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Problem Sheet 1

C2.6: Introduction to Schemes

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Question 1

i) For ring R, and radical ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$, prove that $\mathfrak{a} \subseteq \mathfrak{b} \iff \mathbb{V}(\mathfrak{a}) \supset \mathbb{V}(\mathfrak{b})$.

ii) Show that the presheaf of constant real functions is not a sheaf on X when $X = 2$ points with the discrete topology.

iii) Show that the sheafification of the presheaf of constant functions is the sheaf of locally constant functions.

(Optional: What happens in the case $X = \mathbb{Q}$?)

iv) Show that a scheme X is irreducible $\iff$ every non-empty open subset is dense.

v) Show that R is Noetherian $\implies$ every subset of $\operatorname{Spec} R$ is quasi-compact.

Proof. i) The direction " $\implies$ " is trivial. For the " $\impliedby$ " direction, suppose that $\mathbb{V}(\mathfrak{b}) \subseteq \mathbb{V}(\mathfrak{a})$ and $a \not\subseteq b$. Then we pick $r \in \mathfrak{a}$ such that $r \notin \mathfrak{b}$. Then $r \neq 0 \in R/\mathfrak{b}$. Since $\mathfrak{b}$ is a radical ideal, the nilradical $\operatorname{Nil}(R/\mathfrak{b}) = \{0\}$. So there exists $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{b} \subseteq \mathfrak{p}$ and $r \notin \mathfrak{p}$. The condition $\mathbb{V}(\mathfrak{b}) \subseteq \mathbb{V}(\mathfrak{a})$ implies that $\mathfrak{a} \subseteq \mathfrak{p}$. So $r \in \mathfrak{a}$ implies that $r \in \mathfrak{p}$. We obtain a contradication. Therefore we have $\mathfrak{a} \subseteq \mathfrak{b}$.

ii) Let $F : \mathsf{Top}X \to \mathsf{Ab}$ be the presheaf of constant functions on X. Let $s_1 \in F(\{1\})$ and $s_2 \in F(\{2\})$ be the real-valued functions such that $s_1(1) = 1 \neq 2 = s_2(2)$. Then $\{s_1, s_2\}$ agrees on the overlap $\{1\} \cap \{2\} = \varnothing$. But if there exists $s \in F(X) = F(\{1\} \cup \{2\})$ such that $s|_1 = s_1$ and $s|_2 = s_2$, then s is not constant on $\{1, 2\}$. So F is not a sheaf.

iii) Let $F : \mathsf{Top}X \to \mathsf{Ab}$ be the presheaf of constant functions on X. Let F^+ be the sheafification of F. Let U be open in X and $s : U \to \bigsqcup_{x \in U} F_x$ in $F^+(U)$. By definition, for any $x \in U$, there exist $V \subseteq U$ with $x \in V$ and $t \in F(V)$ such that $s(y) = t_y \in F_y$ for all $y \in V$. Since $t \in F(V)$ is a constant function, $t : V \to \mathbb{R}$ is given by $t(y) = c_V$ for some $c_V \in \mathbb{R}$. Hence $s(y) = t_y = c_V$ for $y \in V$. That is, s is constant on V, which is an open neighbourhood of x. That proves that s is a locally constant function. We deduce that F^+ is the sheaf of locally constant functions.

iv) The statement is true for any topological space X.

If X is reducible, then $X = X_1 \cup X_2$ for closed sets $X_1, X_2 \subsetneq X$. Then $X \setminus X_1$ is a non-empty open set of X. The closure

$$\overline{X \setminus X_1} \subseteq X_2 \subsetneq X$$

since X_2 is closed. So $X \setminus X_1$ is not dense in X. Conversely, suppose that X has a non-empty non-dense open subset U. Then $\overline{U} \neq X$. We have $X = \overline{U} \cup (X \setminus U)$. So X is reducible.

v) Let $\{\mathfrak{p}_i\}_{i \in I} \subseteq \operatorname{Spec} R$. We claim that there exists $i_1, \ldots, i_n \in I$ such that

$$\sum_{i \in I} \mathfrak{p}_i = \sum_{k=1}^{n} \mathfrak{p}_{i_k}$$

Since R is Noetherian, $\sum_{i \in I} \mathfrak{p}_i$ is finitely generated. Let $r_1, \ldots, r_n \in R$ be the generators. For each r_k, since $r_k \in \sum_{i \in I} \mathfrak{p}_i$, we have $r_k = \sum_{j=1}^{\ell_k} s_j$, where $s_j \in \mathfrak{p}_j$ for some $j \in I$. Therefore, by relabeling, we may assume that $r_1, \ldots, r_n$ generates $\sum_{i \in I} \mathfrak{p}_i$, where each $r_k \in \mathfrak{p}_{i_k}$. This proves our claim.

