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**Problem Sheet 1**

# C7.6: General Relativity II

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# Section A: Introductory

## Question 1. Tensor Field

Let $M$ be a smooth manifold and recall that $\mathfrak{X}^\infty(M)$ denotes the space of vector fields and $\Omega^1(M)$ the space of covector fields (1-forms). Show that a map

$$\tau : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \times \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \to C^\infty(M)$$

is induced by a $(k, \ell)$-tensor field if, and only if, it is multilinear over $C^\infty(M)$. Similarly a map

$$\tau : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \times \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \to \mathfrak{X}^\infty(M)$$

is induced by a $(k+1, \ell)$-tensor field if, and only if, it is multilinear over $C^\infty(M)$.

## Question 2. Lie Bracket

Let $M$ be a smooth manifold.

(a) Let $X, Y, Z \in \mathfrak{X}^\infty(M)$. Prove the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(b) Let $\nabla$ be an affine connection on $M$. Show that the torsion $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is a $(1, 2)$-tensor field. Here $X, Y \in \mathfrak{X}^\infty(M)$.

(c) (c) Is the Lie bracket $[\cdot, \cdot] : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$ an affine connection?

*Proof.* (a) The Lie bracket is the commutator of vector fields: $[X, Y] = X \circ Y - Y \circ X$. So

$$\begin{aligned}
[X, [Y, Z]] + [Y, [Z, X]] &= [X, YZ - ZY] + [Y, ZX - XZ] \\
&= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY \\
&= XYZ + ZYX - YXZ - ZXY \\
&= [[X, Y], Z]
\end{aligned}$$

which implies the Jacobi identity.

(b) It suffice to show that $T(X, Y)$ is a $\mathrm{C}^\infty(M)$-linear. Since $T(X, Y) = -T(Y, X)$, it suffices to show this in the first slot. For $f \in \mathrm{C}^\infty(M)$,

$$\begin{aligned}
T(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\
&= f\nabla_X Y - f\nabla_Y X - Y(f) \cdot X - (fX \circ Y - Y(f) \cdot X - fY \circ X) \\
&= f(\nabla_X Y - \nabla_Y X - [X, Y]) = fT(X, Y)
\end{aligned}$$

So $T : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$ is induced by the type-$(1, 2)$ tensor field $T$.

(c) Suppose that the Lie bracket is indeed an affine connection. Then for any $f \in \mathrm{C}^\infty(M)$ and $X, Y \in \Gamma(\mathrm{T}M)$,

$$f\mathcal{L}_X Y = \mathcal{L}_{fX} Y = -\mathcal{L}_Y (fX) = -(f\mathcal{L}_Y X + Y(f) \cdot X) = f\mathcal{L}_X Y - Y(f) \cdot X$$

which implies that $Y(f)\cdot X = 0$. This is absurd (just take $X = Y = \partial_x$, $f(x) = x$ on $M = \mathbb{R}$). $\square$

## Section B: Core

### Question 3. Riemann Tensor

Let $X, Y, Z \in \mathfrak{X}^\infty(M)$ be smooth vector fields on a Lorentzian manifold $(M, g)$. Show that

$$(\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

where $\nabla$ is the Levi-Civita connection. I.e. the left- and right-hand sides are equivalent definitions of the Riemann curvature tensor $R(X,Y)Z$. In (abstract) index notation, this can be written as

$$\nabla_\mu\nabla_\nu Z^\lambda - \nabla_\nu\nabla_\mu Z^\lambda = R^\lambda_{\rho\mu\nu}Z^\rho$$

*Proof.* Let $\mathrm{T}^r_s M := \underbrace{\mathrm{T}M\otimes\cdots\otimes\mathrm{T}M}_{r}\otimes\underbrace{\mathrm{T}^*M\otimes\cdots\otimes\mathrm{T}^*M}_{s}$ be the type-$(r,s)$ tensor bundle, and $\Gamma(\mathrm{T}^r_s M)$ be the algebra of its sections.

