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Problem Sheet 4

B8.1: Probability, Measure & Martingales

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Remark.

Section 1

Question 1

Let $(Y_n)_{n\geqslant 0}$ be a *supermartingale* on a filtered probability space $\left(\Omega,\mathscr{F},(\mathscr{F}_n)_{n\geqslant 0},\mathbb{P}\right)$, $(V_n)_{n\geqslant 1}$ a nonnegative predictable process and let $X_0 = 0$ and

$$X_n = \sum_{k=1}^{n} V_k\left(Y_k - Y_{k-1}\right), \quad n \geqslant 1$$

Show that if X_n is integrable, $n \geqslant 0$, then $\mathbf{X}$ is a supermartingale. What happens when V is negative? And/or Y is a submartingale?

Proof. For $k \leqslant n$, all Y_k and V_k are $\mathscr{F}_n$-measurable, so X_n is $\mathscr{F}_n$-measurable. We have:

$$\mathbb{E}[X_{n+1} - X_n \mid \mathscr{F}_n] = \mathbb{E}[V_{n+1}(Y_{n+1} - Y_n) \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} V_{n+1}\mathbb{E}[Y_{n+1} - Y_n \mid \mathscr{F}_n]$$

Since $\mathbf{Y}$ is a supermartingale, $\mathbb{E}[Y_{n+1} - Y_n \mid \mathscr{F}_n] \leqslant 0$. Since V_{n+1} is non-negative, we deduce that $\mathbb{E}[X_{n+1} - X_n \mid \mathscr{F}_n] \leqslant 0$ and hence $\mathbf{X}$ is a supermartingale.

Alternatively, if $\mathbf{Y}$ is a submartingale, then $\mathbf{X}$ is also a submartingale. If V_n is negative, the result has a change in sign: $\mathbf{X}$ is a submartingale if $\mathbf{Y}$ is a supermartingale, and $\mathbf{X}$ is a supermartingale if $\mathbf{Y}$ is a submartingale. □

Question 2

Suppose $X_n, n \geqslant 1$ are i.i.d. with finite mean. Suppose N is a stopping time relative to the natural filtration generated by the sequence $(X_n)_{n\geqslant 1}$, with $\mathbb{E}[N] < \infty$. Show that

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[N]\mathbb{E}[X_1]$$

(This is known as *Wald's equation* and it improves upon *random sum theorem* in Part A Probability.)

Proof. Let $\mathscr{F}_n := \sigma(X_1,\ldots,X_n)$ be the natural filtration generated by (X_n). Define $M_n := \sum_{i=1}^{n} X_i - n\mathbb{E}[X_1]$. It is clear that M_n is $\mathscr{F}_n$-measurable. And

$$\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} \mathbb{E}[M_n \mid \mathscr{F}_n] + \mathbb{E}[X_{n+1} \mid \mathscr{F}_n] - \mathbb{E}[\mathbb{E}[X_1] \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} M_n + \mathbb{E}[X_{n+1}] - \mathbb{E}[X_1] = M_n$$

Hence (M_n) is a martingale with respect to $(\mathscr{F}_n)$. Moreover, note that

$$\mathbb{E}[|M_{n+1} - M_n| \mid \mathscr{F}_n] = \mathbb{E}[|X_{n+1} - \mathbb{E}[X_1]| \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} \mathbb{E}[|X_{n+1} - \mathbb{E}[X_1]|] \leqslant \mathbb{E}[|X_{n+1}|] + |\mathbb{E}[X_1]| = \mathbb{E}[|X_1|] + |\mathbb{E}[X_1]|$$

Since X_1 has finite mean, $\mathbb{E}[|X_1|], |\mathbb{E}[X_1]| < \infty$. By optional stopping theorem (Corollary 8.19) with condition (ii) satisfied, we have

$$\mathbb{E}\left[M_N \mathbf{1}_{\{N>\infty\}}\right] = \mathbb{E}[M_1]$$

Hence

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = 0 + \mathbb{E}[N\mathbb{E}[X_1]] = \mathbb{E}[N]\mathbb{E}[X_1]$$ □

Question 3. Martingale formulation of Bellman's Optimality Principle

Your winnings per unit stake on a certain game are ε_n in round n, where $(\varepsilon_n)_{n\geqslant 1}$ is an i.i.d. sequence of random variables with

$$\mathbb{P}[\varepsilon_n = +1] = p, \quad \mathbb{P}[\varepsilon_n = -1] = q, \quad \text{where } 1/2 < p = 1 - q < 1$$