Let $A \subseteq \operatorname{Spec} R$. We shall prove that A is compact[1]. Suppose that $\{U_i\}_{i \in I}$ is an open cover of A. Let $\mathbb{V}(\mathfrak{p}_i) := \operatorname{Spec} R \setminus U_i$. Then $\bigcap_{i \in I} \mathbb{V}(\mathfrak{p}_i) \cap A = \varnothing$. But

$$\bigcap_{i \in I} \mathbb{V}(\mathfrak{p}_i) = \mathbb{V}\left(\sum_{i \in I} \mathfrak{p}_i\right) = \mathbb{V}\left(\sum_{k=1}^{n} \mathfrak{p}_{i_k}\right) = \bigcap_{k=1}^{n} \mathbb{V}(\mathfrak{p}_{i_k})$$

So $\bigcap_{k=1}^{n} \mathbb{V}(\mathfrak{p}_{i_k}) \cap A = \varnothing$. In order words, $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover. Hence A is compact. $\square$

[1] I shall use compact for quasi-compact as these words mean the same thing (Hausdorff property is not a part of the definition of compactness.)

yes but sometimes in geometry over $\mathbb{C}$ people don't specify how they're thinking about a given space: algebraic vs analytic. Using quasi-compact helps.

Question 2

Let $(X, \mathscr{O}_X)$ be a scheme. For $s \in \mathscr{O}_X(U)$, show that $s_x = 0 \in \mathscr{O}_{X,x}$ for all x implies that $s = 0$. Prove that X is reduced $\iff$ all stalks $\mathscr{O}_{X,x}$ are reduced.

Proof. For each $x \in U$, since $s_x = 0 \in \mathscr{O}_{X,x}$, there exists open $W \subseteq U$ with $x \in W$ such that $s|_W = 0|_W = 0$. The local-to-global condition of sheaf $\mathscr{O}_X$ suggests that $s = 0$.

Suppose that X is reduced. Then for each open set U, $\mathscr{O}_X(U)$ is a reduced ring. The stalk at $x \in X$ is the ring

$$\mathscr{O}_{X,x} = \varinjlim_{U \ni x} \mathscr{O}_X(U)$$

To prove that $\mathscr{O}_{X,x}$ is reduced, we simply need to prove that the direct limit of a directed system of reduced rings is reduced. This follows from Exercise 2.22 of *Atiyah & MacDonald*. Let us write out the details.

For $U \ni x$, let $\varphi_U : \mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ be the natural map. Suppose that $\alpha \in \mathrm{Nil}(\mathscr{O}_{X,x})$. Then there exists open $U \subseteq X$ and $\beta \in \mathscr{O}_X(U)$ such that $\alpha = \varphi_U(\beta)$. If $\alpha^n = 0$, then $\varphi(\beta^n) = 0$. Hence there exists open $V \subseteq U$ such that the restriction map $\rho_{UV} : \mathscr{O}_X(U) \to \mathscr{O}_X(V)$ satisfies that $\rho_{UV}(\beta^n) = \rho_{UV}(\beta)^n = 0$. Since $\mathrm{Nil}(\mathscr{O}_X(V)) = \{0\}$, we have $\rho_{UV}(\beta) = 0$. But this implies that $\alpha = \varphi_U(\beta) = \varphi_V \circ \rho_{UV}(\beta) = 0$. Hence $\mathrm{Nil}(\mathscr{O}_{X,x}) = \{0\}$. $\mathscr{O}_{X,x}$ is reduced as claim.

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, "\varphi_U"'] & & \mathscr{O}_X(V) \arrow[dl, "\varphi_V"] \\ & \mathscr{O}_{X,x} & \end{tikzcd}$$

Conversely, suppose that all stalks of $\mathscr{O}_X$ are reduced. Let $U \subseteq X$ be an open set, and let $s \in \mathrm{Nil}(\mathscr{O}_X(U))$. Then $s^n = 0$ for some $n \in \mathbb{N}$. Under the natural map we have $s_x^n = \varphi_{U,x}(s^n) = 0$ for all $x \in U$. Since $\mathscr{O}_{X,x}$ is reduced, we have $s_x = 0$ for all $x \in U$. By the first part we have $s = 0$. Hence $\mathscr{O}_X(U)$ is reduced. We conclude that the scheme X is reduced. □

Question 3

Let $X = \operatorname{Spec} R$. Prove that

i) X is irreducible $\iff$ R has a unique minimal prime $\mathfrak{p}$ (*hint: nilradical*) $\iff$ X has a unique generic point $\mathfrak{p}$.

ii) X is reduced and irreducible $\iff$ R is an integral domain (*you may assume as known that localisation preserves the "reduced" property*)

Proof. i) We prove the first equivalence. Let $\mathrm{Nil}(R)$ be the nilradical of R. By definition it is contained in every prime ideal of R. Then R has a unique minimal prime ideal if and only if $\mathrm{Nil}(R)$ is prime.

