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Problem Sheet 2
B1.1: Logic

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Question 1 α

Prove that for any $\phi, \phi_i, \psi, \chi \in \mathrm{Form}(\mathcal{L})$ and any $\Gamma \subseteq \mathrm{Form}(\mathcal{L})$

(a) $((\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi)))$ is a tautology.

(b) $\Gamma \cup \{\psi\} \vDash \phi$ if and only if $\Gamma \vDash (\psi \to \phi)$.

(c) $\neg \bigwedge_{i=1}^{n} \phi_i$ is logically equivalent to $\bigvee_{i=1}^{n} \neg \phi_i$.

(d) $(\phi \vee \psi)$ is logically equivalent to $((\phi \to \psi) \to \psi)$.

Proof. (a) We argue by contradiction. Note that by truth table, $v(\phi \to \psi) = F$ if and only if $v(\phi) = T$ and $v(\psi) = F$. Suppose that the formula is not a tautology. Then there exists $v : \{\phi, \psi, \chi\} \to \{T, F\}$ such that

$$v((\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi))) = F.$$

You should add tilde to indicate that this is not the original valuation function. Usually we treat valuation of propositional variables and valuation of formulas as two different functions

Then

$$v(\phi \to \psi) = T \text{ and } v((\phi \to (\psi \to \chi)) \to (\phi \to \chi)) = F.$$

The latter implies that $v(\phi \to (\psi \to \chi)) = T$ and $v(\phi \to \chi) = F$. $v(\phi \to \chi) = F$ implies that $v(\phi) = T$ and $v(\chi) = F$. Since $v(\phi) = T$ and $v(\phi \to \psi) = T$, we deduce that $v(\psi) = T$. But this implies that $v(\phi \to (\psi \to \chi)) = F$, which is contradictory.

Hence $((\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi)))$ is a tautology. ✓

(b) "$\Longrightarrow$": Suppose that $\Gamma \cup \{\psi\} \vDash \phi$. For any valuation v such that $\tilde{v}(\gamma) = T$ for all $\gamma \in \Gamma$: if $\tilde{v}(\psi) = T$, then by assumption we have $\tilde{v}(\phi) = T$, which implies that $\tilde{v}(\psi \to \phi) = T$; if $\tilde{v}(\psi) = F$, then by truth table of $\to$ we know that $\tilde{v}(\psi \to \phi) = T$. In conclusion, we have $\Gamma \vDash (\psi \to \phi)$. ✓

"$\Longleftarrow$": Suppose that $\Gamma \vDash (\psi \to \phi)$. For any valuation v such that $\tilde{v}(\gamma) = T$ for all $\gamma \in \Gamma$ and $\tilde{v}(\psi) = T$, we have $\tilde{v}(\psi \to \phi) = T$ by assumption. From $\tilde{v}(\psi) = T$ we deduce that $\tilde{v}(\phi) = T$. Hence $\Gamma \cup \{\psi\} \vDash \phi$. ✓

(c) We use induction on n. Base case: When $n = 1$, it is trivial that $\phi_1 \vDash\Dashv \phi_1$.

You mean ¬ϕ1 ≡ ¬ϕ1 rather?

Induction case: Suppose that it is true for $n-1$. Then $\psi := \neg \bigwedge_{i=1}^{n-1} \phi_i$ is logically equivalent to $\bigvee_{i=1}^{n-1} \neg \phi_i$. By checking the true table we can prove that $\neg(\neg\psi \wedge \phi_n) \vDash\Dashv (\psi \vee \neg\phi_n)$. Hence

$$\neg \bigwedge_{i=1}^{n} \phi_i \vDash\Dashv \neg(\neg\psi \wedge \phi_n) \vDash\Dashv (\psi \vee \neg\phi_n) \vDash\Dashv \bigvee_{i=1}^{n-1} \neg\phi_i \vee \neg\phi_n \vDash\Dashv \bigvee_{i=1}^{n} \phi_i$$ ✓

(d) We list the truth table as follows:

ϕ	ψ	$(\phi \vee \psi)$	$(\phi \to \psi)$	$((\phi \to \psi) \to \psi)$
T	T	T	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

Hence $(\phi \vee \psi) \vDash\Dashv ((\phi \to \psi) \to \psi)$. ✓ □

Question 2 α

Let ϕ be the formula $((p_0 \to p_1) \wedge (p_1 \to p_2))$. Find a formula in DNF logically equivalent to ϕ which is a disjunct of just three conjunctive clauses.