Let $\mathscr{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$. Assume that your stake V_n on game n is $\mathscr{F}_{n-1}$ -measurable (i.e., only depends on the outcome of the game up to time $n-1$), and that V_n must lie strictly between 0 and Z_{n-1} where Z_{n-1} is your fortune at time $n-1$. Your object is to maximize the expected "interest rate" at a certain integer time horizon N, i.e. $\mathbb{E}[\log(Z_N/Z_0)]$, given the constant $Z_0 > 0$, your capital at time 0 . Prove that, if $Z_n > 0$

$$\mathbb{E}[\log(Z_{n+1}/Z_n) \mid \mathscr{F}_n] = f(V_{n+1}/Z_n)$$

where $f(x) := p\log(1+x) + q\log(1-x)$.

Deduce that, if (V_n) is any (predictable) strategy, then $(\log Z_n - n\alpha)_{n\geqslant 0}$ is a supermartingale, where

$$\alpha := p\log p + q\log q + \log 2$$

α is sometimes called the "entropy". Conclude that

$$\mathbb{E}[\log(Z_N/Z_0)] \leqslant N\alpha$$

Describe explicitly the strategy (V_n) for which $(\log Z_n - n\alpha)_{n\geqslant 0}$ becomes a martingale and the inequality becomes an equality.

Proof. The captial (Z_n) satisfies

$$Z_{n+1} = Z_n + V_{n+1}\varepsilon_{n+1}$$

Hence

$$\log\frac{Z_{n+1}}{Z_n} = \log\left(1 + \frac{V_{n+1}}{Z_n}\varepsilon_{n+1}\right) = \log\left(1+\frac{V_{n+1}}{Z_n}\right)\mathbf{1}_{\{\varepsilon_{n+1}=1\}} + \log\left(1-\frac{V_{n+1}}{Z_n}\right)\mathbf{1}_{\{\varepsilon_{n+1}=-1\}}$$

Since (V_n) is predictable with respect to $(\mathscr{F}_n)$, then V_{n+1}/Z_n is $\mathscr{F}_n$-measurable. Since $\sigma(\varepsilon_{n+1})$ is independent of $\mathscr{F}_n = \sigma(\varepsilon_1, ..., \varepsilon_n)$, we have

$$\begin{aligned}
\mathbb{E}\left[\log\frac{Z_{n+1}}{Z_n} \,\middle|\, \mathscr{F}_n\right] &\overset{\text{a.s.}}{=\!=} \mathbb{E}\left[\log\left(1+\frac{V_{n+1}}{Z_n}\right)\mathbf{1}_{\{\varepsilon_{n+1}=1\}} \,\middle|\, \mathscr{F}_n\right] + \mathbb{E}\left[\log\left(1-\frac{V_{n+1}}{Z_n}\right)\mathbf{1}_{\{\varepsilon_{n+1}=-1\}} \,\middle|\, \mathscr{F}_n\right] \\
&\overset{\text{a.s.}}{=\!=} \log\left(1+\frac{V_{n+1}}{Z_n}\right)\mathbb{P}(\varepsilon_{n+1}=1) + \log\left(1-\frac{V_{n+1}}{Z_n}\right)\mathbb{P}(\varepsilon_{n+1}=-1) \\
&= p\log\left(1+\frac{V_{n+1}}{Z_n}\right) + q\log\left(1-\frac{V_{n+1}}{Z_n}\right) = f(V_{n+1}/Z_n)
\end{aligned}$$

Since Z_n is $\mathscr{F}_n$-measurable, so is $(\log Z_n - n\alpha)$. We have

$$\mathbb{E}[\log Z_{n+1} - (n+1)\alpha \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} \mathbb{E}\left[\log Z_n - n\alpha + \log\frac{Z_{n+1}}{Z_n} - \alpha \,\middle|\, \mathscr{F}_n\right] \overset{\text{a.s.}}{=\!=} \log Z_n - n\alpha + f(V_{n+1}/Z_n) - \alpha$$

Note that $f(x) \to -\infty$ as $x \to \pm 1$. f has a unique maximum at $x_0 \in (-1, 1)$, where

$$f'(x_0) = 0 \implies \frac{p}{1+x_0} - \frac{1-p}{1-x_0} = 0 \implies x_0 = 2p-1 \implies f(x_0) = p\log(2p) + q\log(2q) = \alpha$$