Suppose that $\mathrm{Nil}(R)$ is not prime. Then there exist $r, s \notin \mathrm{Nil}(R)$ such that $rs \in \mathrm{Nil}(R)$. Then $\langle rs \rangle \subseteq \mathrm{Nil}(R)$, and

$$\operatorname{Spec} R = \mathbb{V}(\langle rs \rangle) = \mathbb{V}(\langle r \rangle \langle s \rangle) = \mathbb{V}(r) \cup \mathbb{V}(s)$$

Since $\mathbb{V}(r)$ and $\mathbb{V}(s)$ are proper closed subsets of $\operatorname{Spec} R$, $\operatorname{Spec} R$ is reducible.

Conversely, suppose that $\operatorname{Spec} R$ is reducible. Then there exist $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} R$ such that $X = \mathbb{V}(\mathfrak{p}_1) \cup \mathbb{V}(\mathfrak{p}_2)$ and $\mathbb{V}(\mathfrak{p}_1), \mathbb{V}(\mathfrak{p}_2) \neq X$. So there exist $r_1 \in \mathfrak{p}_1$ and $r_2 \in \mathfrak{p}_2$ such that $r_1, r_2 \notin \mathrm{Nil}(R)$. But note that

$$X = \mathbb{V}(\mathfrak{p}_1) \cup \mathbb{V}(\mathfrak{p}_2) = \mathbb{V}(\mathfrak{p}_1 \cap \mathfrak{p}_2) \implies \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathrm{Nil}(R)$$

So $r_1 r_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathrm{Nil}(R)$. Hence $\mathrm{Nil}(R)$ is not prime.

The second equivalence is tautological:

$$\begin{aligned} X \text{ has a unique generic point } \mathfrak{p} &\iff \mathbb{V}(\mathfrak{p}) = X \\ &\iff \forall\, \mathfrak{q} \in \operatorname{Spec} R\colon\ \mathfrak{p} \subseteq \mathfrak{q} \\ &\iff R \text{ has a unique minimal prime ideal } \mathfrak{p} \end{aligned}$$

ii) First we note that reducedness is a local property. That is, R is reduced if and only if $R_{\mathfrak{p}}$ is reduced for each $\mathfrak{p} \in \operatorname{Spec} R$. Let us prove this as a revision of commutative algebra.

Special case of exercise 2.

Suppose that R is not reduced. Let $x \in \operatorname{Nil}(R) \setminus \{0\}$. Consider the annihilator $\operatorname{Ann}_R(x) := \{y \in R\colon\ xy = 0\}$ which is an ideal of R. Since $1 \notin \operatorname{Ann}_R(x)$, there exists a maximal ideal $\mathfrak{m}$ such that $\operatorname{Ann}_R(x) \subseteq \mathfrak{m}$. Let $\varphi : R \to R_{\mathfrak{m}}$ be the natural map sending r to $r/1$. Suppose that $\varphi(x) = 0$. There there exists $y \notin \mathfrak{m}$ such that $xy = 0$, which is impossible by our construction. Hence $\varphi(x) \neq 0$. Since x is nilpotent, $x^n = 0$ for some $n \in \mathbb{N}$. Hence $\varphi(x)^n = 0$ and $\varphi(x) \in \operatorname{Nil}(R_{\mathfrak{m}}) \setminus \{0\}$. The localisation $R_{\mathfrak{m}}$ is not reduced.

Conversely, suppose that R is reduced. For each $\mathfrak{p} \in \operatorname{Spec} R$, let $\varphi : R \to R_{\mathfrak{p}}$ be the natural map. By Corollary 3.3 of *Atiyah & MacDonald*, $\operatorname{Nil}(R_{\mathfrak{p}}) = \varphi(\operatorname{Nil}(R)) = \varphi(\{0\}) = \{0\}$. Hence $R_{\mathfrak{p}}$ is reduced for all $\mathfrak{p} \in \operatorname{Spec} R$.

Return to the question. From Question 2 we have

$$X \text{ is reduced} \iff \mathcal{O}_{X,\mathfrak{p}} \text{ is reduced for all } \mathfrak{p} \in X \iff R_{\mathfrak{p}} \text{ is reduced for all } \mathfrak{p} \in \operatorname{Spec} R \iff R \text{ is reduced}$$

Now we note from part (i) that X is reduced and irreducible if and only if $\operatorname{Nil}(R) = \{0\}$ and it is prime. But since $R \cong R/\{0\}$, then R is an integral domain if and only if $\{0\}$ is a prime ideal. We thus finish the proof. □

Question 4

Let $(X, \mathcal{O})$ be a scheme.

i) If R is a local ring, show that

$$\operatorname{Mor}(\operatorname{Spec} R, X) \overset{1:1}{\longleftrightarrow} \bigsqcup_{x \in X} \operatorname{Hom}_{\text{local rings}}(\mathcal{O}_x, R)$$

Hint: for $\varphi : m \mapsto x$ *show that* $x \in \overline{\varphi(p)}$ *for any* p.

ii) If K is a field, show that $\operatorname{Mor}(\operatorname{Spec} K, X)$ and

$$\operatorname{Mor}(\operatorname{Spec} K, X) \overset{1:1}{\longleftrightarrow} \bigsqcup_{x \in X} \{\text{field extensions } \kappa(x) \hookrightarrow K\}$$

