Total 9

Great Work !!

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Problem Sheet 2

B3.2: Geometry of Surfaces

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Question 1 a

Let $f : X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 1.

(i) Show that f has no ramification points.

(ii) Show that f is a homeomorphism.

(iii) Show that f^{-1} is holomorphic.

Proof. (i) Recall that in lecture we are shown that

$$\deg f = \sum_{x \in f^{-1}(\{y\})} \nu_f(x)$$

for any $y \in Y$, where $\nu_f(x)$ is the ramification index of $x \in X$. $x \in X$ is a ramification point of and only if $\nu_f(x) > 1$. Therefore $\deg f = 1$ implies that f has no ramification points. You should spell out the obvious here.

(ii) $\deg f = 1$ also implies that $f^{-1}(y)$ is a singleton for any $y \in Y$. Hence f is a bijection. For $x_0 \in X$, let $y_0 \in Y$. Let (U, φ) and (V, ψ) be local charts such that $x_0 \in U$ and $y_0 \in V$. Then $\tilde{f} := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is a injective holomorphic function onto its image. By inverse function theorem in complex analysis, its inverse function $\tilde{f}^{-1} = \varphi \circ f^{-1} \circ \psi^{-1}$ is also holomorphic. Hence $f^{-1} = \varphi^{-1} \circ \tilde{f}^{-1} \circ \psi$ is holomorphic near $y_0 \in Y$. Then $f^{-1} : Y \to X$ is a holomorphic map. In particular, $f : X \to Y$ is a biholomorphism. So it is a homeomorphism.

(iii) This is shown in (ii). □

Question 2 a

Let $f : X \to Y$ be a nonconstant holomorphic map of compact connected Riemann surfaces, where X is the Riemann sphere. Use the general form of the Riemann-Hurwitz formula to deduce that Y is homeomorphic to X.

Proof. Riemann-Hurwitz formula:

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{x \in X} (\nu_f(x) - 1)$$

Since Y is a compact connected Riemann surface, it is orientable. So by classification theorem of compact surfaces, Y is homeomorphic to some connected sum of tori. $\chi(Y) = 2 - 2n$ for some $n \in \mathbb{N}$.

Suppose that $n \geq 1$. Then

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{x \in X} (\nu_f(x) - 1) \leq \deg f \cdot \chi(Y) \leq 0$$

But we know that X is the Riemann sphere, so $\chi(X) = 2$. This is a contradiction. Hence $n = 0$ and $Y \cong S^2$. We deduce that Y is homeomorphic to X. □

Great!!

Question 3

The *Korteweg-de Vries equation* which describes shallow water waves is

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x} = 0$$

(i) A solution with a fixed wave form is given by $\phi(x, t) = f(x - ct)$. Show that f satisfies the equation

$$-cf' + f''' + 6ff' = 0$$

(ii) Using the relation $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ find constants a, b such that $f = a\wp + b$ satisfies this equation where $\wp$ is the Weierstrass $\wp$-function.

Can you describe the sort of wave this corresponds to?

Proof. (i) Let $f(x - ct) = \phi(x, t)$. Then $\partial\phi/\partial x = f'(x - ct)$, $\partial^3\phi/\partial x^3 = f'''(x - ct)$ and $\partial\phi/\partial t = -cf'(x - ct)$. Hence the Korteweg-de Vries equation is transformed into

$$-cf' + f''' + 6ff' = 0$$

(ii) Note that $6ff' = (3f^2)'$. We integrate the equation and obtain

$$3f^2 - cf + f'' = \text{const}$$

Substituting $f = a\wp + b$, we have

$$3a\wp^2 + (6b - c)\wp + \wp'' = \text{const}$$

We differentiate with respect to the formula $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ to obtain

$$\wp'' = 2\big((\wp - e_1)(\wp - e_2) + (\wp - e_2)(\wp - e_3) + (\wp - e_3)(\wp - e_1)\big)$$

Hence

$$3a\wp^2 + (6b - c)\wp + 2\big((\wp - e_1)(\wp - e_2) + (\wp - e_2)(\wp - e_3) + (\wp - e_3)(\wp - e_1)\big) = \text{const}$$

$$(3a + 6)\wp^2 + (6b - c - 4(e_1 + e_2 + e_3))\wp = \text{const}$$

The integrating constant can be chosen such that the right hand side of the equation vanishes.