Hence $f(V_{n+1}/Z_n) \leqslant \alpha$ for all $\omega \in \Omega$ such that $\frac{V_{n+1}}{Z_n}(\omega) \in (-1,1)$, with equality holds if and only if $\frac{V_{n+1}}{Z_n}(\omega) = 2p-1$. In particular,

$$\mathbb{E}[\log Z_{n+1} - (n+1)\alpha \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} \log Z_n - n\alpha + f(V_{n+1}/Z_n) - \alpha \leqslant \log Z_n - n\alpha$$

Hence $(\log Z_n - n\alpha)$ is a supermartingale with respect to $(\mathscr{F}_n)$.

Finally, since $(\log Z_n - n\alpha)$ is a supermartingale, we have

$$\mathbb{E}[\log Z_N - N\alpha] \leqslant \mathbb{E}[\log Z_0] \implies \mathbb{E}\left[\log\frac{Z_N}{Z_0}\right] \leqslant N\alpha$$

The best strategy is that $V_{n+1} = (2p-1)Z_n$. In this case $f(V_{n+1}/Z_n) = \alpha$ and hence $(\log Z_n - n\alpha)$ becomes a martingale. Now we have $\mathbb{E}\left[\log\frac{Z_N}{Z_0}\right] = N\alpha$. □

Question 4

Let (Z_n) be a Galton-Watson branching process with offspring distribution X, with $\mu = \mathbb{E}[X] > 1$ and $\sigma^2 = \mathrm{Var}[X] < \infty$. Set $\mathscr{F}_n = \sigma(Z_k : k \leqslant n)$ and $M_n = Z_n/\mu^n$. Recall that M is an $(\mathscr{F}_n)$ martingale. Compute $\mathbb{E}\left[Z_{n+1}^2 \mid \mathscr{F}_n\right]$

(*You may reason conditionally on* $\{Z_n = k\}$ *since these form a countable partition of* Ω.)

Using induction, find a formula for $\mathbb{E}\left[Z_n^2\right]$ in terms of n, μ and σ. Deduce that (M_n) is bounded in $\mathscr{L}^2$, and converges in $\mathscr{L}^2$ and in $\mathscr{L}^1$.

Proof. We know that $Z_{n+1} = \sum_{r=1}^{Z_n} X_{n+1,r}$. Hence

$$\begin{aligned}
\mathbb{E}\left[Z_{n+1}^2 \mid \mathscr{F}_n\right] &= \mathbb{E}\left[\left(\sum_{r=1}^{Z_n} X_{n+1,r}\right)^2 \,\middle|\, \mathscr{F}_n\right] \overset{\text{a.s.}}{=} \mathbb{E}\left[\sum_{r=1}^{Z_n} X_{n+1,r}^2 \,\middle|\, \mathscr{F}_n\right] + 2\mathbb{E}\left[\sum_{r=1}^{Z_n}\sum_{s=1}^{r-1} X_{n+1,r}X_{n+1,s} \,\middle|\, \mathscr{F}_n\right] \\
&= \mathbb{E}\left[\sum_{r=1}^{\infty} X_{n+1,r}^2 \mathbf{1}_{\{Z_n \geqslant r\}} \,\middle|\, \mathscr{F}_n\right] + 2\mathbb{E}\left[\sum_{r=1}^{\infty}\sum_{s=1}^{r-1} X_{n+1,r}X_{n+1,s}\mathbf{1}_{\{Z_n \geqslant r\}} \,\middle|\, \mathscr{F}_n\right] \\
&\overset{\text{a.s.}}{=} \sum_{r=1}^{\infty} \mathbb{E}\left[X_{n+1,r}^2 \mathbf{1}_{\{Z_n \geqslant r\}} \mid \mathscr{F}_n\right] + \sum_{r=1}^{\infty}\sum_{s=1}^{r-1} \mathbb{E}\left[X_{n+1,r}X_{n+1,s}\mathbf{1}_{\{Z_n \geqslant r\}} \mid \mathscr{F}_n\right] \qquad \text{(cMCT)} \\
&\overset{\text{a.s.}}{=} \sum_{r=1}^{\infty} \mathbb{E}\left[X_{n+1,r}^2\right] \mathbf{1}_{\{Z_n \geqslant r\}} + \sum_{r=1}^{\infty}\sum_{s=1}^{r-1} \mathbb{E}\left[X_{n+1,r}\right]\mathbb{E}\left[X_{n+1,s}\right]\mathbf{1}_{\{Z_n \geqslant r\}} \\
&= \sum_{r=1}^{Z_n} (\sigma^2 + \mu^2) + \sum_{r=1}^{Z_n}\sum_{s=1}^{r-1} \mu^2 \\
&= \sigma^2 Z_n + \mu^2 Z_n^2
\end{aligned}$$