Solution. Assuming that $\tilde{v}((p_0 \to p_1) \wedge (p_1 \to p_2)) = T$. We must have $\tilde{v}(p_0 \to p_1) = T$ and $\tilde{v}(p_1 \to p_2) = T$. We can list all the possibilities as follows:

If $v(p_0) = T$, then $v(p_1) = T$. Then $v(p_1) = T$.

If $v(p_0) = F$:

If $v(p_1) = T$, then $v(p_2) = T$.

If $v(p_1) = F$, then $v(p_2)$ could be T or F.

Hence $((p_0 \to p_1) \wedge (p_1 \to p_2))$ is logically equivalent to the disjunctive normal form:

$$(p_0 \wedge p_1 \wedge p_2) \vee (\neg p_0 \wedge p_1 \wedge p_2) \vee (\neg p_0 \wedge \neg p_1) \quad \checkmark \qquad \square$$

Question 3 α-

(i) Prove that every formula is logically equivalent to one in conjunctive normal form.

(ii) Let v_0 be the valuation that assigns the value T to every propositional variable. Prove that a formula ϕ is logically equivalent to one built up from propositional variables using just the connectives $\wedge$ and $\to$ if and only if $\tilde{v}_0(\phi) = T$.

Proof. (i) It has been proven in the lectures that every formula is logically equivalent to one in disjunctive normal form. For $\phi \in \text{Form}(\mathcal{L})$, we have

$$\neg\phi \vDash\Dashv \bigvee_{k=1}^{n} \bigwedge_{i_k=1}^{j_k} \chi_{i_k}$$

Shouldn't you give χ two subscripts (both k and i)? In your way, χ1 may need to be both p and ¬p for different conjunctions

where $\chi_{i_k} = p_m$ or $\neg p_m$ for some $m \in \mathbb{N}$. By de Morgan's Law,

$$\phi \vDash\Dashv \neg\neg\phi \vDash\Dashv \neg \bigvee_{k=1}^{n} \bigwedge_{i_k=1}^{j_k} \chi_{i_k} \vDash\Dashv \bigwedge_{k=1}^{n} \neg \bigwedge_{i_k=1}^{j_k} \chi_{i_k} \vDash\Dashv \bigwedge_{k=1}^{n} \bigvee_{i_k=1}^{j_k} \neg\chi_{i_k}$$

You should mention that by double negation elimination, when χ is ¬p, ¬χ i.e. ¬¬p is equivalent to p

Hence every formula is logically equivalent to one in conjunctive normal form. ✓

(ii) "$\Longrightarrow$": We use induction on the length of the formula.

Base case: If ϕ has length 1, then $\phi = p_i$ for some $i \in \mathbb{N}$. It is clear that if $v_0(p_i) = T$ then $\tilde{v}_0(\phi) = T$.

Induction case: Suppose that the result holds for all formulae of length less than n. Let ϕ be a formula of length n. Then $\phi = (\psi_1 \wedge \psi_2)$ or $(\psi_1 \to \psi_2)$ for some formulae ψ_1, ψ_2. By induction hypothesis, $\tilde{v}_0(\psi_1) = \tilde{v}_0(\psi_2) = T$. By truth table of $\wedge$ and $\to$, $\tilde{v}_0(\psi_1 \wedge \psi_2) = \tilde{v}_0(\psi_1 \to \psi_2) = T$. Hence $\tilde{v}_0(\phi) = T$. ✓

"$\Longleftarrow$": For a formula $\phi \in \text{Form}(\mathcal{L})$, we can put it into a conjunctive normal form:

$$\phi \vDash\Dashv \bigwedge_{k=1}^{n} \bigvee_{i_k=1}^{j_k} \chi_{i_k}$$

where $\chi_{i_k} = p_m$ or $\neg p_m$ for some $m \in \mathbb{N}$.

If $v_0(\phi) = T$, then (by induction on n) $v_0\left(\bigvee_{i_k=1}^{j_k} \chi_{i_k}\right) = T$ for all $k \in \{1, \dots, n\}$. For each k, by induction on j_k, we have $v_0(\chi_{i_k}) = T$ for some $i_k \in \{1, \dots, j_k\}$. In other words, $\chi_{i_k} = p_m$ for some i_k and some m. Then we have

$$\bigvee_{i_k=1}^{j_k} \chi_{i_k} \vDash\Dashv \bigvee_{i_k=1}^{\ell_k} p_{m_{i_k}} \vee \bigvee_{i_k=\ell_k+1}^{j_k} \neg p_{m_{i_k}}$$

where $\ell_k \in \{1, \dots, j_k\}$.

