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Problem Sheet 4

B3.2: Geometry of Surfaces

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Question 1

The smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = \cos 2\pi x + \cos 2\pi y$. Determine and classify the critical points of f.

A torus T is formed by identifying opposite edges of $[0,1] \times [0,1]$ so that f induces a smooth function on T. Use it to verify that $\chi(T) = 0$.

Proof. Note that f is doubly periodic:

$$\forall\, x, y \in \mathbb{R} \quad f(x+1, y) = f(x, y+1) = f(x, y)$$

Hence f induces a smooth function on $T^2 = [0,1]^2/\sim$.

The gradient of f is $\nabla f = (-2\pi \sin 2\pi x, -2\pi \sin 2\pi y)$. The Hessian matrix is

$$H(x, y) = \begin{pmatrix} -4\pi^2 \cos 2\pi x & 0 \\ 0 & -4\pi^2 \cos 2\pi y \end{pmatrix}$$

At the critical points,

$$\nabla f = 0 \iff \sin 2\pi x = \sin 2\pi y = 0 \iff x = \frac{n}{2} \wedge y = \frac{m}{2},\ n, m \in \mathbb{Z}$$

Hence there are 4 critical points in $[0,1)^2$: $(0,0)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$. The Hessian matrix at each point is given respectively by

$$H(0,0) = -4\pi^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H\left(0, \frac{1}{2}\right) = -4\pi^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H\left(\frac{1}{2}, 0\right) = -4\pi^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H\left(\frac{1}{2}, \frac{1}{2}\right) = -4\pi^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ are local extrema. $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$ are saddles.

Therefore the Euler characteristic of the torus is $\chi(T^2) = 1 - 2 + 1 = 0$. □

Question 2

Prove that along a geodesic γ on a surface of revolution the product $\rho \sin\varphi$ is constant, where $\rho(s)$ is the distance from $\gamma(s)$ to the axis of revolution, and $\varphi(s)$ is the angle between $\gamma'(s)$ and the meridian through $\gamma(s)$. Prove that on the ellipsoid of revolution obtained by rotating $x^2/a^2 + y^2/b^2 = 1$ about the x-axis, every geodesic which is not a meridian remains always between two parallels of latitude.

[On a surface of revolution $\boldsymbol{r}(u, v) = (u, f(u)\cos v, f(u)\sin v)$ the meridians are given by v = constant and the parallels of latitude by u = constant].

Proof. From a physical approach, we note that the geodesic locally minimises the distance between two points on the surface. By the principle of least action, the geodesic satisfies the Euler-Lagrange equation, so it is the path of a free particle on the surface. Since the surface is azimuthally symmetric, by Noether's Theorem the angular momentum L_x along the axis of revolution is conserved. From classical mechanics we know that

$$L_x = (m\gamma(s) \times \gamma'(s)) \cdot \mathbf{e}_x = m(\mathbf{e}_x \times \gamma(s)) \cdot \gamma'(s) = m\rho \sin\varphi$$

The last equality follows from that $\mathbf{e}_x \times \gamma(s)$ is along the parallel of latitude through $\gamma(s)$. Hence $\rho \sin\varphi$ is constant along the geodesic γ.

The constant paths at the north and the south pole are parallels of latitude. So it suffices the prove that any geodesic passing through one of the poles is a meridian. But this is trivially true by the uniqueness of geodesic. □

Question 3

Let $\mathscr{H}$ be the upper half plane model of the hyperbolic plane and let L be a geodesic in $\mathscr{H}$. Find the locus of all points equidistant from L.

[Hint: First consider the geodesic $\{(0, e^{-t}) : t \in \mathbb{R}\}$ and find the images of a point P with respect to all isometries mapping the geodesic to itself.]

Proof. Recall that the isometry group $\mathrm{Isom}(\mathbb{H})$ of the hyperbolic plane $\mathbb{H}$ is generated by the Möbius transformations

$$\mathrm{M\ddot{o}b}(\mathbb{H}) := \left\{ z \mapsto \frac{az+b}{cz+d} : \ a,b,c,d \in \mathbb{R}, \ ad-bc = 1 \right\}$$

and the reflection $z \mapsto -\overline{z}$. Recall also that the geodesics on the hyperbolic plane are (in the Euclidean sense) lines perpendicular to the x-axis and semi-circles centred on the x-axis.

First we consider the geodesic $L_0 = \{(0, y) : y > 0\}$. We need to determine the stabliser of L_0 under the action of $\mathrm{Isom}(\mathbb{H})$ on the set of geodesics of $\mathbb{H}$.

Suppose that $z \mapsto \dfrac{az+b}{cz+d}$ fixes L_0. Then

$$\forall\, y > 0 \ \mathrm{Re}\left(\frac{ay\mathrm{i}+b}{cy\mathrm{i}+d}\right) = 0 \implies acy^2 + bd = 0 \implies ac = bd = 0$$

Since $ad - bc = 1$, we have either $a = d = 0$ and $bc = -1$, or $b = c = 0$ and $ad = 1$. Therefore $\mathrm{Stab}(L_0) \cap \mathrm{M\ddot{o}b}(\mathbb{H})$ is generated by

$$d_a : z \mapsto az \ (a > 0), \quad i : z \mapsto -\frac{1}{z}$$

In addition, the reflection $r : z \mapsto -\overline{z}$ clearly fixes L_0. Therefore $\mathrm{Stab}(L_0)$ is generated by the two above transformations and the reflection.