Recall that $\nabla Z \in \Gamma(\mathrm{T}^1_1 M)$ is given by $\nabla Z(X,\eta) = \eta(\nabla_X Z)$ for any $X \in \Gamma(\mathrm{T}M)$ and $\eta \in \Gamma(\mathrm{T}^*M)$. Then $\nabla\nabla Z$ is a type-$(1,2)$ tensor field in the following way:

$$\begin{aligned}(\nabla\nabla Z)(X,Y) &:= \nabla_X(\nabla Z)(Y) = \mathrm{tr}(\nabla_X(\nabla Z)\otimes Y)\\ &= \mathrm{tr}(\nabla_X(\nabla Z\otimes Y) - \nabla Z\otimes\nabla_X Y)\\ &= \nabla_X\nabla_Y Z - \mathrm{tr}(\nabla Z\otimes\nabla_X Y)\end{aligned}$$

where tr is the contraction of indices (in an unambiguous way). Therefore we have

$$\begin{aligned}(\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X) &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \mathrm{tr}(\nabla Z\otimes(\nabla_X Y - \nabla_Y X))\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \mathrm{tr}(\nabla Z\otimes[X,Y]) \qquad (\nabla\text{ is torsion-free})\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\end{aligned}$$

*I cannot follow the logic of abstract index equation here. How can we just throw away the Lie bracket without choosing an orthonormal system of local coordinates?*[1]

If we work in a local chart $\{x^0, ..., x^n\}$ such that $g(\partial_i, \partial_j) = \eta_{ij}$. Then $[\partial_i, \partial_j] = 0$. And we have the Rieman curvature in terms of the components:

$$\nabla_\mu\nabla_\nu Z^\lambda - \nabla_\nu\nabla_\mu Z^\lambda = R^\lambda_{\rho\mu\nu}Z^\rho$$

$\square$

### Question 4. Parallel Transport

Let $(M, g)$ be a Lorentzian or Riemannian manifold, consider a point $p \in M$ and let $x^\mu$ be a local coordinate system centred at $p$ (i.e. $x^\mu(p) = 0$ ). Let $X, Y, Z \in T_pM$ be three tangent vectors. Let $0 < \varepsilon, \delta \ll 1$ be very small. We first parallelly propagate $Z$ along the curve $\gamma$, i.e. first from 0 along the straight coordinate line to $\varepsilon X^\mu$ and then along the straight coordinate line to $\varepsilon X^\mu + \delta Y^\mu$ to obtain the vector $Z_\gamma(\varepsilon X + \delta Y)$. Next we parallelly propagate $Z$ along the curve $\gamma'$, i.e. first from 0 to $\delta Y^\mu$ and then to $\varepsilon X^\mu + \delta Y^\mu$, both along straight coordinate lines. Denote the resulting vector by $Z_{\gamma'}(\varepsilon X + \delta Y)$.

[1] My suspicion is supported by this wiki page: `www.wikiwand.com/en/Tetrad_formalism#/Manipulation_of_indices`

Show that to leading order in $\varepsilon$ and $\delta$ we have

$$Z^\lambda_\gamma(\varepsilon X + \delta Y) - Z^\lambda_{\gamma'}(\varepsilon X + \delta Y) = -\varepsilon\delta R^\lambda_{\rho\alpha\beta}(0) Z^\rho X^\alpha Y^\beta$$

thus giving another interpretation of curvature. [*Hint: Can you justify that the parallel transport of $Z$ from 0 to $\varepsilon X^\mu$ is to leading order $Z^\mu - \varepsilon\Gamma^\mu_{\nu\rho}(0)X^\nu Z^\rho$ ?*]