Taking expectation on both sides we obtain

$$\mathbb{E}\left[Z_{n+1}^2\right] = \mathbb{E}\left[\mathbb{E}\left[Z_{n+1}^2 \mid \mathscr{F}_n\right]\right] = \sigma^2 \mathbb{E}[Z_n] + \mu^2 \mathbb{E}\left[Z_n^2\right]$$

From Example 8.31 we know that $\mathbb{E}[Z_n] = \mu^n$. Hence

$$\mathbb{E}\left[Z_{n+1}^2\right] = \sigma^2 \mu^n + \mu^2 \mathbb{E}\left[Z_n^2\right]$$

With $\mathbb{E}\left[Z_0^2\right] = 1$, inductively we have

$$\mathbb{E}\left[Z_n^2\right] = \sigma^2 \sum_{k=n-1}^{2n-2} \mu^k + \mu^{2n} = \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} + \mu^{2n}$$

Therefore

$$\mathbb{E}\left[M_n^2\right] = \mathbb{E}\left[\frac{Z_n^2}{\mu^{2n}}\right] = \sigma^2 \frac{\mu^n - 1}{\mu^{n+1}(\mu - 1)} + 1 \to 1$$

as $n \to \infty$. In particular $(\mathbb{E}\left[M_n^2\right])$ is a bounded sequence. Since M_n converges in L^2, and the probability measure is finite, by Lemma 5.13, M_n also converges in L^1. □

Question 5

On a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n \geqslant 0}, \mathbb{P}\right)$, let $(M_n)_{n \geqslant 0}$ be an adapted and integrable process. Show that M is a martingale if and only if $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ for all bounded stopping times τ.

[*Hint: consider* $A \in \mathscr{F}_n$ *and two stopping times* $\tau = n+1$ *and* $\tau = n + \mathbf{1}_{A^c}$]

Proof. The forward direction is exactly the optional sampling theorem (Theorem 8.17). We only prove the backward direction. Suppose that $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ for all bounded stopping times τ.

Fix $A \in \mathscr{F}_n$. Let $\tau = n + \mathbf{1}_{A^c}$. Note that $\{\tau = n\} = A \in \mathscr{F}_n$ and $\{\tau = n+1\} = A^c \in \mathscr{F}_n \subseteq \mathscr{F}_{n+1}$. Hence τ is a bounded stopping time. We have

$$\mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mathbf{1}_A] + \mathbb{E}[M_\tau \mathbf{1}_{A^c}] = \mathbb{E}[M_n \mathbf{1}_A] + \mathbb{E}[M_{n+1}\mathbf{1}_{A^c}]$$

On the other hand, we also have

$$\mathbb{E}[M_0] = \mathbb{E}[M_{n+1}] = \mathbb{E}[M_{n+1}\mathbf{1}_A] + \mathbb{E}[M_{n+1}\mathbf{1}_{A^c}]$$

Hence $\mathbb{E}[M_n\mathbf{1}_A] = \mathbb{E}[M_{n+1}\mathbf{1}_A]$. Since M_n is $\mathscr{F}_n$-measurable, we deduce that

$$\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] \overset{\text{a.s.}}{=\!=} M_n$$

Therefore (M_n) is a martingale with respect to $(\mathscr{F}_n)$. □

Question 6

Let $A \subseteq \mathbb{Z}^2$ be a finite set of points in the square lattice, and let B (the boundary of A) be the set of points in $\mathbb{Z}^2 \setminus A$ with at least one (horizontal or vertical) neighbour in A. Given any function $g : B \to \mathbb{R}$, construct a function $f : (A \cup B) \to \mathbb{R}$ such that $f\big|_B = g$ and, for every $v \in A$

$$f(v) = \frac{1}{4}\sum_{w\sim v} f(w)$$

where the sum is over the 4 neighbours of w; f is called a discrete harmonic function with boundary condition g. Let $(X_n)_{n\geqslant 0}$ be a simple symmetric random walk on $\mathbb{Z}^2$, $X_0 \in A$. Denote $\mathfrak{h}_B = \inf\{n \geqslant 0 : X_n \in B\}$ the first hitting time of the boundary. Show that $f\left(X_{n\wedge\mathfrak{h}_B}\right), n \geqslant 0$ is a martingale relative to the natural filtration of $\mathbf{X}$.