($\kappa(x) := \mathcal{O}_x/\mathfrak{m}_x$, *where* $\mathfrak{m}_x$ *is the unique maximal ideal of* $\mathcal{O}_x$.)

iii) The **Zariski tangent space** at x is defined as $\mathrm{T}_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ (*dual over the field* $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$). Let X be a scheme *over a field* k, meaning that we are given a morphism $X \to \operatorname{Spec} k$. Convince yourself that this means that locally X is the Spec of a k-algebra, not just the Spec of a ring. Show that

$$\operatorname{Mor}(\operatorname{Spec}(k[\varepsilon]/\varepsilon^2), X) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\substack{x \in X\colon \kappa(x) \cong k \\ \text{as } k\text{-algebras}}} \mathrm{T}_x$$

(*On the LHS, we mean morphisms of schemes over* k, *so they commute with maps to* $\operatorname{Spec} k$, *and maps of sheaves are* k*-algebra homomorphisms.*)

Remark. If locally X *is the Spec of finitely generated* k*-algebras, and* k *is algebraically closed, then* $\kappa(x) \cong k$ *at closed points* $x \in X$.

Comment on what happens for $X = \operatorname{Spec}(k[x]/x^2)$. (*Compare Sec. 0.2 of Notes.*)

Proof. i) Recall that a morphism of schemes from $\operatorname{Spec} R$ to X is $(f, f^\#)$, where $f : \operatorname{Spec} R \to X$ is continuous, and $f^\# : \mathcal{O} \to f_*\mathcal{O}_{\operatorname{Spec} R}$ is a morphism of sheaves of rings, and for each $\mathfrak{p} \in \operatorname{Spec} R$, the map $f^\#_\mathfrak{p} : \mathcal{O}_{f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}} = R_\mathfrak{p}$ is a homomorphism of local rings.

Let $\mathfrak{m}$ be the maximal ideal of R. First we note that the natural map $\varphi : R \to R_\mathfrak{m}$ is a local ring isomorphism, because $R \setminus \mathfrak{m}$ is the set of units of R. If $(f, f^\#)$ is a scheme morphism from $\operatorname{Spec} R$ to X, then at $f(\mathfrak{m})$, we have the local ring homomorphism $f^\#_\mathfrak{m} : \mathcal{O}_{f(\mathfrak{m})} \to R_\mathfrak{m} \cong R$. Hence $(f, f^\#)$ determines $(f(\mathfrak{m}), f^\#_\mathfrak{m}) \in \bigsqcup_{x \in X} \operatorname{Hom}_{\text{local rings}}(\mathcal{O}_x, R)$. We denote the map $(f, f^\#) \mapsto ((f(\mathfrak{m}), f^\#_\mathfrak{m}))$ by $\mathscr{F}$.

On the other hand, we fix $x \in X$ and let $\rho : \mathcal{O}_x \to R$ be a local ring homomorphism. Let U be an open neighbourhood of x. Since $(X, \mathcal{O})$ is a scheme, $(U, \mathcal{O}|_U) \cong (S, \operatorname{Spec} S)$ as locally ringed spaces for some ring S. On the stalk of x we have the local ring isomorphism $\sigma^\#_x : S_\mathfrak{q} \to \mathcal{O}_x$ where $\mathfrak{q} \in \operatorname{Spec} S$. Consider the composition ring homomorphism ψ: suppress $S \xrightarrow{\varphi} S_\mathfrak{q} \xrightarrow{\sigma^\#_x} \mathcal{O}_x \xrightarrow{\rho} R$. Then ψ induces a morphism of locally ringed spaces $(\tilde{f}, \tilde{f}^\#)$ from $\operatorname{Spec} R$ to $U \subseteq X$. We denote the map $(x, \rho) \mapsto (\tilde{f}, \tilde{f}^\#)$ by $\mathscr{G}$.

It remains to show that $\mathscr{F} \circ \mathscr{G} = \mathrm{id}$ and $\mathscr{G} \circ \mathscr{F} = \mathrm{id}$.

Let $(\tilde{f}(\mathfrak{m}), \tilde{f}^\#_\mathfrak{m}) = \mathscr{F} \circ \mathscr{G}(x, \rho)$. We shall show that $\tilde{f}(\mathfrak{m}) = x$ and $\tilde{f}^\#_\mathfrak{m} = \rho$. Note that the homeomorphism $\sigma : U \to \operatorname{Spec} S$ sends x to $\mathfrak{q}$. On the stalks, $\tilde{f} = \sigma^{-1} \circ \operatorname{Spec} \psi$ becomes the map $\rho \circ \sigma_x : S_\mathfrak{q} \to R \cong R_\mathfrak{m}$. So

$$\tilde{f}(\mathfrak{m}) = \sigma^{-1} \circ (\operatorname{Spec} \psi)(\mathfrak{m}) = \sigma^{-1}(\mathfrak{q}) = x$$

And

$$\tilde{f}^\#_\mathfrak{m} = \psi^\#_\mathfrak{m} \circ (\sigma^{-1})^\#_\mathfrak{q} = \rho \circ \sigma^\#_x \circ (\sigma^{-1})^\#_\mathfrak{q} = \rho$$