Furthermore we shall prove that $e_1 + e_2 + e_3 = 0$ with some complex analysis techniques.

Starting from

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

Note that

$$\frac{1}{(1-w)^2} = \sum_{\ell=0}^{\infty} (\ell+1) w^\ell, \qquad \text{for } |w| < 1$$

which is obtained by differentiating the geometric series. Hence for $|z| < |\omega|$:

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2} \sum_{\ell=0}^{\infty} (\ell+1)\left(\frac{z}{\omega}\right)^\ell$$

Therefore we obtain the Laurent expansion of $\wp$ near $z = 0$:

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{\omega^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^\ell - \frac{1}{\omega^2}\right) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \sum_{\ell=1}^{\infty} (\ell+1)\frac{z^\ell}{\omega^{\ell+2}} \\ &= \frac{1}{z^2} + \sum_{\ell=1}^{\infty} (\ell+1) z^\ell \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(\ell+2)}\end{aligned}$$

By symmetry, replacing (m, n) with $(-m, -n)$, we see that the following series is zero for odd ℓ.

$$\sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(\ell+2)} = \sum_{(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-(\ell+2)}$$

Hence

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) z^{2k} \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(2k+2)} = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_k z^{2k}$$

From the Laurent expansion we find that

$$\wp = z^{-2} + c_1 z^2 + c_2 z^4 + O\left(z^6\right)$$

$$\wp^3 = z^{-6} + 3c_1 z^{-2} + 3c_2 + O\left(z^2\right)$$

$$(\wp')^2 = 4z^{-6} - 8c_1 z^{-2} - 16c_2 + O\left(z^4\right)$$

Hence $g = (\wp')^2 - 4\wp^3 + 20c_1\wp + 28c_2 = O\left(z^2\right)$ near $z = 0$. So g can be extended to a holomorphic function near $z = 0$. Since g is doubly periodic, the image $\operatorname{im} g$ is compact in $\mathbb{C}$. Hence by Liouville's Theorem g is constant. $g(z) = 0$ implies that

$$(\wp')^2 = 4\wp^3 - 20c_1\wp - 28c_2$$

But we also know that

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

By Vieta's Theorem we deduce that $e_1 + e_2 + e_3 = 0$.

✓

Great Work!!

Hence

$$(3a+6)\wp^2 + (6b-c)\wp = \text{const}$$

We deduce that when $a=-2$ and $b=c/6$, $f=a\wp+b$ satisfies the Korteweg-de Vries equation. $\square$

a

Question 4

Let $f: X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 2. Show that there is a non-trivial holomorphic homeomorphism $\sigma: X \to X$ such that $f \circ \sigma = f$ and σ^2 is the identity map. How many fixed points does your map have?

Proof. Since $\deg f = 2$, $f^{-1}(\{y\})$ is a singleton if y is a branch point or a doubleton if y is not a branch point. We define σ as follows: For $x \in X$, if x is a ramification point, then define $\sigma(x) = x$; if x is not a ramification point, then there exists a unique $x' \in X \backslash \{x\}$ such that $f(x) = f(x')$ and we define $\sigma(x) = x'$. Great!!!

From the definition it is clear that $f \circ \sigma = f$ and $\sigma^2 = \text{id}$. The latter implies that σ is self-inverse. It remains to show that σ is holomorphic. Yes!

If we remove the ramification points and branch points, f is a covering map from X to Y of degree 2, and σ is a covering transformation. At a non-ramification point x, there exists a open neighbourhood $U \subseteq X$ of x on which f is injective. $f|_U : U \to f(U)$ and $f|_{\sigma(U)} : \sigma(U) \to f(U)$ are biholomorphisms. So σ maps neighbourhood U of x biholomorphically to neighbourhood $\sigma(U)$ of $\sigma(x)$. σ is holomorphic at x. ✓

At a ramification point $x \in X$, there exists local charts (U, φ) and (V, ψ), where $x \in U$ and $f(x) \in V$, such that $\tilde{f} = \psi \circ f \circ \varphi^{-1}$ is given by $\tilde{f}(z) = z^2$. Then $\tilde{\sigma} = \varphi \circ \sigma \circ \varphi^{-1}$ satisfies $\tilde{f} \circ \tilde{\sigma} = \tilde{f}$. Since $\tilde{\sigma} \neq \text{id}$, we have $\tilde{\sigma}(z) = -z$. In particular $\tilde{\sigma}$ is holomorphic at $z = 0$. Hence σ is holomorphic at x. ✓

We conclude that σ is a biholomorphism. The fixed points of σ are exactly the ramification points of X. ✓ $\square$

Can you compute how many in terms of topological data?