*Proof.* Let $\gamma : (-t_0, t_0) \to M$ be the curve with $x^\mu \circ \gamma(\varepsilon) = 0 + \varepsilon X^\mu + \mathcal{O}(\varepsilon^2)$. Then $\dot{x}^\mu \circ \gamma(\varepsilon) = X^\mu + \mathcal{O}(\varepsilon)$. Suppose that $Z(t)$ is a parallel tangent vector field along $\gamma$. Then the covariant derivative is given by:

$$0 = \nabla_{\dot\gamma} Z = \left(\frac{\mathrm{d}Z^\lambda[\gamma(t)]}{\mathrm{d}t} + \Gamma^\lambda_{\mu\nu} X^\mu Z^\nu\right)\partial_\lambda$$

In an asymptotic expansion $Z(\varepsilon) = Z + \varepsilon Z' + \mathcal{O}(\varepsilon^2)$, we find that at $\mathcal{O}(1)$ we have $(Z')^\lambda = -\Gamma^\lambda_{\mu\nu}(0)X^\mu Z^\nu$. Therefore the $Z$ is parallelly transported to $(Z^\lambda - \varepsilon\Gamma^\lambda_{\mu\nu}(0)X^\mu Z^\nu)\partial_\lambda$.

Next we can compute the parallel transport of $Z$ along the two prescribed curves (to the second order):

$$\begin{aligned}
Z^\lambda_\gamma(\varepsilon X + \delta Y) &= Z^\lambda(\varepsilon X) - \delta\Gamma^\lambda_{\mu\nu}(\varepsilon X) Y^\mu(\varepsilon X) Z^\nu(\varepsilon X) \\
&= Z^\lambda - \varepsilon\Gamma^\lambda_{\mu\nu}(0)X^\mu Z^\nu - \delta(\Gamma^\lambda_{\mu\nu}(0) + \varepsilon X^\eta \partial_\eta \Gamma^\lambda_{\mu\nu}(0))(Y^\mu - \varepsilon\Gamma^\mu_{\alpha\beta}(0)X^\alpha Y^\beta)\left(Z^\nu - \varepsilon\Gamma^\nu_{\sigma\rho}(0)X^\sigma Z^\rho\right) \\
&= Z^\lambda - (\varepsilon X^\mu + \delta Y^\mu)\Gamma^\lambda_{\mu\nu}(0) Z^\nu + \varepsilon\delta\Gamma^\lambda_{\mu\nu}(0)\left(\Gamma^\nu_{\sigma\rho}(0) X^\sigma Y^\mu Z^\rho + \Gamma^\mu_{\sigma\rho}(0) X^\sigma Y^\rho Z^\nu\right) \\
&\quad + \varepsilon\delta X^\eta Y^\mu Z^\nu \partial_\eta \Gamma^\lambda_{\mu\nu}(0) \\
&= Z^\lambda - (\varepsilon X^\mu + \delta Y^\mu)\Gamma^\lambda_{\mu\nu}(0) Z^\nu + \varepsilon\delta X^\mu Y^\nu Z^\rho\left(\Gamma^\lambda_{\nu\sigma}(0)\Gamma^\sigma_{\mu\rho}(0) + \Gamma^\lambda_{\sigma\rho}(0)\Gamma^\sigma_{\mu\nu}(0) + \partial_\mu \Gamma^\lambda_{\nu\rho}(0)\right)
\end{aligned}$$

For simplicity we suppress the dependence on the coordinates. To the second order, we have

$$\begin{aligned}
Z^\lambda_\gamma(\varepsilon X + \delta Y) - Z^\lambda_{\gamma'}(\varepsilon X + \delta Y) &= \varepsilon\delta(X^\mu Y^\nu - X^\nu Y^\mu) Z^\rho \left(\Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} + \partial_\mu \Gamma^\lambda_{\nu\rho}\right) \\
&= \varepsilon\delta X^\mu Y^\nu Z^\rho \left(\Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} + \partial_\mu\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho}\right) \quad (\text{torsion-free: } \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu})
\end{aligned}$$