Let $(\tilde{f}, \tilde{f}^\#) = \mathscr{G} \circ \mathscr{F}(f, f^\#)$. We shall show that $f = \tilde{f}$ and $f^\# = \tilde{f}^\#$. Let $x = f(\mathfrak{m})$. For any $\mathfrak{p} \in \operatorname{Spec} R$, since $\mathfrak{m} \in \overline{\{\mathfrak{p}\}} = \mathbb{V}(\mathfrak{p})$, we have

$$x = f(\mathfrak{m}) \in f(\overline{\{\mathfrak{p}\}}) \subseteq \overline{\{f(\mathfrak{p})\}}$$

We claim that $f(\operatorname{Spec} R) \subseteq U$ for any open neighbourhood U of x. Suppose not. There is $\mathfrak{p} \in \operatorname{Spec} R$ such that $f(\mathfrak{p}) \in X \setminus U$. Note that $X \setminus U$ is closed. Therefore

$$x \in \overline{\{f(\mathfrak{p})\}} \subseteq X \setminus U$$

which is a contradiction. Next we fix U as in the definition of $\tilde{f}$ (so we also fix an isomorphism $\sigma : U \to \operatorname{Spec} S$.) Then $\sigma \circ f$ is a morphism of locally ringed spaces from $\operatorname{Spec} R$ to $\operatorname{Spec} S$. Since the functor $\operatorname{Spec} : \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Aff}$ is full, there exists a ring homomorphism $\xi : S \to R$ such that $\sigma \circ f = \operatorname{Spec} \xi$. We claim that $\xi = \psi$. Note that we have defined ψ to be $f^\#_\mathfrak{m} \circ \sigma^\#_x \circ \varphi = (\sigma \circ f)^\#_\mathfrak{m} \circ \varphi = (\operatorname{Spec} \xi)^\#_\mathfrak{m} \circ \varphi$. It remains to prove that $(\operatorname{Spec} \xi)^\#_\mathfrak{m} \circ \varphi = \xi$. This is easy: for $s \in S$,

$$(\operatorname{Spec} \xi)^\#_\mathfrak{m} \circ \varphi(s) = (\operatorname{Spec} \xi)^\#_\mathfrak{m}\left(\frac{s}{1}\right) = \frac{\xi(s)}{\xi(1)} = \xi(s)$$

We conclude that there is a bijective correspondence:

$$\operatorname{Mor}(\operatorname{Spec} R, X) \xleftrightarrow{1:1} \bigsqcup_{x \in X} \operatorname{Hom}_{\text{local rings}}(\mathcal{O}_x, R)$$

ii) We note that for a field K, the prime specturm $\operatorname{Spec} K = \{\{0\}\}$. K is a local ring with maximal ideal $\{0\}$. Suppose that $\varphi : \mathcal{O}_x \to K$ is a local ring homomorphism. Then by definition $\ker \varphi = \varphi^{-1}(\{0\}) = \mathfrak{m}_x$. By the universal property of quotient ring, there exists a unique ring homomorphism $\psi : \kappa(x) := \mathcal{O}_x / \mathfrak{m}_x \to K$ such that $\psi \circ \pi = \varphi$.

$$\begin{tikzcd}
\mathcal{O}_x \arrow[r, "\varphi"] \arrow[d, "\pi"'] & K \\
\kappa(x) \arrow[ur, dashed, "\exists ! \psi"'] &
\end{tikzcd}$$

Since $\ker\varphi \neq \mathcal{O}_x$, $\psi \neq 0$. Hence $\psi : \kappa(x) \to K$ is a field extension. Furthermore, the assignment $\varphi \mapsto \psi$ is bijective. Hence we have the bijective correspondence

$$\coprod_{x\in X} \mathrm{Hom}_{\text{local rings}}(\mathcal{O}_x, K) \xleftrightarrow{1:1} \coprod_{x\in X} \{\text{field extensions } \kappa(x) \hookrightarrow K\}$$

Combining with the result of (i), we obtain the claimed correspondence of (ii).

iii) First, we claim that $R := k[\varepsilon]/\langle \varepsilon^2 \rangle$ is a local ring with maximal ideal $\langle \varepsilon \rangle$. Let $\mathfrak{p} \in \operatorname{Spec} k[\varepsilon]$ such that $\pi(\mathfrak{p}) \neq 0 \in R$. Then $\langle \varepsilon^2 \rangle \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $\varepsilon \in \mathfrak{p}$. Hence $\langle \varepsilon \rangle \subseteq \pi(\mathfrak{p})$. But $\langle \varepsilon \rangle$ is maximal in R, because $R/\langle \varepsilon \rangle \cong k$ is a field. Hence $\mathfrak{p} = \langle \varepsilon \rangle \in \operatorname{Spec} R$. We deduce that R is local, and $\operatorname{Spec} R = \{\langle \varepsilon \rangle\}$.