If $\ell_k = j_k$, then by Question 1.(d) and induction, we have

$$\bigvee_{i_k=1}^{j_k} \chi_{i_k} \vDash\Dashv \bigvee_{i_k=1}^{\ell_k} p_{m_{i_k}} \vDash\Dashv (((\cdots((p_{m_1} \to p_{m_2}) \to p_{m_2})\cdots) \to p_{m_{j_k}}) \to p_{m_{j_k}})$$

If $\ell_k \neq j_k$, then by Question 1.(c), we have

$$\bigvee_{i_k=\ell_k+1}^{j_k} \neg p_{m_{i_k}} \vDash\Dashv \neg \bigwedge_{i_k=\ell_k+1}^{j_k} p_{m_{i_k}}$$

Therefore

$$\bigvee_{i_k=1}^{\ell_k} \chi_{i_k} \vDash\Dashv \neg \bigwedge_{i_k=\ell_k+1}^{j_k} p_{m_{i_k}} \vee (((\cdots((p_{m_1} \to p_{m_2}) \to p_{m_2})\cdots) \to p_{m_{\ell_k}}) \to p_{m_{\ell_k}})$$

$$\vDash\Dashv \left(\bigwedge_{i_k=\ell_k+1}^{j_k} p_{m_{i_k}} \to (((\cdots((p_{m_1} \to p_{m_2}) \to p_{m_2})\cdots) \to p_{m_{\ell_k}}) \to p_{m_{\ell_k}}) \right)$$

Now we can conclude that ϕ is logically equivalent to a formula which has binary connectives $\wedge$ and $\to$ only. □ ✓

Question 4 α-

(i) Find the truth tables for all binary connectives $\star$ having the property that $\{\star\}$ is adequate. Justify your answer.

(ii) Show that there is no adequate unary connective.

Proof. (i) We claim that the only adequate binary connectives are nor and nand, whose truth table are given by:

ϕ	ψ	$\phi \downarrow \psi$
T	T	F
T	F	F
F	T	F
F	F	T

ϕ	ψ	$\phi \uparrow \psi$
T	T	F
T	F	T
F	T	T
F	F	T

For $\phi, \psi \in \text{Form}(\mathcal{L})$, we have $\neg\phi \vDash\Dashv \phi \downarrow \phi$ and $\phi \vee \psi \vDash\Dashv \neg(\phi \downarrow \psi)$. Since we know that $\{\neg, \vee\}$ is adequate, $\{\downarrow\}$ is also adequate.

Similarly, for $\phi, \psi \in \text{Form}(\mathcal{L})$, we have $\neg\phi \vDash\Dashv \phi \uparrow \phi$ and $\phi \wedge \psi \vDash\Dashv \neg(\phi \uparrow \psi)$. Since we know that $\{\neg, \wedge\}$ is adequate, $\{\uparrow\}$ is also adequate. ✓

Next, we shall show that there are no other adequate binary operators. To begin with, we list all other 14 binary connectives as follows:

ϕ	ψ	$\perp$	$\top$	$\phi \vee \psi$	$\phi \wedge \psi$	$\phi \leftrightarrow \psi$	$\phi \not\leftrightarrow \psi$	ϕ	$\neg\phi$	ψ	$\neg\psi$	$\phi \to \psi$	$\phi \not\to \psi$	$\phi \leftarrow \psi$	$\phi \not\leftarrow \psi$
T	T	T	F	T	T	T	F	T	F	T	F	T	F	T	F
T	F	T	F	T	F	F	T	T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T	F	T	T	F	T	F	F	T
F	F	T	F	F	F	T	F	F	T	F	T	T	F	T	F

We say that $\star$ is a **truth-preserving** connective, if $\tilde{v}(\phi \star \psi) = T$ whenever $\tilde{v}(\phi) = T$ and $\tilde{v}(\psi) = T$. Similarly, we say that $\star$ is a **falsity-preserving** connective, if $\tilde{v}(\phi \star \psi) = F$ whenever $\tilde{v}(\phi) = F$ and $\tilde{v}(\psi) = F$.

By induction on the length, we can show that a formula built up from a truth-preserving connective is also truth-preserving. Consider the truth function $J : \{T, F\}^2 \to \{T, F\}$ such that $J(T, T) = F$. It is clear that no $\chi \in \text{Form}(\mathcal{L}[\{\star\}])$ such that $J = J_\chi$. Therefore no truth-preserving connective is adequate. Similarly, no falsity-preserving connective is adequate.

You can simply consider one single propositional variable and J : {T, F} → {T, F} as J(T) = F, J(F) = T

The only connectives which are neither truth-preserving nor falsity-preserving are $\neg\phi$, $\neg\psi$, $\uparrow$, and $\downarrow$.

