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Problem Sheet 4

A10: Fluids and Waves

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Question 1

The free surface of a fluid moving in two dimensions is given parametrically by $\boldsymbol{r}(x,t) = (x, \eta(x,t))$. Show that a unit normal to the surface is

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1),$$

and deduce that the velocity of the surface normal to itself is given by

$$\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{n} = \frac{\eta_t}{\sqrt{1+\eta_x^2}}.$$

Hence show that the kinematic condition that *the velocity of the fluid normal to the surface equals the velocity of the surface normal to itself* leads to the boundary condition

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{on } y = \eta.$$

Deduce that *fluid particles on the free surface stay on the free surface*.

Proof. Let $y = \eta(x,t)$. The tangent vector:

$$\boldsymbol{\tau} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right) = \left(\frac{\mathrm{d}x}{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}}, \frac{\mathrm{d}y}{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}}\right) = \left(\frac{1}{\sqrt{1+\eta_x^2}}, \frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)$$

The normal vector (on the LHS of the tangent vector):

$$\boldsymbol{n} \cdot \boldsymbol{\tau} = 0 \implies \boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)$$

The normal velocity of the surface:

$$v_n = \frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{n} = (0, \eta_t) \cdot \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1) = \frac{\eta_t}{\sqrt{1+\eta_x^2}}$$

By the kinematic condition, we must have $v_n = \boldsymbol{u} \cdot \boldsymbol{n}$ on $y = \eta$, where

$$\boldsymbol{u} \cdot \boldsymbol{n} = (u, v) \cdot \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1) = \frac{-\eta_x u + v}{\sqrt{1+\eta_x^2}}$$

Combining the two equations, we have:

$$v = \eta_x u + \eta_t = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{on } y = \eta.$$

Changing the partial derivative to convective derivative via $\dfrac{\partial}{\partial t} = \dfrac{\mathrm{D}}{\mathrm{D}t} - \boldsymbol{u} \cdot \boldsymbol{\nabla}$, we have:

$$\frac{\mathrm{D}y}{\mathrm{D}t} = v = \frac{\mathrm{D}\eta}{\mathrm{D}t} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\eta + u\frac{\partial \eta}{\partial x} = \frac{\mathrm{D}\eta}{\mathrm{D}t} \quad \implies \quad \frac{\mathrm{D}}{\mathrm{D}t}(y - \eta) = 0$$

Hence the particle with $y = \eta$ at some time will stay on the free surface forever. □

Question 2

Consider small two-dimensional water waves on the free surface of an incompressible irrotational fluid with a velocity potential $\phi(x,y,t)$, which satisfies Laplace's equation. Suppose that the free surface has equation $y = \eta(x,t)$, the water has depth h, and the bottom is at $y = -h$. Show that we can choose ϕ such that the boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0$$

are satisfied on the free surface $y = \eta$. Show that, when the problem is linearized by neglecting quadratic terms, these boundary conditions are simplified to

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \qquad \frac{\partial \phi}{\partial t} + g\eta = 0$$

on $y = 0$. Show that travelling harmonic waves, with $\eta = A\cos(kx - \omega t)$ and $\phi = f(y)\sin(kx - \omega t)$, are possible provided $\omega^2 = gk\tanh(kh)$. Find and sketch the particle paths.

Proof. The velocity potential ϕ satisfies that $\dfrac{\partial \phi}{\partial x} = u$ and $\dfrac{\partial \phi}{\partial y} = v$. The kinematic boundary condition on $y = \eta$ is given by:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$$

Since the fluid is incompressible and irrotational, it satisfies the Bernoulli Theorem everywhere:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + gy = F(t)$$

for some scalar field $F(t)$. At $y = \eta$, $p = p_{\text{atm}}$ is the atmospheric pressure. We choose $F(t) = \dfrac{p_{\text{atm}}}{\rho}$. The dynamic boundary condition becomes:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0$$