[*Hint. For $v \in A$ consider a random walk starting at v and consider the value of $g\left(X_{\mathfrak{h}_B}\right)$.*]

Proof. For $v, w \in \mathbb{Z}^2$, we write $v \sim w$ if and only if $\|v - w\| = 1$. Let $A = \{v_1, \dots, v_n\}$. Given $g : B \to \mathbb{R}$, the condition that for all $v \in A$,

$$f(v) = \frac{1}{4}\sum_{w\sim v} f(w)$$

gives n linear equations:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where

$$a_{ij} = \frac{1}{4}\mathbf{1}_{\{v_i\sim v_j\}} - \delta_{ij}, \qquad b_i = -\frac{1}{4}\sum_{B\ni w\sim v_i} g(w)$$

We claim that the coefficient matrix has full rank. (*I don't know how to prove this. It seems to require some techniques in Graph Theory.*) Therefore $(f(v_1), \dots, f(v_n))$ has a unique solution. It uniquely determines a function $f : A \cup B \to \mathbb{R}$ such that $f|_B = g$.

Let $(\mathscr{F}_n)$ be the filtration generated by (X_n). It is clear that $f(X_n)$ is $\mathscr{F}_n$-measurable.
Let $Y_n := X_n - X_{n-1} \in \{(1,1), (1,-1), (-1,1), (-1,-1)\} =: \Lambda$. We have

$$\mathbb{E}\big[f(X_{n+1}) \mid \mathscr{F}_n\big] \overset{\text{a.s.}}{=\!=} \sum_{\lambda\in\Lambda} \mathbb{E}\big[f(X_n+\lambda)\mathbf{1}_{\{Y_{n+1}=\lambda\}} \mid \mathscr{F}_n\big] \overset{\text{a.s.}}{=\!=} \sum_{\lambda\in\Lambda} f(X_n+\lambda)\mathbb{P}(Y_{n+1}=\lambda) = \frac{1}{4}\sum_{\lambda\in\Lambda} f(X_n+\lambda) = f(X_n)$$

Hence $(f(X_n))_{n=1}^{\infty}$ is a martingale with respect to $(\mathscr{F}_n)$.

For $n \in \mathbb{N}$,

$$\{\mathfrak{h}_B > n\} = \bigcap_{k=0}^{n} \{X_n \in A\} \in \mathscr{F}_n$$

Hence $\mathfrak{h}_B$ is a stopping time. $(f(X_{n\wedge\mathfrak{h}_B}))$ is a stopped martingale with respect to its natural filtration.

Given $g : B \to \mathbb{R}$, we can define $f : A \cup B \to \mathbb{R}$ as follows: for $v \in A \cup B$, we define $f(v) := \mathbb{E}\big[g(X_{\mathfrak{h}_B})\big]$, where (X_n) is a simple symmetric random walk with $X_0 = v$. Then f is a discrete harmonic function with $f|_B = g$. □

Question 7. Gambler's Ruin

Let $(X_n)_n \geqslant 1$ be an i.i.d. sequence of random variables with

$$\mathbb{P}[X_1 = 1] = p, \quad \mathbb{P}[X_1 = -1] = q, \quad \text{where } 0 < p = 1 - q < 1$$

and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + \ldots + X_n, \quad \tau := \inf\{n : S_n = 0 \text{ or } S_n = b\}$$

Let $\mathscr{F}_n := \sigma(X_1, \ldots, X_n)$ and check that τ is a stopping time. Prove that

$$M_n := \left(\frac{q}{p}\right)^{S_n} \text{ and } N_n := S_n - n(p - q)$$

are martingales w.r.t. $(\mathscr{F}_n)$. Deduce the values of $\mathbb{P}[S_\tau = 0]$ and $\mathbb{E}[\tau]$. (Recall that in Prelims, you may have found these values by solving linear recurrence relations.)

[*Hint: Carefully argue the use of Doob's optional stopping theorem. You could consider $\tau \wedge m$ and take limits, or use* Q9 *on Pb Sheet* 3.]