Let $\mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R)$ be the set of maps $\varphi : \mathcal{O}_x \to R$ which are both k-algebra homomorphisms and local ring homomorphisms. Let $x \in X$ such that $\kappa(x) \cong k$ as k-algebras. We claim that there is a bijective correspondence:

$$\mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R) \longleftrightarrow \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)) =: (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$$

Let $\varphi \in \mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R)$. by definition $\varphi(\mathfrak{m}_x) \subseteq \varepsilon$. Then $\varphi(\mathfrak{m}_x^2) \subseteq \langle \varepsilon \rangle^2 = \langle \varepsilon^2 \rangle = \{0\}$. Hence φ restricting on $\mathfrak{m}_x$ induces the map $\widetilde{\varphi} \in \mathrm{Hom}^k_{\mathrm{loc}}(\mathfrak{m}_x/\mathfrak{m}_x^2, R)$ with $\operatorname{im}\widetilde{\varphi} \subseteq \langle \varepsilon \rangle / \langle \varepsilon^2 \rangle$. Let $\sigma : R \twoheadrightarrow k$ be the k-linear map given by $a + b\varepsilon \mapsto b$. Then $\sigma \circ \widetilde{\varphi} \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ as a k-linear map.

Conversely, let $f \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. Consider the k-linear map $g : \mathfrak{m}_x \to k\varepsilon$ given by the composition

$$\mathfrak{m}_x \xrightarrow{\pi} \mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{f} \kappa(x) \xrightarrow{\cong} k\varepsilon$$

Consider the short exact sequence of k-vector spaces:

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_x/\mathfrak{m}_x \longrightarrow 0$$

Since k is a field, the sequence splits. So there exists a retraction $r : \mathcal{O}_x \to \mathfrak{m}_x$. Hence we can construct a k-linear map $\psi : \mathcal{O}_x \to k \oplus k\varepsilon \cong k[\varepsilon]/\langle \varepsilon^2 \rangle$ such that the following diagram commutes and hence gives a splitting of the short exact sequence above.

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_x \arrow[r, shift right, "\iota"'] \arrow[d, "g"'] & \mathcal{O}_x \arrow[l, shift right, "r"'] \arrow[r, "\pi_x"] \arrow[d, "\psi"] & \mathcal{O}_x/\mathfrak{m}_x \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & k\varepsilon \arrow[r] & k \oplus k\varepsilon \arrow[r] \arrow[d, dash, dashed, "\cong"] & k \arrow[r] & 0 \\
 & & k[\varepsilon]/\langle \varepsilon^2 \rangle & &
\end{tikzcd}$$

More explicitly, $\psi(s) = \pi_x(s) + g \circ r(s)\varepsilon$. But we note that by construction ψ is also a ring homomorphism, and $\psi(\mathfrak{m}_x) \subseteq k\varepsilon = \langle \varepsilon \rangle$. Therefore $\psi \in \mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R)$.

We can explicitly verify that we have constructed mutually inverse maps $\varphi \mapsto \sigma \circ \widetilde{\varphi}$ and $f \mapsto \psi$. Therefore our claim is proven.

Let $x \in X$ such that $\kappa(x) \ncong k$. Then the only k-algebra homomorphism from $\kappa(x)$ to k is the zero map. Suppose there is $\varphi \in \mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R)$. Then we can construct a k-algebra homomorphism ψ as in the following diagram:

$$\begin{tikzcd}
\mathcal{O}_x \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & R \arrow[r, "\pi"] & k \\
\kappa(x) \arrow[urr, dashed, "\exists!\psi"'] & &
\end{tikzcd}$$

Therefore $\psi = 0$. We have $\operatorname{im}\varphi \subseteq \ker\pi = \langle \varepsilon \rangle$. This is impossible as $1 \in \operatorname{im}\varphi$. Hence $\mathrm{Hom}^k_{\mathrm{loc}}(\mathcal{O}_x, R) = \varnothing$. In summary, we have the bijective correspondence

$$\bigsqcup_{x\in X} \mathrm{Hom}^k_{\mathrm{loc}}(\mathscr{O}_x, R) \xleftrightarrow{1:1} \bigsqcup_{\substack{x\in X:\, \kappa(x)\cong k \\ \text{as } k\text{-algebras}}} \mathrm{T}_x$$

Finally, the essentially same proof as in (i) shows that there is a bijective correspondence

$$\mathrm{Mor}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2), X) \xleftrightarrow{1:1} \bigsqcup_{x\in X} \mathrm{Hom}^k_{\mathrm{loc}}(\mathscr{O}_x, R)$$

which establishes the claimed result in (iii). □

Question 5. A non-affine scheme

In topology, a classic example of a non-Hausdorff space that locally looks Euclidean is the line with two origins:

$$(\mathbb{R}\times\{1\}\sqcup\mathbb{R}\times\{2\})/((x,1)\sim(x,2) \text{ for } x\neq 0)$$

Note that $\sigma_1 = (0,1) \neq (0,2) = \sigma_2$ are two origins, but the space near σ_i is still homeomorphic to $\mathbb{R}$ via $\mathbb{R}\times\{i\}$. It is not Hausdorff since any two neighbourhoods of σ_1, σ_2 intersect.