Question 5. The classification of elliptic curves. c

There are bijections

$$\frac{\{\text{Riemann surfaces homeomorphic to a torus}\}}{\text{biholomorphisms}} \longleftrightarrow \frac{\{\text{Quotients } \mathbb{C}/\Lambda \text{ for } \Lambda \text{ a lattice in } \mathbb{C}\}}{\text{biholomorphisms}}$$

$$\longleftrightarrow \frac{\{\text{Quotients } \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \text{ with } \tau \in \mathbb{H}\}}{\text{biholomorphisms}} \longleftrightarrow \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$$

Here $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ is the upper half-plane in $\mathbb{C}$. The second map comes by writing $\Lambda = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$, choosing τ to be whichever of ω_2/ω_1 or $-\omega_2/\omega_1$ lies in $\mathbb{H}$ and noting that $\Lambda = \omega_1 \cdot (\mathbb{Z}+\mathbb{Z}\tau)$ so $\mathbb{C}/\Lambda \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$

The third map is $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \leftrightarrow [\tau]$, and $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ by Möbius transformations, that is , $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{Z})$ acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \dfrac{az+b}{cz+d}$.

Although we will not need the following fact, some easy group theory shows that $\mathrm{SL}(2,\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The corresponding Möbius maps $S(z) = -1/z$ and $T(z) = z+1$ are rather useful in this exercise.

(a) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{Z})$, show that $\operatorname{Im}(Az) = \dfrac{1}{|cz+d|^2} \cdot \operatorname{Im}(z)$. Deduce that, given a constant K, only finitely many $c, d \in \mathbb{Z}$ satisfy $\operatorname{Im}(Az) > K$.

(b) Show that $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ is a topological space homeomorphic to $\mathbb{C}$, by first showing each point of $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ has a representative inside the "strip"

$$\{\tau \in \mathbb{H} : |\operatorname{Re}(\tau)| \le 1/2, |\tau| \ge 1\}$$

and then checking that the only remaining identifications are on the boundary of the strip.

Hint. Try to maximize the imaginary part for the orbit of z under the action.

(c) Show that $\mathrm{PSL}(2,\mathbb{Z})$ acts freely on $\mathbb{H}$ except at the points in the $\mathrm{PSL}(2,\mathbb{Z})$-orbits of $e^{\pi i/3}$ and of i, and show that the stabilisers of those points are respectively $\mathbb{Z}/3$ and $\mathbb{Z}/2$.

(d) Briefly comment on why the natural local complex coordinate from $\mathbb{H}$ makes $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ into a Riemann surface except at $e^{\pi i/3}$ and i.

Proof. (a) By direct calculation,

$$\operatorname{Im}(Az) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{1}{|cz+d|^2}\operatorname{Im}(ac|z|^2+bd+adz+bc\bar{z}) = \frac{ad-bc}{|cz+d|^2}\operatorname{Im} z = \frac{1}{|cz+d|^2}\operatorname{Im} z$$

Then

$$\operatorname{Im}(Az) > K \iff \frac{1}{|cz+d|^2}\operatorname{Im} z > K \iff |cz+d| < \sqrt{\frac{\operatorname{Im} z}{K}}$$

For $z \in \mathbb{H}$, 1 and z are linearly independent over $\mathbb{R}$. So $\Gamma = \{cz+d \in \mathbb{C} : c,d \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}$. So there are finitely many $(c,d) \in \mathbb{Z}^2$ such that $|cz+d| < \sqrt{\frac{\operatorname{Im} z}{K}}$ for any fixed constant $K \in \mathbb{R}_+$. Hence there are finitely many $(c,d) \in \mathbb{Z}^2$ such that $\operatorname{Im}(Az) > K$.