Proof. First, it is clear that S_n is $\mathscr{F}_n$-measurable. Note that

$$\{\tau > n\} = \bigcap_{k=1}^{n} \{S_k \in (0, b)\} \in \mathscr{F}_n$$

Hence τ is a stopping time.

M_n and N_n are $\mathscr{F}_n$-measurable by definition.

$$\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] = \mathbb{E}\left[M_n \left(\frac{q}{p}\right)^{X_{n+1}} \,\middle|\, \mathscr{F}_n\right] \overset{\text{a.s.}}{=\!=} M_n \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] = M_n \left(p \cdot \frac{q}{p} + q \cdot \frac{p}{q}\right) = M_n$$

$$\mathbb{E}[N_{n+1} \mid \mathscr{F}_n] = \mathbb{E}\big[N_n + X_{n+1} - (p - q) \mid \mathscr{F}_n\big] \overset{\text{a.s.}}{=\!=} N_n + \mathbb{E}\big[X_{n+1} - (p - q)\big] = N_n + (p - q) - (p - q) = N_n$$

Hence (M_n) and (N_n) are martingales with repsect to $(\mathscr{F}_n)$.

Note that

$$\bigcap_{i=1}^{b} \{X_{n+i} = 1\} \subseteq \{S_n < 0\} \cup \{S_n \geqslant 0 \wedge S_{n+b} \geqslant b\} \subseteq \{\tau \leqslant n + b\} \implies \prod_{i=1}^{b} \mathbf{1}_{\{X_{n+i}=1\}} \leqslant \mathbf{1}_{\{\tau \leqslant n+b\}}$$

Hence for all $n \in \mathbb{Z}_+$,

$$\mathbb{P}(\tau \leqslant n + b \mid \mathscr{F}_n) = \mathbb{E}\big[\mathbf{1}_{\{\tau \leqslant n+b\}} \mid \mathscr{F}_n\big] \geqslant \mathbb{E}\left[\prod_{i=1}^{b} \mathbf{1}_{\{X_{n+i}=1\}} \,\middle|\, \mathscr{F}_n\right] = \prod_{i=1}^{b} \mathbb{P}(X_{n+i} = 1) = p^b$$

By Question 9 of Sheet 3, we deduce that $\mathbb{E}[\tau] < \infty$. In particular $\tau < \infty$ almost surely.

Consider the stopped martingale $(M_{\tau \wedge n})$. It is uniformly bounded:

$$|M_{\tau \wedge n}| = \left(\frac{q}{p}\right)^{S_{\tau \wedge n}} \leqslant 1 \vee \left(\frac{q}{p}\right)^b$$

Hence by optional stopping theorem, $\mathbb{E}[M_{\tau \wedge n}] = \mathbb{E}[M_0]$. Taking $n \to \infty$, we obtain $\mathbb{E}\big[M_\tau \mathbf{1}_{\{\tau < \infty\}}\big] = \mathbb{E}[M_0]$.

Let $\alpha = \mathbb{P}(S_\tau = 0)$. Then $1 - \alpha = \mathbb{P}(S_\tau = b)$. We have

$$\left(\frac{q}{p}\right)^a = \left(\frac{q}{p}\right)^{S_0} = M_0 = \mathbb{E}\big[M_\tau \mathbf{1}_{\{\tau < \infty\}}\big] = \alpha + (1 - \alpha)\left(\frac{q}{p}\right)^b$$

The solution is

$$\alpha = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^b}$$

Next, note that

$$|N_{n+1} - N_n| = |X_{n+1} - (p-q)| \leqslant 1 + |p-q|$$

By optional stopping theorem, $\mathbb{E}\left[N_\tau \mathbf{1}_{\{\tau<\infty\}}\right] = \mathbb{E}[N_0]$. Hence

$$\mathbb{E}[S_\tau] - \mathbb{E}[\tau](p-q) = a$$

where

$$\mathbb{E}[S_\tau] = (1-\alpha)b = b\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^b}$$

Therefore

$$\mathbb{E}[\tau] = \frac{\mathbb{E}[S_\tau]-a}{p-q} = \frac{b\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^b} - a}{p-q}$$

□

Question 8

Consider a sequence of i.i.d. tosses of a coin with X_i denoting the outcome of the i^{th} toss. These random variables are defined on some $(\Omega, \mathscr{F})$ on which we have two probability measures $\mathbb{P}_A$ and $\mathbb{P}_B$. Under hypothesis $A, \mathbb{P}_A$ is the true measure, and the probability of a head on any toss if $p = a$. Under hypothesis B, the measure is $\mathbb{P}_B$ and $p = b$, for some $a, b \in (0,1)$