In algebraic geometry, $\mathrm{Spec}\, k[x]$ is the line k with the Zariski topology and $\mathrm{Spec}\, k[x]_{(x)}$ is the germ of the line at $0 \in k$.

i) Let $R := k[x]_{(x)}$. Show that $\mathrm{Spec}\, R = \{(0),(x)\}$ with

$$\mathscr{O}_{\mathrm{Spec}\,R}: \quad \begin{aligned} \varnothing &\longrightarrow 0 \\ \mathrm{Spec}\,R &\longrightarrow R \\ D_x = \{(0)\} &\longrightarrow k(x) = \mathrm{Frac}\,R \end{aligned}$$

ii) Let $X = \{\sigma_1,\sigma_2,\ell\}$ with the basis of open sets $D_1 = \{\sigma_1,\ell\}$, $D_2 = \{\sigma_2,\ell\}$, $D_{12} = \{\ell\}$. Define the presheaf $\mathscr{O}$ by $\mathscr{O}(X) = \mathscr{O}(D_1) = \mathscr{O}(D_2) = k[x]_{(x)}$, $\mathscr{O}(D_{12}) = k(x) = \mathrm{Frac}(k[x]_{(x)})$, $\mathscr{O}(\varnothing) = 0$, and the restriction homomorphisms are given by $\mathrm{id}: \mathscr{O}(X)\to\mathscr{O}(D_i)$ and inclusion $\mathscr{O}(X)\to\mathscr{O}(D_{12})$. Show that $(X,\mathscr{O})$ is a scheme that is not affine.

Proof. i) Since $k[x]/\langle x\rangle \cong k$ is a field, $\langle x\rangle$ is a maximal ideal of $k[x]$. Hence $R = k[x]_{\langle x\rangle}$ is a local ring. We know that $\mathrm{Spec}\,R = \{\varphi(\mathfrak{p}) : \mathfrak{p}\in\mathrm{Spec}\,k[x],\ \mathfrak{p}\subseteq\langle x\rangle\}$. For $\mathfrak{p}\in\mathrm{Spec}\,k[x]$ such that $\mathfrak{p}\subsetneq\langle x\rangle$, let $f(x)\in\mathfrak{p}$. Then $f(x) = xg(x)$ for some $g\in k[x]$. Since $x\notin\mathfrak{p}$ and $\mathfrak{p}$ is prime, $g\in\mathfrak{p}$. Hence $f(x) = x^2h(x)$ for some $h\in k[x]$. Therefore we must have $f = 0$. Hence $\mathfrak{p} = \{0\}$. We have thus shown that $\mathrm{Spec}\,R = \{\{0\},\langle x\rangle\}$. The Zariski open subsets are $D_0 = \varnothing$, $D_1 = \mathrm{Spec}\,R$, and $D_x = \{\{0\}\}$. By the construction of $\mathscr{O}_{\mathrm{Spec}\,R}$, we have

$$\mathscr{O}_{\mathrm{Spec}\,R}(\varnothing) = R_0 = \{0\}, \qquad \mathscr{O}_{\mathrm{Spec}\,R}(\mathrm{Spec}\,R) = R_1 = R, \qquad \mathscr{O}_{\mathrm{Spec}\,R}(D_x) = R_x$$

We claim that $R_x \cong k(x)$. For $f\in k[x]\setminus\{0\}$, let $f(x) = x^m g(x)$, where $\gcd(g(x),x) = 1$. Then $1/g(x)\in R = k[x]_{\langle x\rangle}$. And $1/f(x) = (1/g(x))/x^m \in R_x$. Hence R_x is the field $\mathrm{Frac}\,k[x] = k(x)$.

ii) It is straightforward to verify that $\mathscr{O}$ is a sheaf and $(X,\mathscr{O})$ is a locally ringed space. By (i), we note that on the open set D_i, $i = 1,2$, $(D_i,\mathscr{O}|_{D_i}) \cong (\mathrm{Spec}\,R,\mathscr{O}_{\mathrm{Spec}\,R})$ as locally ringed spaces. Therefore $(X,\mathscr{O})$ is a scheme.

Suppose that X is an affine scheme. Then $(X,\mathscr{O}) \cong (\mathrm{Spec}\,S,\mathscr{O}_{\mathrm{Spec}\,S})$ for some ring S. Then $k[x]_{\langle x\rangle} = \mathscr{O}(X) \cong S$. But then $\mathrm{Spec}\,S$ has only two elements, whereas X has three elements. They cannot be homeomorphic topological spaces. Contradiction. So X is not an affine scheme. □

Question 6

Let A,B be Abelian categories.

i) Show that $h^X := \mathrm{Hom}_{\mathsf{A}}(X,-) : \mathsf{A}\to\mathsf{Ab}$ is a left exact functor.

ii) Show that if $h^X(A) \to h^X(B) \to h^X(C)$ is exact at $h^X(B)$ for all $X \in \mathsf{A}$, then $A \to B \to C$ is exact at B.

iii) Show that $h^\bullet : \mathsf{A} \to \mathsf{Ab}^{\mathsf{A}}$ is a fully faithful contravariant functor, called **contravariant Yoneda embedding**.

iv) Let $F : \mathsf{A} \to \mathsf{B}$ be a left adjoint functor to $G : \mathsf{B} \to \mathsf{A}$. Prove that F is right exact and G is left exact.