If we take the component equation obtained in Question 3, and expand all covariant derivatives (which is as painful as the calculations above), then we shall find the Riemann curvature in terms of the Christoffel symbols:

$$R^\lambda{}_{\rho\nu\mu} = 2\left(\partial_{[\nu}\Gamma^\lambda_{\mu]\rho} - \Gamma^\lambda_{\sigma[\nu}\Gamma^\sigma_{\mu]\rho}\right) = \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} + \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\mu\Gamma^\lambda_{\nu\rho}$$

Therefore we deduce that

$$Z^\lambda_\gamma(\varepsilon X + \delta Y) - Z^\lambda_{\gamma'}(\varepsilon X + \delta Y) = -\varepsilon\delta R^\lambda{}_{\rho\nu\mu}(0) X^\mu Y^\nu Z^\rho$$ □

**Question 5. Lie Derivative**

Let $M$ be a smooth manifold and let $X, Y$ be smooth vector fields. Show that for a general $(k, \ell)$-tensor field $T$ we have the following identity for the Lie derivative:

$$\mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) = \mathcal{L}_{[X,Y]} T.$$

*Proof.* If we can invoke the result that $\mathcal{L}_X Y = [X, Y]$, which is proven in the lectures, then the equation we want is just Jacobi identity in disguise: Yes, this is true for vector field T. We want to prove it for T of general (k, ℓ) type.

$$[X,[Y,T]] + [Y,[T,X]] + [T,[X,Y]] = 0 \iff [X,[Y,T]] - [Y,[X,T]] = [[X,Y],T]$$
$$\iff \mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) = \mathcal{L}_{[X,Y]} T$$ □

For 1-form, $\mathcal{L}_X(df) = d(X(f))$. Use vector fields and 1-forms to locally build (k, ℓ) tensor and then show:

$\mathcal{L}_X(\mathcal{L}_Y(T\otimes S)) - \mathcal{L}_Y(\mathcal{L}_X(T\otimes S)) - \mathcal{L}_{[X,Y]}(T\otimes S) = ((\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X - \mathcal{L}_{[X,Y]})T)\otimes S + T\otimes(\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X - \mathcal{L}_{[X,Y]})S.$

**Question 6. Killing Vectors**

(a) Let $(M, g)$ be an $n$-dimensional Lorentzian (or Riemannian) manifold. Use the equation

$$\nabla_a \nabla_b K_c = R^d_{abc} K_d \tag{3}$$

from the lectures to show that the maximum number of linearly independent Killing vector fields (KVF) on $M$ is $\dfrac{n(n+1)}{2}$.

[*Hint: Derive a system of ODEs with initial data given at a point $p$ in $M$.*]

(b) Consider 4-dimensional Minkowski spacetime. Write down equation (3) in standard Cartesian coordinates and derive the 10-dimensional space of Killing vectors on Minkowski spacetime. Choose the basis vectors such that they form the infinitesimal generators of translations and Lorentz transformations.

*Proof.* (a) *The phrasing of the question is a bit confusing, as we can only have $n$ $C^\infty(M)$-linearly independent vector fields on an $n$-dimensional manifold. I think it really asks to find $\frac{1}{2}n(n+1)$ $\mathbb{R}$-linearly independent Killing vector fields.*

Fix $p \in M$. Suppose that $K$ is a Killing vector field. We shall prove that, if $K(p) = 0$ and $\nabla K|_p = 0$, then $K = 0$ on the connected component of $p$ in $M$.[2]

The Killing equation (3) in coordinate-independent form is given by

$$(\nabla\nabla K)(X, Y) = R(X, K)Y$$

We define on the bundle $\mathrm{T}M \oplus \mathrm{End}(\mathrm{T}M) \cong \mathrm{T}M \oplus \mathrm{T}^1_1 M$ a new connection $D$:

$$D_X(Y, \sigma) := (\nabla_X Y - \sigma(X), \nabla_X \sigma - R(X, Y))$$

Note that by Killing equation, we have

$$D_X(K, \nabla K) = (\nabla_X K - \nabla_X K, \nabla_X \nabla K - R(X, K)) = 0$$

for all $X \in \Gamma(\mathrm{T}M)$. Then $(K, \nabla K)$ is a parallel section of $D$.