Let $P_A(x_1, \dots, x_n)$ denote the probability of a sequence of outcomes $(x_1, \dots, x_n)$ under the hypothesis A, i.e.

$$P_A(x_1, \dots, x_n) = \mathbb{P}_A(X_1 = x_1, \dots, X_n = x_n)$$

with the analogous definition for P_B. Show that

$$Z_n = \frac{P_A(X_1, X_2, \dots, X_n)}{P_B(X_1, X_2, \dots, X_n)}$$

is a martingale under $\mathbb{P}_B$, relative to the filtration generated by the tosses, $\mathscr{F}_n = \sigma(X_k : k \leqslant n)$ Conclude that $Z_\infty := \lim_{n\to\infty} Z_n$ exists with probability 1. If $b \neq a$, what is the distribution of Z_∞?

Deduce that $1/Z_n$ is a $\mathbb{P}_A$-martingale and comment on its convergence.

This is a special case of the *consistency of the likelihood ratio test* in Statistics.

Proof. It is clear from definition that Z_n is $\mathscr{F}_n$-measurable. By independence, $Z_{n+1} = Z_n \frac{P_A(X_{n+1})}{P_B(X_{n+1})}$. Therefore

$$\mathbb{E}_B[Z_{n+1} \mid \mathscr{F}_n] = \mathbb{E}\left[Z_n \frac{P_A(X_{n+1})}{P_B(X_{n+1})} \;\middle|\; \mathscr{F}_n \right] \overset{\text{a.s.}}{=} Z_n \mathbb{E}_B\left[\frac{P_A(X_{n+1})}{P_B(X_{n+1})}\right] = Z_n\left(b\frac{P_A(H)}{P_B(H)} + (1-b)\frac{P_A(T)}{P_B(T)}\right) = Z_n\left(b\frac{a}{b} + (1-b)\frac{1-a}{1-b}\right) = Z_n$$

Hence (Z_n) is a $\mathbb{P}_B$-martingale with respect to $(\mathscr{F}_n)$. Since (Z_n) is a non-negative martingale, by Corollary 8.30, $Z_\infty := \lim_{n\to\infty} Z_n$ exists almost everywhere and is $\mathbb{P}_B$-integrable. (*I don't know what the distribution of Z_∞ is.*)

By swapping A and B, we see that $1/Z_n$ is a $\mathbb{P}_A$-martingale. It converges almost surely to some $\mathbb{P}_A$-integrable random variable.

□

Question 9

Let $M_n, n \geqslant 0$ be a martingale with $M_0 = 0$. Which of the following are possible?

(a) For some $n, \mathbb{E}[M_n] > 0$.

(b) For some a.s. finite stopping time $\tau, \mathbb{E}[M_\tau] > 0$.

(c) $M_n \to \infty$ as $n \to \infty$ with probability 1.

What about if in addition M is bounded in $\mathscr{L}^1$? What about if in addition M is uniformly integrable?

Proof. (a) This is never possible. By the property of martingale, $\mathbb{E}[M_n] = \mathbb{E}[M_0] = M_0 = 0$.

(b) This is possible. Example 8.18 is an example.

If (M_n) is uniformly bounded in L^1, I suspect that this is still possible. but I cannot give an example. (The martingale in Example 8.18 is not uniformly bounded in L^1.)

If (M_n) is uniformly integrable, then by optional stopping theorem, $\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0$.

(c) This is possible. Consider independent random variables $X_1, X_2, \ldots$ such that

$$\mathbb{P}(X_n = 1) = 1 - 2^{-n}, \quad \mathbb{P}\big(X_n = 1 - 2^n\big) = 2^{-n}$$

Let $Y_n := \sum_{k=1}^n X_k$ and $(\mathscr{F}_n)$ be the natural filtration generated by (Y_n). We have

$$\mathbb{E}[Y_{n+1} \mid \mathscr{F}_n] \overset{\text{a.s.}}{=} Y_n + \mathbb{E}[X_{n+1}] = Y_n$$

Hence (Y_n) is a martingale with respect to its natural filtration.