(b) By (a), the set $\{(c,d) \in \mathbb{Z}^2 : \operatorname{Im}(Az) \geqslant \operatorname{Im} z\}$ is finite. Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be the map such that $\operatorname{Im}(Az)$ is maximal. Let n be the closet integer to $\operatorname{Re}(Az)$. Then we find that $w := Az - n$ is in the region $\{\tau \in \mathbb{H} : |\operatorname{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$. Because $|\operatorname{Re} w| = |\operatorname{Re}(Az) - n| \leqslant 1/2$ and

$$|\operatorname{Im} w| = |\operatorname{Im}(Az)| \leqslant \left|\operatorname{Im}\left(\frac{1}{w}\right)\right| = \frac{|\operatorname{Im} w|}{|w|^2} \implies |w| \geqslant 1$$

Which PSL elements do the transformations you used correspond to ?

For $z, w \in \{\tau \in \mathbb{H} : |\operatorname{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$, let $A \in \mathrm{PSL}(2,\mathbb{Z})$ such that $Az = w$. Without loss of generality we assume that $\operatorname{Im}(w) > \operatorname{Im}(z)$. Then

$$\operatorname{Im} w = \operatorname{Im}(Az) = \frac{\operatorname{Im} z}{|cz+d|^2} \geqslant \operatorname{Im}(z) \implies |cz+d| \leqslant 1$$

Note that

$$|c \operatorname{Im} z| = |\operatorname{Im}(cz+d)| \leqslant |cz+d| \leqslant 1$$

Hence $|c| = 1/|\operatorname{Im} z|$. But $\operatorname{Im} z \geqslant \sqrt{3}/2$. Hence $|c| < 2/\sqrt{3}$. As $c \in \mathbb{Z}$, the only possibilities are $c \in \{0, \pm 1\}$. In addition, since $|\operatorname{Re} z| \leqslant 1/2$, we have

$$|d| - \frac{1}{2} \leqslant |d + c\operatorname{Re} z| \leqslant |cz+d| \leqslant 1$$

Hence $d \in \{0, \pm 1\}$.

- $c = 0$:

  Since $ad - bc = 1$, $c = 0$ implies that $ad = 1$. Hence $A = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. $w = z + b$. Since $\operatorname{Re} z, \operatorname{Re} w \in [-1/2, 1/2]$, $b \in \{0, \pm 1\}$. If $b = 0$, then $z = w$. If $b = \pm 1$, then z and w are on the boundary lines $\operatorname{Re} z = \pm 1/2$ respectively. Hence we should identify $\operatorname{Re} z = 1/2$ with $\operatorname{Re} z = -1/2$.

- $c = \pm 1$ and $d = 0$:

  $d = 0$ implies that $bc = -1$. Hence $A = \pm\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$. $w = a - \frac{1}{z}$. In this case $|z| = |cz| \leqslant |cz+d| \leqslant 1$. But we also have $|z| \geqslant 1$. Hence $|z| = 1$. Since $\operatorname{Re} z, \operatorname{Re} w \in [-1/2, 1/2]$, $a \in \{0, \pm 1\}$. If $a = 0$, then $w = -1/z$. In particular $|w| = 1$ and $\operatorname{Re} w = -\operatorname{Re} z$. Hence we should identify the arc $\{z \in \mathbb{C} : \operatorname{Re} z \in [-1/2, 0], |z| = 1\}$ with the arc $\{z \in \mathbb{C} : \operatorname{Re} z \in [0, 1/2], |z| = 1\}$ by symmetry. If $a = \pm 1$, then z and w are the vertices $e^{\pi i/3}$ and $e^{2\pi i/3}$.

- $c = d = \pm 1$: $|cz+d| \leqslant 1$ implies that $|z+1| = 1$. The only point is $z = e^{2\pi i/3}$.

- $c = -d = \pm 1$: $|cz+d| \leqslant 1$ implies that $|z-1| = 1$. The only point is $z = e^{\pi i/3}$.

(c) It suffices to consider the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\{\tau \in \mathbb{H} : |\operatorname{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$ (with left half of the boundary removed). Let $A \in \mathrm{PSL}(2,\mathbb{Z})$ such that $Az = z$. From the analysis in (b) we can quickly identify the possible cases:

- $c = 0$ and $b = 0$: A is the identity map.