To linearize the boundary conditions, we first use non-dimensionalization:

$$\eta = A\eta^* \qquad x = \lambda x^* \qquad u = A\omega u^* \qquad v = A\omega v^* \qquad \phi = A\omega\lambda\phi^* = 2\pi Ac\phi^* \qquad t^* = t/\omega$$

where A is the amplitude of the wave, ω is the frequency, and λ is the wave length. For smal disturbance, we may assume that $A/\lambda \ll 1$. The boundary conditions become:

$$A\omega\frac{\partial \phi^*}{\partial y^*} = A\omega\frac{\partial \eta^*}{\partial t^*} + A\omega\cdot\frac{A}{\lambda}\cdot\frac{\partial \phi^*}{\partial x^*}\frac{\partial \eta^*}{\partial x^*}, \qquad A\omega^2\lambda\frac{\partial \phi^*}{\partial t^*} + A^2\omega^2\frac{1}{2}|\nabla\phi^*|^2 + Ag\eta^* = 0$$

Negelcting the terms of order $O(A^2)$, we obtain the boundary conditions on $y = \eta$:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \qquad \frac{\partial \phi}{\partial t} + g\eta = 0$$

Furthermore, by expanding all quantities into Taylor series of η and neglecting the non-linear terms, the boundary conditions becomes

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on } y = 0$$

We substitute the travelling harmonic wave solutions $\eta = A\cos(kx - \omega t)$ and $\phi = f(y)\sin(kx - \omega t)$ (they can be obtained by separation of variables) into the Laplace equation:

$$f'' - k^2 f = 0$$

and into the boundary conditions:

$$f'(0) = A\omega \qquad \omega f(0) = Ag$$

In addition the boundary condition at the base needs to be satisfied:

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = -h \qquad \Longrightarrow \qquad f'(-h) = 0$$

The solution is $f(y) = B\cosh(k(y + h))$ for some constant B. The boundary conditions on the free surface give the linear equations:

$$\begin{pmatrix} \omega & -k\sinh(kh) \\ g & -\omega\cosh(kh) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations admit non-trivial solutions if the determinant is zero. From this we obtain the dispersion relation:

$$\omega^2 = gk\tanh(kh)$$

The particles on the surface moves vertically, whose paths are describe by $y(t) = \eta(x,t) = A\cos(kx - \omega t)$. □

Question 3

Inviscid incompressible fluid of density ρ_2 occupies the region $y > 0$ and lies vertically above a simliar fluid of greater density ρ_1 in $y < 0$. Small ampltiude waves perturb the interface between the fluids so that its equation becomes $y = \eta(x,t)$. Assuming η and the fluid velocites to be small, derive three boundary conditions relating η and the velocity potentials ϕ_1, ϕ_2 of the two fluids at $y = 0$. If $\eta(x,t) = A\cos(kx - \omega t)$, with $k > 0$, show that

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) gk$$

Proof. Suppose that ϕ_1 and ϕ_2 are the velocity potential of the two fluids. Kinematic boundary condition:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi_1}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi_2}{\partial x}\frac{\partial \eta}{\partial x} \quad \text{on } y = \eta$$

After linearization, it becomes

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{on } y = 0$$

By Bernoulli's Theorem,

$$\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2}|\nabla\phi_1|^2 + gy = F_1(t) \qquad \frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} + \frac{1}{2}|\nabla\phi_2|^2 + gy = F_2(t)$$

On $y = \eta$, the pressure $p_1 = p_2$. We have

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2}|\nabla\phi_1|^2 + g\eta - F_1(t)\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}|\nabla\phi_2|^2 + g\eta - F_2(t)\right)$$

We can choose the scalar fields $F_1(t), F_2(t)$ such that $\rho_1 F_1(t) = \rho_2 F_2(t)$. After linearization, the dynamic boundary condition becomes

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) \quad \text{on } y = 0$$

In addition, if the fluid is sufficiently deep, we can employ the far field approximation:

$$\nabla\phi_1 \to 0 \quad \text{as } y \to -\infty \qquad \nabla\phi_2 \to 0 \quad \text{as } y \to +\infty$$

These are the boundary conditions satisifed by η, ϕ_1, and ϕ_2.

If $\eta(x,t) = A\cos(kx - \omega t)$, then the velocity potentials have the form $\phi_1(x,y,t) = Be^{ky}\sin(kx - \omega t)$, $\phi_2(x,y,t) = Ce^{-ky}\sin(kx - \omega t)$. Substitute them into the boundary conditions:

$$\begin{cases} A\omega = Bk = -Ck \\ \rho_1(-\omega B + gA) = \rho_2(-\omega C + gA) \end{cases}$$

or

$$\begin{pmatrix} \omega & -k \\ -(\rho_1 - \rho_2)g & (\rho_1 + \rho_2)\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations admit non-trivial solutions if the determinant is zero. From this we obtain the dispersion relation:

$$\omega^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, gk$$ □

Question 4

Suppose now that there is a surface tension T between the two fluids of question 3 and that $\rho_1 < \rho_2$. Derive the linearized boundary conditions to be satisfied at $y = 0$. Show that the frequency ω is now related to the wave number k by the equation

$$(\rho_1 + \rho_2)\omega^2 = k(Tk^2 - (\rho_2 - \rho_1)g).$$

Deduce that the waves are unstable if their wavelength λ exceeds a critical value

$$\lambda_c = 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)g}}.$$

Proof. Suppose that the pressure of each fluid on the surface is p_1 and p_2 respectively. We need to modify the dynamic boundary condition by taking the suface tension T into account. The balance of forces on the surface:

$$\int_{x=a}^{x=b} (p_2 - p_1)\boldsymbol{n}\, \mathrm{d}s = (T\boldsymbol{\tau})_{x=a}^{x=b}$$

where

$$\mathrm{d}s = \sqrt{1 + \eta_x^2}\, \mathrm{d}x \qquad \boldsymbol{\tau} = \frac{1}{\sqrt{1+\eta_x^2}} \begin{pmatrix} 1 \\ \eta_x \end{pmatrix} \qquad \boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}} \begin{pmatrix} -\eta_x \\ 1 \end{pmatrix}$$

Taking the derivative with respect to x at both sides:

$$(p_2 - p_1)\boldsymbol{n} = \frac{T}{\sqrt{1+\eta_x^2}} \frac{\partial \boldsymbol{\tau}}{\partial x}$$

By definition of the curvature κ, $\kappa\boldsymbol{n} = \mathrm{d}\boldsymbol{\tau}/\mathrm{d}s$. We obtain:

$$p_2 - p_1 = T\kappa$$

After linearizationm $\kappa \approx \eta_{xx}$. The dynamic boundary boundary condition becomes $p_2 - p_1 = T\eta_{xx}$. Combine this with Bernoulli's Theorem we obtain:

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) = T\eta_{xx} \quad \text{on } y = 0$$

We substitute the travelling harmonic wave $\eta(x,t) = A\cos(kx - \omega t)$, $\phi_1(x,y,t) = B\mathrm{e}^{ky}\sin(kx - \omega t)$, and $\phi_2(x,y,t) = -B\mathrm{e}^{-ky}\sin(kx - \omega t)$ into the equation:

$$\rho_1(-\omega B + gA) - \rho_2(\omega B + gA) = -TAk^2$$

Hence

$$\begin{pmatrix} \omega & -k \\ Tk^2 + (\rho_1 - \rho_2)g & -(\rho_1 + \rho_2)\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations admit non-trivial solutions if the determinant is zero. From this we obtain the dispersion relation:

$$(\rho_1 + \rho_2)\omega^2 = k(Tk^2 - (\rho_2 - \rho_1)g)$$

The waves are unstable if ω is imaginary. In this case $k(Tk^2 - (\rho_1 - \rho_2)g) < 0$. Hence

$$\lambda > \lambda_c = \frac{2\pi}{k} = 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)g}}$$ □

Question 5

Water flows steadily with speed U over a corrugated bed $y = -h + \varepsilon\cos(kx)$, where $\varepsilon \ll h$, so that there is a time-independent disturbance $\eta(x)$ to the free surface, which would be at $y = 0$ but for the corrugations. By writing the velocity components as

$$u = U + \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y},$$

where $\phi(x, y)$ denotes the velocity potential of the disturbance to the uniform flow, show that the linearized boundary conditions are

$$\frac{\partial\phi}{\partial y} = U\frac{\mathrm{d}\eta}{\mathrm{d}x}, \qquad U\frac{\partial\phi}{\partial x} + g\eta = 0 \qquad \text{on } y = 0,$$
$$\frac{\partial\phi}{\partial y} = -Uk\varepsilon\sin(kx) \qquad \text{on } y = -h,$$

and hence find $\eta(x)$, Deduce that crests on the free surface occur immediately above troughs on the bed if

$$U^2 < \frac{g}{k}\tanh(kh),$$

but that crests on the surface overlie the crests on the bed if this inequality is reversed.