Since

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n < 0) = \sum_{n=1}^{\infty} 2^{-n} < \infty$$

by the first Borel-Cantelli Lemma, we have

$$\mathbb{P}\big(X_n < 0 \text{ for infinitely many } n\big) = 0$$

Hence

$$\mathbb{P}\Big(\lim_{n\to\infty} Y_n = +\infty\Big) \geqslant \mathbb{P}\big(X_n > 0 \text{ for all but finitely many } n\big) = 1$$

If (M_n) is uniformly bounded in L^1, then by forward convergence theorem (Theorem 8.29), $M_n \to M_\infty$ as $n \to \infty$ with probability 1, where M_∞ is integrable and hence is finite almost everywhere.

If (M_n) is uniformly integrable, then in particular it is uniformly bounded in L^1. The previous result also holds. □

Question 10. Polya's urn model

At time 0 we have an urn with two balls, one white and one black. At each successive time, we draw at random one ball from the urn and return it back along with another ball of the same colour. This way, at time n, we have $n+2$ balls in the urn of which B_n are black and $W_n = n+2-B_n$ are white. Note that $B_n \in \{1, \ldots, n+1\}$.

Using induction, or otherwise, show that $\mathbb{P}(B_n = k) = \frac{1}{n+1}, k = 1, \ldots, n+1$.

Show that $M_n = \frac{B_n}{n+2}$ is a martingale relative to the filtration you should specify. Conclude that M_n converges to some M_∞ and specify in what sense. What is the distribution of M_∞?

Proof. We use induction on n. Base case: When $n = 1$, $\mathbb{P}(B_1 = 1) = \mathbb{P}(B_1 = 2) = 1/2$.

Induction case: For $k > 1$,

$$\begin{aligned}\mathbb{P}(B_{n+1} = k) &= \mathbb{P}(B_{n+1} = k \wedge B_n = k-1) + \mathbb{P}(B_{n+1} = k \wedge B_n = k) \\ &= \mathbb{P}(B_{n+1} = k \mid B_n = k-1)\mathbb{P}(B_n = k-1) + \mathbb{P}(B_{n+1} = k \mid B_n = k)\mathbb{P}(B_n = k)\end{aligned}$$

$$
\begin{aligned}
&= \frac{k-1}{n+2}\frac{1}{n+1} + \frac{n+2-k}{n+2}\frac{1}{n+1} \\
&= \frac{1}{n+2}
\end{aligned}
$$

For $k = 1$, $\mathbb{P}(B_{n+1} = 1) = \mathbb{P}(B_{n+1} = 1 \mid B_n = 1)\mathbb{P}(B_n = 1) = \frac{n+1}{n+2}\frac{1}{n+1} = \frac{1}{n+2}$. This completes the induction.

Let X_n represents the colour of the n-th draw, such that $X_n = 1$ if the n-th draw is black and $X_n = 0$ if the n-th draw is white. Let $(\mathscr{F}_n)$ be the filtration generated by (X_n). Then $M_n = \frac{B_n}{n+2} = \frac{1+\sum_{k=1}^{n} X_k}{n+2}$ is $\mathscr{F}_n$-measurable.

$$
\begin{aligned}
\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] &\overset{\text{a.s.}}{=\!=} \mathbb{E}\left[\frac{B_n+1}{n+3}\mathbf{1}_{\{X_{n+1}=1\}} \,\middle|\, \mathscr{F}_n \right] + \mathbb{E}\left[\frac{B_n}{n+3}\mathbf{1}_{\{X_{n+1}=0\}} \,\middle|\, \mathscr{F}_n \right] \\
&\overset{\text{a.s.}}{=\!=} \frac{B_n+1}{n+3}\mathbb{P}(X_{n+1} = 1 \mid \mathscr{F}_n) + \frac{B_n}{n+3}\mathbb{P}(X_{n+1} = 0 \mid \mathscr{F}_n) \\
&\overset{\text{a.s.}}{=\!=} \frac{B_n+1}{n+3}\frac{B_n}{n+2} + \frac{B_n}{n+3}\frac{n+2-B_n}{n+2} \\
&= \frac{B_n}{n+2} = M_n
\end{aligned}
$$

Hence (M_n) is a martingale with respect to $(\mathscr{F}_n)$. Since (M_n) is non-negative, by Corollary 8.30 it converges almost surely to M_∞, which is integrable.

Note that M_n is in fact the uniform distribution in $\left\{\frac{1}{n+2}, \ldots, \frac{n+1}{n+2}\right\}$. Therefore M_∞ is the uniform distribution in $[0,1]$. □