Remark: (iii) and (iv) also hold if Ab *is replaced by* Set, *except the last statement about exactness becomes: F preserves colimits and G preserves limits.*

Proof. i) Suppose that we have a short exact sequence in A:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

In a general Abelian category, the exactness at B is equivalent to that $g \circ f = 0$ and $\operatorname{coker} f \circ \ker g = 0$. The sequence above is exact if and only if $f = \ker g$ and $g = \operatorname{coker} f$.

Applying the functor h^X, we need to show the exactness of

$$0 \longrightarrow \operatorname{Hom}(X, A) \xrightarrow{h^X f} \operatorname{Hom}(X, B) \xrightarrow{h^X g} \operatorname{Hom}(X, C)$$

Since f is a monomorphism, $h^X f(\alpha) = f \circ \alpha = 0$ implies that $\alpha = 0$. Hence $h^X f$ is injective. Hence $h^X f$ is a monomorphism. The sequence is exact at $\operatorname{Hom}(X, A)$.

Since $g \circ f = 0$, $h^X g \circ h^X f = h^X(g \circ f) = 0$. Let $\alpha \in \operatorname{Hom}(X, B)$ with $h^X g(\alpha) = g \circ \alpha = 0$. By the universal property of $f = \ker g : A \to B$, there exists a unique map $\tilde{\alpha} : X \to A$ such that $\alpha = f \circ \tilde{\alpha} = h^X f(\tilde{\alpha})$. Hence $\ker h^X g = \operatorname{im} h^X f$. The sequence is exact at $\operatorname{Hom}(X, B)$.

We deduce that h^X is left exact.

ii) First we take $X = A$ and $\mathrm{id} \in \operatorname{Hom}(A, A)$. By exactness at $\operatorname{Hom}(A, B)$, we have $h^X(g) \circ h^X(f) = h^X(g \circ f) = 0$. Hence $g \circ f \circ \mathrm{id} = h^X(g \circ f) = 0$. Hence $g \circ f = 0$.

Let $\ker g : K \to B$ be the kernel of $g : B \to C$, and let $\operatorname{coker} f : B \to Z$ be the cokernel of $f : A \to B$. Since $g \circ \ker g = h^K g(\ker g) = 0$, by exactness at $\operatorname{Hom}(K, B)$, there exists $\alpha \in \operatorname{Hom}(K, A)$ such that $\ker g = h^K(\alpha) = f \circ \alpha$. On the other hand, since $g \circ f = 0$, by the universal property of $\operatorname{coker} f$, there exists a unique $\beta \in \operatorname{Hom}(C, Z)$ such that $\beta \circ g = \operatorname{coker} f$, as shown in the following diagram:

$$\begin{tikzcd}
 & K \arrow[dl, "\alpha"'] \arrow[d, "\ker g"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\operatorname{coker} f"'] & C \arrow[dl, dashed, "\exists! \beta"] \\
 & Z &
\end{tikzcd}$$

Therefore $\operatorname{coker} f \circ \ker g = \beta \circ g \circ f \circ \alpha = 0$. We deduce that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B.

iii)

iv) For $X \in \operatorname{Obj}(\mathsf{A})$ and $Y \in \operatorname{Obj}(\mathsf{B})$, the adjunction $F \dashv G$ implies the isomorphism of Abelian groups

$$\operatorname{Hom}_{\mathsf{A}}(F(X), Y) = \operatorname{Hom}_{\mathsf{B}}(X, G(Y))$$

Suppose that the following is a short exact sequence in B:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

For each X, we apply the left exact functor $h^{F(X)}$. So we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}(F(X), A) \xrightarrow{h^{F(X)} f} \mathrm{Hom}(F(X), B) \xrightarrow{h^{F(X)} g} \mathrm{Hom}(F(X), C)$$

By functoriality of adjunction, this implies that the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}(X, G(A)) \xrightarrow{h^X G(f)} \mathrm{Hom}(X, G(B)) \xrightarrow{h^X G(g)} \mathrm{Hom}(X, G(C))$$ ✓

By (ii) we deduce that the following sequence is exact:

$$0 \longrightarrow G(A) \xrightarrow{G(f)} G(B) \xrightarrow{G(g)} G(C) \longrightarrow 0$$ ✓

Hence G is left exact.

From the adjunction $F \dashv G$ we have the adjunction $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$. Hence $F^{\mathrm{op}} : B^{\mathrm{op}} \to A^{\mathrm{op}}$ is a right adjoint, which is left exact. Therefore F is right exact. $\square$

perfect.