We take $q \in M$ lying in the same path component of $p$, and a curve $\gamma$ joining $p$ and $q$. Then $D_{\dot\gamma}(K, \nabla K) = 0$ on $\gamma$. This is a system of first-order ODEs with the initial condition $(K, \nabla K)|_p = 0$. By Picard's theorem it admits the unique solution $(K, \nabla K) = 0$ on $\gamma$. Hence $K(q) = 0$. This shows that $\{q \in M : K(q) = 0\}$ is an open and closed set, and hence it is the connected component of $p$ in $M$.

Finally, we take a local trivialisation $U \oplus \mathbb{R}^n \oplus \mathbb{R}^{n^2}$ of $\mathrm{T}M \oplus \mathrm{End}(\mathrm{T}M)$ in a neighbourhood $U$ of $p \in M$. If $K$ is a Killing field, then $\nabla_\mu K^\nu = -\nabla_\nu K^\mu$. So the local matrix of $\nabla K$ at $p$ is anti-symmetric. It follows that

$$V := \operatorname{span}\left\{(K, \nabla K)|_p \in U \oplus \mathbb{R}^n \oplus \mathbb{R}^{n^2} : \mathcal{L}_K g = 0\right\}, \quad \dim V \leqslant n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$$

Let $m = \frac{1}{2}n(n+1)$. Suppose that $K_1, \ldots, K_m$ are $\mathbb{R}$-linearly independent Killing vector fields on $M$. Then they form a basis of $V$ at $p \in M$. If $K_0$ is another Killing vector field, then there exists $a_1, \ldots, a_m \in \mathbb{R}$ such that $(K_m, \nabla K_m)|_p = \sum_{i=1}^m a_i (K_i, \nabla K_i)|_p$. The previous argument shows that $K_0 = \sum_{i=1}^m a_i K_i$ on the connected component of $M$. Moreover, if $M$ is connected, then it has at most

[2] I found this lovely coordinate-independent proof from `math.stackexchange.com/questions/374054`.

$m = \frac{1}{2}n(n+1)$ linearly independent Killing vector fields.

(b) The Minkowski spacetime $(M^4, \eta)$ is flat. In the standard Cartesian coordinates, the Killing equations become $\partial_\mu \partial_\nu K_\lambda = 0$. So $K_\mu = c_\mu + a^\nu{}_\mu x_\nu$, where $c_\mu, a^\nu{}_\mu \in \mathbb{R}$, and $a^\mu{}_\nu = -a^\nu{}_\mu$. They span a 10-dimensional vector bundle of Killing vector fields.

The four translations are generated by $K_\mu = \partial_\mu$ for $\mu \in \{0,1,2,3\}$. The six generators of the Lorentz algebra are given by $L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$ for $0 \leqslant \mu < \nu \leqslant 3$ ($L_{01}, L_{02}, L_{03}$ generate the Lorentz boosts, and $L_{12}, L_{13}, L_{23}$ generate the rigid rotations). □

**Question 7. Locally Inertial Coordinates**

This problem introduces and discusses locally inertial coordinates, which can be used by a freely falling observer to make contact with special relativity. We will make use of them later in the course when we discuss the observation of gravitational waves.

Let $(M, g)$ be a $(3+1)$-dimensional Lorentzian manifold.