$\neg\phi$ cannot be adequate because it has no dependence on ψ. Consider a truth function $J : \{T, F\}^2 \to \{T, F\}$ such that $J(T, T) \neq J(T, F)$. It is clear that no $\chi \in \text{Form}(\mathcal{L}[\{\neg\phi\}])$ such that $J = J_\chi$. Similarly, $\neg\psi$ is not adequate either.

This is not rigorous because we can have a formula with the second propositional variable occurring as the first operand of the connective. More rigorously you should enforce both J(T, T) ≠ J(T, F) and J(T, T) ≠ J(F, T)

(ii) All unary connectives are binary connectives. We have shown that none of them forms an adequate set. □

Do not say this, because you have "negation of first operand" and "negation of second operand" as binary connectives, and you cannot say that one of them *is* the unary negation. Rather, you can say that "binary tautology" and "negation of one operand" are binary connectives expressible by the two unary connectives respectively, so some of them should be adequate if any unary connective is adequate

Question 5 α

Prove that for any formulae α, β of $\mathcal{L}_0$, the following formulae are theorems of the system L_0. You may use the deduction theorem and, that for α, β,

$$\text{if} \vdash (\alpha \to \beta) \text{ and} \vdash (\neg\alpha \to \beta) \text{ then} \vdash \beta.$$

(i) $(\neg\alpha \to (\alpha \to \beta))$,

(ii) $(\neg\neg\alpha \to \alpha)$,

(iii) $(\alpha \to \neg\neg\alpha)$,

(iv) $((\neg\alpha \to \alpha) \to \alpha)$.

Proof. (i) We first prove that $\{\alpha, \neg\alpha\} \vdash \beta$:

$$\begin{array}{lll} \alpha_1: & (\neg\alpha \to (\neg\beta \to \neg\alpha)) & [\text{A1}] \\ \alpha_2: & \neg\alpha & [\text{Premise}] \\ \alpha_3: & (\neg\beta \to \neg\alpha) & [\text{MP } \alpha_1, \alpha_2] \\ \alpha_4: & ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)) & [\text{A3}] \\ \alpha_5: & (\alpha \to \beta) & [\text{MP } \alpha_3, \alpha_4] \\ \alpha_6: & \alpha & [\text{Premise}] \\ \alpha_7: & \beta & [\text{MP } \alpha_5, \alpha_6] \end{array}$$

By Deduction Theorem, $\neg\alpha \vdash (\alpha \to \beta)$. Again by Deduction Theorem, $\vdash (\neg\alpha \to (\alpha \to \beta))$. ✓

(ii) By Deduction Theorem, it suffices to prove that $\neg\neg\alpha \vdash \alpha$.

$$\begin{array}{lll} \alpha_1: & (\neg\neg\alpha \to (\neg\neg\neg\neg\alpha \to \neg\neg\alpha)) & [\text{A1}] \\ \alpha_2: & \neg\neg\alpha & [\text{Premise}] \\ \alpha_3: & (\neg\neg\neg\neg\alpha \to \neg\neg\alpha) & [\text{MP } \alpha_1, \alpha_2] \\ \alpha_4: & ((\neg\neg\neg\neg\alpha \to \neg\neg\alpha) \to (\neg\alpha \to \neg\neg\neg\alpha)) & [\text{A3}] \\ \alpha_5: & (\neg\alpha \to \neg\neg\neg\alpha) & [\text{MP } \alpha_3, \alpha_4] \\ \alpha_6: & ((\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)) & [\text{A3}] \\ \alpha_7: & (\neg\neg\alpha \to \alpha) & [\text{MP } \alpha_5, \alpha_6] \\ \alpha_8: & \alpha & [\text{MP } \alpha_2, \alpha_7] \end{array}$$

✓

(iii) Using the result in part (ii),

$$\begin{array}{lll} \alpha_1: & (\neg\neg\neg\alpha \to \neg\alpha) & [\text{Part (ii)}] \\ \alpha_2: & ((\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)) & [\text{A3}] \\ \alpha_3: & (\alpha \to \neg\neg\alpha) & [\text{MP } \alpha_1, \alpha_2] \end{array}$$

✓

(iv) Note that $\{\neg\alpha, (\neg\alpha \to \alpha)\} \vdash \alpha$ by modus ponens and that $\{\alpha, (\neg\alpha \to \alpha)\} \vdash \alpha$ since $\alpha \in \{\alpha, (\neg\alpha \to \alpha)\}$. By Deduction Theorem, we have $\vdash (\alpha \to ((\neg\alpha \to \alpha) \to \alpha))$ and $\vdash (\neg\alpha \to ((\neg\alpha \to \alpha) \to \alpha))$. By the metatheorem given in the question, we deduce that $\vdash ((\neg\alpha \to \alpha) \to \alpha)$. ✓ □