, we have $\mathrm{id}_{D^2} \simeq c_1$ (We let $F(1) = x_0$) and $F = F \circ \mathrm{id}_{D^2} \simeq F \circ c_1 = c_{x_0}$.

Hence $\tilde{f} \simeq c_{x_0}$ and $f \simeq c_{x_0}$. Why? Need to check this is true relative to ∂I

Every based loop is null-homotopic. Hence $\pi_1(X, x_0) = \{e\}$ for all $x_0 \in X$ and $\pi_1(X) = \{e\}$.

(iii) $\Rightarrow$ (i): Suppose $f: S^1 \to X$ is a continuous map.

$f(1) = x_0 \in X$. By (iii), $\pi_1(X, x_0)$ is trivial.

~~Let $g: S^1 \to I$ defined by $g(e^{2\pi i t}) = t$.~~

Let g defined as the previous part. Then $f \circ g: [0,1] \to X$

X is simply-connected $\Leftrightarrow$ X is path-connected + (i) $\overset{\text{why?}}{\Leftrightarrow}$ all $f: S^1 \to X$ are homotopic

is a based loop at x_0. Since $\pi_1(X, x_0) = \{e\}$, we have $f \circ g \simeq c_{x_0}$. Hence f is null-homotopic. relative

2. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection maps. By Proposition 3.18, they induce group homomorphisms:
$\pi_X^* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and
$\pi_Y^* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$.
We shall show that $\theta := \pi_X^* \times \pi_Y^*$ is a group isomorphism between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$:
It is clear from definition that θ is a group homomorphism.
Injectivity: Suppose $f : [0,1] \to X \times Y$ is a loop based at (x_0, y_0) such that $[f] \in \operatorname{Ker} \theta$.

$$\begin{tikzcd} & \pi_1(X\times Y,(x_0,y_0)) \arrow[dl, "p_1^*"'] \arrow[dr, "p_2^*"] \arrow[dd, "\theta"] & \\ \pi_1(X,x_0) & & \pi_1(Y,y_0) \\ & \pi_1(X,x_0)\times\pi_1(Y,y_0) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then $\theta([f]) = \left(\pi_X^*([f]), \pi_Y^*([f])\right) = id$
$\Rightarrow \pi_X^*([f]) = id_{\pi_1(X, x_0)}$, $\pi_Y^*([f]) = id_{\pi_1(Y, y_0)}$
$\Rightarrow \pi_X \circ f \simeq c_{x_0}$, $\pi_Y \circ f \simeq c_{y_0}$.
Suppose the respective homotopies are given by F_X and F_Y.
Then $f \simeq c_{(x_0, y_0)}$ via the homotopy $F_X \times F_Y$. Great
It shows that $\operatorname{Ker} \theta$ is trivial and hence θ is injective.
Surjectivity: Let $g : [0,1] \to X$ be a loop based at x_0 and $h : [0,1] \to Y$ a loop based at y_0.
$f : [0,1] \to X \times Y$ defined by $f := g \times h$ is a loop based at (x_0, y_0). Moreover we have $\theta([f]) = ([g], [h])$.
Hence θ is surjective.
In conclusion, θ is a group isomorphism and
$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_2(Y, y_0)$.

(This isomorphism is so canonical that I believe there exists an elegant categorical proof of this result.) You have shown that the functor π_1 preserves products.

The torus $T^2 := S^1 \times S^1$. We have:

$\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

Great A^+

3. (i) Suppose there exists a retraction $r: D^2 \to S^1$, i.e. $r \circ i = id_{S^1}$ where $i: S^1 \hookrightarrow D^2$ is the inclusion.
They induce group homomorphisms $r_*: \pi_1(D^2) \to \pi_1(S^1)$ and $i_*: \pi_1(S^1) \to \pi_1(D^2)$ such that $r_* \circ i_* = id_{\pi_1(S^1)}$.
This is impossible, because $\pi_1(D^2)$ is trivial, so that $r_* \circ i_*$ is trivial. $\pi_1(S^1) \cong \mathbb{Z}$ is not trivial.

(ii) Suppose f has no fixed points. Then $g: D^2 \to S^1$ given by $g(x) = \frac{x - f(x)}{\|x - f(x)\|}$ is well-defined and continuous.
We shall prove that $g \circ i \simeq id_{S^1}$ where $i: S^1 \hookrightarrow D^2$ is the inclusion.
Suppose $\exists x_0 \in S^1 : g(x_0) = -x_0$.

$\Rightarrow \frac{x_0 - f(x_0)}{\|x_0 - f(x_0)\|} = -x_0 \Rightarrow f(x_0) = x_0(1 + \|x_0 - f(x_0)\|)$

$\Rightarrow \|f(x_0)\| = \|x_0\| \cdot (1 + \|x_0 - f(x_0)\|) = 1 + \|x_0 - f(x_0)\| > 1$

contradicting that f maps D^2 onto D^2.