(a) Let $y^\mu$ be a coordinate system around a point $p \in M$. Show that

$$\partial_\lambda g_{\mu\nu}(p) = 0 \quad \forall \lambda, \mu, \nu \in \{0, \ldots, 3\} \quad \Longleftrightarrow \quad \Gamma^\lambda_{\mu\nu}(p) = 0 \quad \forall \lambda, \mu, \nu \in \{0, \ldots, 3\}$$

(b) Let $\gamma : I \to M$ be an affinely parameterised timelike geodesic with $g(\dot\gamma, \dot\gamma) = -1$, i.e. the worldline of a freely falling observer, parameterised by proper time. Consider a proper time $s_0 \in I$ and consider an orthonormal Lorentz frame $e_0 := \dot\gamma(s_0), e_1, e_2, e_3 \in T_{\gamma(s_0)}M$, i.e. we have $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Parallelly propagate the Lorentz frame $e_\mu$ along $\gamma$ such that $\nabla_{\dot\gamma} e_\mu = 0$ holds. Introduce the mapping

$$\left(x^0, x^1, x^2, x^3\right) \mapsto \exp_{\gamma(x^0)}\left(x^1 e_1 + x^2 e_2 + x^3 e_3\right) \in M.$$

(*Recall from GR I that the exponential map* $\exp_p : T_pM \supseteq U \to M$ *at basepoint* $p$ *maps the tangent vector* $X$ *to* $\exp_p(X) := \sigma(1)$, *where* $\sigma : [0,1] \to M$ *is the unique geodesic with* $\sigma(0) = p$ *and* $\dot\sigma(0) = X$.)

Show that in a small neighbourhood of $\gamma(s_0)$ these are coordinates and that in these coordinates the metric takes the form

$$g = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \mathcal{O}\left(r^2\right) dx^\mu \otimes dx^\nu,$$

where $r^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$. We call such coordinates locally inertial coordinates.

(c) Let $\gamma : I \to M$ be a timelike curve parametrised by proper time. Assume there exists a coordinate system $x^\mu$ in a neighbourhood of some point $\gamma(s_0)$ such that in these coordinates $\gamma(s) = (s, 0, 0, 0)$ and

$$g = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \mathcal{O}\left(r^2\right) dx^\mu \otimes dx^\nu$$

holds, where $r$ is as above. Show that, in particular, $\gamma$ must be an affinely parametrised geodesic and that $\partial_1, \partial_2, \partial_3$ are parallel along $\gamma$.

*Proof.* (a) Let $\nabla$ be the Levi-Civita connection. The compatibility equation $\nabla g = 0$ in local coordinates:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{$*$}$$

$$\partial_\lambda g_{\mu\nu} = \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} \tag{$**$}$$

So " $\implies$ " follows from $(*)$ and " $\impliedby$ " follows from $(**)$.

(b) Let $\varphi^{-1} : (\mathbb{R}^4, \eta) \cong \mathrm{T}_{\gamma(s_0)}M \to M$ be the map:

$$\varphi^{-1}(x^0, x^1, x^2, x^3) = \exp_{\gamma(x^0)}(x^i e_i) \neq \exp_{\gamma(s_0)}((x^0 - s_0)e_0 + x^i e_i)$$

We claim that $\varphi^{-1}$ is a local diffeomorphism. For a complete proof, see *Theorem 3.7 of C3.11 Riemannian Geometry*. Then the inverse $\varphi : U \to (\mathbb{R}^4, \eta)$ defines a coordinate chart on $M$, with the frame vector fields $\{e_\mu\}$. We need to show that $\partial_\lambda g_{\mu\nu} = 0$ at $\gamma(s_0)$. By (a) it suffices to show that the Christoffel symbols vanishes at $\gamma(s_0)$.