- $c = \pm 1$, $d = 0$ and $a = 0$: $z = Az = -1/z$ implies that $z = i$.

- $c = d = \pm 1$: $z = e^{2\pi i/3}$ (this point is in the orbit of $e^{\pi i/3}$)
- $c = -d = \pm 1$: $z = e^{\pi i/3}$.

In particular, the stabilizers of the orbits:

- $z = i$: $\{\mathrm{id}, S\} \cong \mathbb{Z}/2\mathbb{Z}$;
- $z = e^{\pi i/3}$: $\{\mathrm{id}, T \circ S, T \circ S \circ T \circ S\} \cong \mathbb{Z}/3\mathbb{Z}$
- all other orbits: $\{\mathrm{id}\}$.

(d) Except at the orbits of i and $e^{\pi i/3}$, $\mathrm{PSL}(2,\mathbb{Z})$ acts freely and properly discontinuously on $\mathbb{H}$. The topological space $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ is a Riemann surface by the following construction[1]:

- It is clear that $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ is Hausdorff and second countable.
- We can choose an altas $\{(U_i, \varphi_i) : i \in I\}$ on $\mathbb{H}$ such that $U_i \cap g(U_i) = \varnothing$ for all $i \in I$ and $g \in \mathrm{PSL}(2,\mathbb{Z}) \setminus \{\mathrm{id}\}$. This is possible as we have shown that the action is properly discontinuous. Let $\pi : \mathbb{H} \to \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ be the projection map. Then $V_i = \pi(U_i)$ is open in $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ and $\{V_i : i \in I\}$ covers $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$.
- Since $\pi_i := \pi|_{U_i} : U_i \to V_i$ is a homeomorphism, define $\psi_i := \varphi \circ \pi_i^{-1} : V_i \to \varphi(U_i) \subseteq \mathbb{C}$, which is also a homeomorphism.
- We shall show that the altas $\{(V_i, \psi_i) : i \in I\}$ is compatible. For $V_i \cap V_j \neq \varnothing$,

$$\psi_i(V_i \cap V_j) = \varphi_i \circ \pi_i^{-1}(V_i \cap V_j) = \varphi_i(U_i \cap \pi^{-1}(V_j)) = \varphi_i\left(U_i \cap \bigcup_{g \in \mathrm{PSL}(2,\mathbb{Z})} g(U_j)\right) = \bigcup_{g \in \mathrm{PSL}(2,\mathbb{Z})} \varphi_i(U_i \cap g(U_j))$$

which is a disjoint union of open subsets. For $p \in \psi_i(V_i \cap V_j)$, there exists a unique $g \in \mathrm{PSL}(2,\mathbb{Z})$ such that $p \in W := \varphi_i(U_i \cap g(U_j))$. Then we need to show that $\psi_j \circ \psi_i^{-1}|_W$ is holomorphic. Since

$$\psi_j \circ \psi_i^{-1}\big|_W = \varphi_j \circ \pi_j^{-1} \circ \pi_i \circ \varphi_i^{-1}\Big|_W$$

it suffices to show that $\pi_j^{-1} \circ \pi_i$ is holomorphic on $U_i \cap g(U_j)$. For $q \in U_i \cap g(U_j)$, $q' := \pi_j^{-1} \circ \pi_i(q) \in U_j$. Then $\pi_j(q') = \pi_i(q)$. There exists $g_q \in \mathrm{PSL}(2,\mathbb{Z})$ such that $g_q(q') = q$. Hence $q \in g_q(U_j) \cap g(U_j)$. But it implies that $g_q = g$. Hence $\pi_j^{-1} \circ \pi_i = g^{-1}$ on $U_i \cap g(U_j)$. We deduce that $\pi_j^{-1} \circ \pi_i$ is holomorphic on $U_i \cap g(U_j)$. Hence $\psi_j \circ \psi_i^{-1}$ is holomorphic. The same argument shows that the inverse is also holomorphic. We therefore conclude that the altas is compatible. □

Great Work!

[1] This proof is adapted from Theorem 1.6 of C3.3 Differentiable Manifolds.