$\Rightarrow \forall x \in S^1 : g(x) \neq -x$. By Q1 in Sheet 2, we have $g|_{S^1} \simeq id_{S^1} \Rightarrow g \circ i \simeq id_{S^1}$.

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But $i: S^1 \hookrightarrow D^2$ is null-homotopic as $\pi_1(D^2) = \{e\}$.

$\Rightarrow g \circ i$ is null-homotopic $\Rightarrow id_{S^1}$ is null-homotopic contradicting that $\pi_1(S^1) \cong \mathbb{Z}$.
Hence f has a fixed point in D^2.

V. Good but you are asked to find a retraction.

4. Proving that $\mathbb{R} \not\cong \mathbb{R}^2$ or $\mathbb{R} \not\cong \mathbb{R}^n$ is a standard question in the metric space course.

Suppose $\mathbb{R}$ and $\mathbb{R}^n$ $(n>1)$ are homeomorphic. Then there exists a continuous bijection $f: \mathbb{R}^n \to \mathbb{R}$.

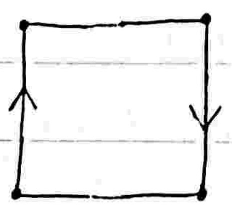
Let $p \in \mathbb{R}$. ~~such that $\inf_{x\in\mathbb{R}^n} f(x) < p < \sup_{x\in\mathbb{R}^n} f(x)$~~

Since f is continuous, $f^{-1}((-\infty, p))$ and $f^{-1}((p, +\infty))$ are open sets in $\mathbb{R}^2$. Moreover, $f^{-1}((-\infty, p)) \neq \emptyset$, $f^{-1}((p, +\infty)) \neq \emptyset$, $f^{-1}((-\infty, p)) \cap f^{-1}((p,+\infty)) = \emptyset$ and $f^{-1}((-\infty, p)) \cup f^{-1}((p,+\infty))$ $= \mathbb{R}^2 \setminus f^{-1}(\{p\})$. This implies that $\mathbb{R}^2 \setminus f^{-1}(\{p\})$ is disconnected. But f is injective, so $f^{-1}(\{p\})$ is a singleton and $\mathbb{R}^2 \setminus f^{-1}(\{p\})$ is connected. Contradiction.

Suppose $\mathbb{R}^2$ and $\mathbb{R}^n$ $(n>2)$ are homeomorphic via the homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^2$.

Let $y_0 = f(0) \in \mathbb{R}^2$. Then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{y_0\}$ are homotopy equivalent. But $\mathbb{R}^2 \setminus \{y_0\} \simeq S^1$ and hence $\pi_1(\mathbb{R}^2 \setminus \{y_0\}) = \mathbb{Z}$, whereas any loop in $\mathbb{R}^n \setminus \{0\}$ is null-homotopic via the straight-line homotopy (with possible indentation around $0 \in \mathbb{R}^n$). So $\pi_1(\mathbb{R}^n \setminus \{0\}) = \{e\} \neq \mathbb{Z} = \pi_1(\mathbb{R}^2 \setminus \{y_0\})$, which is a contradiction. ✓ Great A+

5. We construct the Möbius band by gluing the sides of a square:



Suppose there exists a retraction $r: M \to \partial M$.

Notice that $M \simeq S^1 \simeq \partial M$ by Example 2.14. We have $\pi_1(M) = \pi_1(\partial M) = \mathbb{Z}$.

Since r is a retraction, $r \circ i = id_{\partial M}$ where $i : \partial M \to M$ is the inclusion map. They induce group homomorphisms:
$r_* : \pi_1(M) \to \pi_1(\partial M)$ and $i_* : \pi_1(\partial M) \to \pi_1(M)$ such that $r_* \circ i_* = id_{\pi_1(\partial M)}$.

~~But~~ Suppose $f : [0,1] \to M$ is a loop based at $x_0 \in \partial M$ with ~~$\pi : M \to \partial M$ is the~~ winding number $1 \in \mathbb{Z}$. $r \circ f$ would be a loop with winding number $2 \in \mathbb{Z}$. So $r_* : \mathbb{Z} \to \mathbb{Z}$ is not the identity (which is the unique eqimorphism from $\mathbb{Z}$ to $\mathbb{Z}$) ~~≠~~ and is not surjective. $\Rightarrow r_* \circ i_* \neq id_{\pi_1(\partial M)}$ Contradiction.

There is ~~a~~ no retraction $r : M \to \partial M$.

KOKUYO