Proof. Considering the uniform flow, the modified kinematic boundary condition now reads:

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} + \left(U + \frac{\partial\phi}{\partial x}\right)\frac{\partial\eta}{\partial x} = \left(U + \frac{\partial\phi}{\partial x}\right)\frac{\mathrm{d}\eta}{\mathrm{d}x} \quad \text{on } y = \eta$$

After linearization it becomes:

$$\frac{\partial\phi}{\partial y} = U\frac{\mathrm{d}\eta}{\mathrm{d}x} \quad \text{on } y = 0$$

By Bernoulli's Theorem:

$$\frac{p}{\rho} + \frac{1}{2}|U\boldsymbol{e}_x + \boldsymbol{\nabla}\phi|^2 + g\eta = F \quad \text{on } y = \eta$$

We put $F = \frac{p}{\rho} + \frac{1}{2}U^2$. Neglecting the non-linear terms, we obtain:

$$U\frac{\partial\phi}{\partial x} + g\eta = 0 \quad \text{on } y = 0$$

Lastly, we need to determine the boundary condition imposed by the bed. The equation of the bed is $\gamma(x) = -h + \varepsilon\cos(kx)$. Then the normal vector

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\gamma_x^2}}\begin{pmatrix}-\gamma_x\\1\end{pmatrix} = \frac{1}{\sqrt{1+\varepsilon^2k^2\sin^2(kx)}}\begin{pmatrix}\varepsilon k\sin(kx)\\1\end{pmatrix}$$

On $y=\gamma(x)$, $(U\boldsymbol{e}_x + \boldsymbol{\nabla}\phi)\cdot\boldsymbol{n} = 0$.

$$\left(U + \frac{\partial\phi}{\partial x}\right)\varepsilon k\sin(kx) + \frac{\partial\phi}{\partial y} = 0$$

We may assume that $U \gg \partial\phi/\partial x$. After linearization we obtain

$$\frac{\partial\phi}{\partial y} = -Uk\varepsilon\sin(kx) \quad \text{on } y=-h$$

Suppose that $\eta(x) = A\cos(kx) + B\sin(kx)$ and $\phi(x,y) = (C\mathrm{e}^{ky} + D\mathrm{e}^{-ky})\cos(kx) + (E\mathrm{e}^{ky} + G\mathrm{e}^{-ky})\sin(kx)$. Substituting into the boundary condition at $y=-h$:

$$\frac{\partial\phi}{\partial y} = -Uk\varepsilon\sin(kx) \implies C\mathrm{e}^{-kh} - D\mathrm{e}^{kh} = 0,\ E\mathrm{e}^{-kh} - G\mathrm{e}^{kh} = -U\varepsilon$$

Alternatively we write $\phi(x,y) = C'\cosh(k(y+h))\cos(kx) + (D'\cosh(k(y+h)) - U\varepsilon\sinh(k(y+h)))\sin(kx)$. The boundary conditions at $y=0$:

$$\begin{aligned}
\frac{\partial\phi}{\partial y} = U\frac{\mathrm{d}\eta}{\mathrm{d}x} &\implies C'k\sinh(kh)\cos(kx) + (D'k\sinh(kh) - kU\varepsilon\cosh(kh))\sin(kx) = Uk(-A\sin(kx) + B\cos(kx))\\
&\implies C'\sinh(kh) = UB,\ D'\sinh(kh) - U\varepsilon\cosh(kh) = -UA\\
U\frac{\partial\phi}{\partial x} + g\eta = 0 &\implies -UC'k\cosh(kh)\sin(kx) + Uk(D'\cosh(kh) - U\varepsilon\sinh(kh))\cos(kx) = -gA\cos(kx) - gB\sin(kx)\\
&\implies UC'k\cosh(kh) = gB,\ Uk(D'\cosh(kh) - U\varepsilon\sinh(kh)) = -gA
\end{aligned}$$

Therefore

$$\begin{pmatrix}U & -\sinh(kh)\\ g & -Uk\cosh(kh)\end{pmatrix}\begin{pmatrix}B\\C'\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}U & \sinh(kh)\\ g & Uk\cosh(kh)\end{pmatrix}\begin{pmatrix}A\\D'\end{pmatrix} = \begin{pmatrix}U\varepsilon\cosh(kh)\\ U^2\varepsilon k\sinh(kh)\end{pmatrix}$$

If $U^2 \neq \dfrac{g}{k}\tanh(kh)$, then the system has only trivial solution: $A=B=C'=0$, $D'=U\varepsilon$.

(*There must be some problems in the solution but I cannot find it...*) □