For $\varphi \in \mathrm{Stab}(L_0)$, we know that for $P \in \mathbb{H}$,

$$\mathrm{d}(P, L_0) = \mathrm{d}(\varphi(P), \varphi(L_0)) = \mathrm{d}(\varphi(P), L_0)$$

Therefore the set of points whose distance to L_0 are equal to $\mathrm{d}(P, L_0)$ is exactly $\{\varphi(P) : \varphi \in \mathrm{Stab}(L_0)\}$. Suppose that $P = c\,\mathrm{e}^{\mathrm{i}\theta}$ where $c > 0$ and $\theta \in (0, \pi)$. Then

$$d_a(P) = ac\,\mathrm{e}^{\mathrm{i}\theta}, \quad i(P) = \frac{1}{c}\mathrm{e}^{\mathrm{i}(\pi-\theta)}, \quad r(P) = c\,\mathrm{e}^{\mathrm{i}(\pi-\theta)}$$

Hence $\{\varphi(P) : \varphi \in \mathrm{Stab}(L_0)\}$ is the rays $\{k\mathrm{e}^{\mathrm{i}\theta} : k > 0\}$ and $\{k\mathrm{e}^{\mathrm{i}(\pi-\theta)} : k > 0\}$.

Suppose that the geodesic L is a straight line $\{(a, y) : y > 0\}$ where $a \in \mathbb{R}$. Then $L = \tau_a(L_0)$, where $\tau_a : z \mapsto z + a$ is an isometry of $\mathbb{H}$. Since this is a simple translation, we immediately observe that the loci of all points equidistant from L are the rays in $\mathbb{H}$ starting from $a \in \mathbb{R}$.

Suppose that the geodesic L is a semi-circle $\{z \in \mathbb{H} : |z - a| = r\}$ where $a \in \mathbb{R}$ and $r > 0$. We consider a Möbius transformation ψ such that $\psi(0) = r$ and $\psi(\infty) = -r$. Then

$$\psi(z) = \frac{1}{2r}\frac{z-r}{z+r} \in \mathrm{M\ddot{o}b}(\mathbb{H})$$

Then $\tau_a \circ \psi$ is an isometry of $\mathbb{H}$ which maps L_0 to L. From this Möbius transformation we also observe that all rays starting from the origin are mapped to circular arcs that pass through $a - r$ and $a + r$. We deduce that the loci of all points equidistant from L are the circular arcs in $\mathbb{H}$ passing through $a - r$ and $a + r$. □

Question 4

A hyperbolic triangle has angles α, β, γ, respectively, and opposite sides of lengths a, b, c, respectively. By using the hyperbolic "cos" formula

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$$

applied to relevant right angled triangles, or otherwise, show that

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$$

Proof. Using the cosine rule,

$$\begin{aligned} \sinh^2 a \sinh^2 b \cos^2\gamma &= (\cosh c - \cosh a \cosh b)^2 \\ &= \cosh^2 c + \cosh^2 a \cosh^2 b - 2\cosh a \cosh b \cosh c \\ \implies \sinh^2 a \sinh^2 b \sin^2\gamma &= \sinh^2 a \sinh^2 b - \left(\cosh^2 c + \cosh^2 a \cosh^2 b - 2\cosh a \cosh b \cosh c\right) \\ &= \left(\cosh^2 a - 1\right)\left(\cosh^2 b - 1\right) - \left(\cosh^2 c + \cosh^2 a \cosh^2 b - 2\cosh a \cosh b \cosh c\right) \\ &= 1 - \left(\cosh^2 a + \cosh^2 b + \cosh^2 c + 2\cosh a \cosh b \cosh c\right) \end{aligned}$$

The final formula is symmetric in a, b, c. We deduce that

$$\sinh^2 a \sinh^2 b \sin^2\gamma = \sinh^2 b \sinh^2 c \sin^2\alpha = \sinh^2 c \sinh^2 a \sin^2\beta$$

Hence

$$\frac{\sinh^2 a}{\sin^2\alpha} = \frac{\sinh^2 c}{\sin^2\gamma} = \frac{\sinh^2 b}{\sin^2\beta}$$

Since all sine and hyperbolic sine are positive, we conclude that

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$$

□

Question 5

Show that if a hyperbolic triangle is right-angled, with $\gamma = \pi/2$, then $\cosh c = \cosh a \cosh b$ and use this to prove that in a hyperbolic triangle the length c of the hypotenuse is always longer than the corresponding Euclidean result $\sqrt{a^2+b^2}$.

Proof. By cosine rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$$

Since $\gamma = \pi/2$, the equation reduces to

$$\cosh c = \cosh a \cosh b$$

Next, consider the expansion of $\cosh\sqrt{a^2+b^2}$:

$$\begin{aligned} \cosh\sqrt{a^2+b^2} &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(a^2+b^2)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(2n)!}\binom{n}{k} a^{2k} b^{2(n-k)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(2(n+k))!}\binom{n+k}{k} a^{2k} b^{2n} && \text{(by absolute convergence)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\binom{n+k}{k}}{\binom{2(n+k)}{2k}} \frac{a^{2k}}{(2k)!}\frac{b^{2n}}{(2n)!} \\ &< \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!}\frac{b^{2n}}{(2n)!} && \left(\text{since } \binom{n+k}{k} < \binom{2(n+k)}{2k}\right) \\ &= \cosh a \cosh b \\ &= \cosh c \end{aligned}$$

Since cosh is increasing on $\mathbb{R}_+$, we deduce that $c > \sqrt{a^2+b^2}$. □