For $V = V^\mu e_\mu$, let $\alpha_V : [0, \varepsilon] \to M$ be the unique geodesic such that $\alpha_V(0) = \gamma(s_0)$ and $\dot{\alpha}_V(0) = V$. Then by definition $\alpha_V(s) = \exp_{\gamma(s_0)}(sV)$. Hence $\varphi \circ \alpha_V(s) = (sV^0 + s_0, sV^1, sV^2, sV^3)$. The geodesic equation:

$$\frac{\mathrm{d}^2(\varphi \circ \alpha_V(s))^\lambda}{\mathrm{d}s^2} + \Gamma^\lambda_{\mu\nu}(s)\frac{\mathrm{d}(\varphi \circ \alpha_V(s))^\mu}{\mathrm{d}st}\frac{\mathrm{d}(\varphi \circ \alpha_V(s))^\nu}{\mathrm{d}s} = 0$$

at $s = 0$ gives

$$\Gamma^\lambda_{\mu\nu}(\gamma(s_0))V^\mu V^\nu = 0$$

Since $\Gamma^\lambda_{\mu\nu}$ is symmetric in the lower indices, and $V$ is arbitrary, we deduce that $\Gamma^\lambda_{\mu\nu}(\gamma(s_0)) = 0$. Therefore near $\gamma(s_0)$ the metric takes the form

$$g = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{O}(r^2)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

(c) The Christoffel symbols vanishes at $\gamma(s_0)$.

□

## Section C: Optional

### Question 8. Inner Product of Two-Forms

Let $(M, g)$ be an $n$-dimensional Riemannian manifold.

(a) Let $p \in M$ and denote with $\Lambda^2 T_p^* M$ the space of all 2-covectors at $p$ that are antisymmetric, i.e. all $\omega \in T_p^* M \otimes T_p^* M$ such that $\omega_{ab} = -\omega_{ba}$. Moreover, for $\alpha, \beta \in T_p^* M$ we define the wedge product $\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \in \Lambda^2 T_p^* M$. Let $\alpha^1, \ldots, \alpha^n$ be an orthonormal basis (ONB) for $T_p^* M$. Show that $\alpha^i \wedge \alpha^j$ with $1 \leqslant i < j \leqslant n$ is a basis of $\Lambda^2 T_p^* M$ and thus $\Lambda^2 T_p^* M$ is $\frac{n(n-1)}{2}$ dimensional.

(b) Show that for $\alpha, \beta, \gamma, \delta \in T_p^* M$ the mapping

$$\langle \alpha \wedge \beta, \gamma \wedge \delta \rangle := \det \begin{pmatrix} g^{-1}(\alpha, \gamma) & g^{-1}(\alpha, \delta) \\ g^{-1}(\beta, \gamma) & g^{-1}(\beta, \delta) \end{pmatrix}$$

induces an inner product on $\Lambda^2 T_p^* M$, with respect to which $\alpha^i \wedge \alpha^j, 1 \leqslant i < j \leqslant n$ is an ONB. Also show that for $\omega, \rho \in \Lambda^2 T_p^* M$ one has $\langle \omega, \rho \rangle = g^{ik} g^{jl} \omega_{ij} \rho_{kl}$.

(c) Consider the Riemann curvature tensor as a $(2,2)$-tensor $R_{ij}{}^{kl}$ and show that it is a self-adjoint linear map $\mathbb{R} : \Lambda^2 T_p^* M \to \Lambda^2 T_p^* M$ with respect to the inner product $\langle \cdot, \cdot \rangle$.

(d) Show that if $(M, g)$ is connected and has $\frac{n(n+1)}{2}$ linearly independent Killing vector fields that then the Riemannian curvature tensor is of the form $R_{ijkl} = 2C g_{k[i} g_{j]l} = C\left(g_{ki} g_{jl} - g_{kj} g_{il}\right)$ with $C$ being a constant.

[*Hint: You may use that an isometry* $\phi : M \to M$ *preserves the Riemann tensor, i.e.* $(\phi^* R)_{ijkl} = R_{ijkl}$*.*]

Where in GR I have you encountered such spaces? One can indeed show further that manifolds whose Riemann tensor is of the above form with the same constant $C$ are locally isometric.

*Proof.* (a)

□