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Problem Sheet 2

Equilibrium, Flows, and Orbits in GR

B5: General Relativity

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Question 1. Hydrostatic Equilibrium in GR.

Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -\left(1 - 2GM/rc^2\right)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = R_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See §4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$.

(OPTIONAL: For those who have studied fluids, what quantity is being conserved in the Newtonian limit?)

Proof. For a type $(1,1)$ tensor field T, the covariant divergence is given by

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{|\det g|}} \frac{\partial\left(\sqrt{|\det g|}\, T^\mu_\nu\right)}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda$$

The mixed energy-momentum stress tensor is given by

$$T^\mu_\nu = P\delta^\mu_\nu + \left(\rho + \frac{P}{c^2}\right) U^\mu U_\nu$$

In the hydrostatic equilibrium, the mixed energy-momentum stress tensor is divergenceless. We have

$$\nabla_\mu T^\mu_\nu = \partial_\nu P + \frac{1}{\sqrt{|\det g|}} \partial_\mu \left(\sqrt{|\det g|} \left(\rho + \frac{P}{c^2}\right) U^\mu U_\nu\right) - \Gamma^\lambda_{\mu\nu} \left(\rho + \frac{P}{c^2}\right) U^\mu U_\lambda = 0$$

In static equilibrium, $U^i = 0$ and $g_{0i} = 0$. Hence

$$U^\mu U_\nu = U^0 U_\nu \delta^\mu_0 = U^0 g_{0\nu} U^0 \delta^\mu_0 = g_{00} U^0 U^0 \delta^\mu_0 \delta^0_\nu = U^0 U_0 \delta^\mu_0 \delta^0_\nu = -c^2 \delta^\mu_0 \delta^0_\nu$$

Plugging into the equation:

$$\partial_\nu P - \frac{1}{\sqrt{|\det g|}} \partial_0 \left(\sqrt{|\det g|}\, (\rho c^2 + P)\right) \delta^0_\nu + \Gamma^0_{0\nu} \left(\rho c^2 + P\right) = 0$$

where

$$\Gamma^0_{0\nu} = \frac{g^{0\rho}}{2}\left(\partial_0 g_{\nu\rho} + \partial_\nu g_{0\rho} - \partial_\rho g_{0\nu}\right) = \frac{1}{2} g^{0\rho} \partial_\nu g_{0\rho} = \frac{1}{2} g^{00} \partial_\nu g_{00} = \frac{1}{2g_{00}} \partial_\nu g_{00} = \frac{1}{2} \partial_\nu \left(\ln |g_{00}|\right) = \partial_\nu \left(\ln |g_{00}|^{1/2}\right)$$

We deduce that

$$\partial_i P + (\rho c^2 + P) \partial_i \left(\ln |g_{00}|^{1/2}\right) = 0$$

This is the general hydrostatic equilibrium equation. We seek a spherically symmetric solution with $\rho = \rho(r)$. The equation becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} + (\rho c^2 + P) \frac{\mathrm{d}}{\mathrm{d}r} \left(\ln |g_{00}|^{1/2}\right) = 0$$

Plugging in the expressions of g_{00} and P:

$$K\gamma\rho^{\gamma-1} \frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{2}(\rho c^2 + K\rho^\gamma) \frac{\mathrm{d}}{\mathrm{d}r} \ln\left|1 - \frac{R_S}{r}\right| = 0 \implies \frac{2K\gamma\rho^{\gamma-1}}{\rho c^2 + K\rho^\gamma} \frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r} \ln\left|1 - \frac{R_S}{r}\right| = 0$$

Integrate:

$$\int_{\rho_0}^{\rho} \frac{2K\gamma\rho^{\gamma-1}}{\rho c^2 + K\rho^\gamma} \mathrm{d}\rho + \ln\left|1 - \frac{R_S}{r}\right|_{r_0}^{r} = 0$$

Let us compute the integral...

$$\int_{\rho_0}^{\rho} \frac{2K\gamma\rho^{\gamma-1}}{\rho c^2 + K\rho^\gamma} \mathrm{d}\rho = \int_{\rho_0}^{\rho} \frac{2K\gamma\rho^{\gamma-2}}{c^2 + K\rho^{\gamma-1}} \mathrm{d}\rho = 2K\frac{\gamma}{\gamma-1} \int_{\rho_0}^{\rho} \frac{1}{c^2 + K\rho^{\gamma-1}} \mathrm{d}(\rho^{\gamma-1}) = \frac{2\gamma}{\gamma-1} \ln\left|c^2 + K\rho^{\gamma-1}\right|_{\rho_0}^{\rho} = \frac{2\gamma}{\gamma-1} \ln\left|\frac{c^2 + K\rho^{\gamma-1}}{c^2 + K\rho_0^{\gamma-1}}\right|$$

Therefore

$$\frac{2\gamma}{\gamma-1}\ln\left|\frac{c^2+K\rho^{\gamma-1}}{c^2+K\rho_0^{\gamma-1}}\right|+\ln\left|\frac{1-R_S/r}{1-R_S/r_0}\right|=0$$

Taking exponential:

$$\left(\frac{c^2+K\rho^{\gamma-1}}{c^2+K\rho_0^{\gamma-1}}\right)^{\frac{2\gamma}{\gamma-1}}\left(\frac{1-R_S/r}{1-R_S/r_0}\right)=1 \implies \frac{c^2+K\rho^{\gamma-1}}{c^2+K\rho_0^{\gamma-1}}=\left(\frac{1-R_S/r_0}{1-R_S/r}\right)^{\frac{\gamma-1}{2\gamma}} \implies \frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2}=\left(\frac{1-R_S/r_0}{1-R_S/r}\right)^{\alpha}$$

In the Newtonian limit, we have $R_S/r \ll 1$ and $K\rho^{\gamma-1}/c^2 \ll 1$. We can therefore linearise the above equation:

$$\begin{aligned}
&\frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2}=\left(\frac{1-R_S/r_0}{1-R_S/r}\right)^{\alpha}\\
\implies &\left(1+\frac{K\rho^{\gamma-1}}{c^2}\right)\left(1-\frac{\alpha R_S}{r}\right)=\left(1+\frac{K\rho_0^{\gamma-1}}{c^2}\right)\left(1-\frac{\alpha R_S}{r_0}\right)\\
\implies &\frac{K\rho^{\gamma-1}}{c^2}-\frac{\alpha R_S}{r}=\frac{K\rho_0^{\gamma-1}}{c^2}-\frac{\alpha R_S}{r_0}\\
\implies &\frac{a^2}{\gamma c^2}-\frac{\gamma-1}{2\gamma}\frac{2GM}{rc^2}=\frac{a_0^2}{\gamma c^2}-\frac{\gamma-1}{2\gamma}\frac{2GM}{r_0c^2}\\
\implies &a^2+(\gamma-1)\Phi(r)=a_0^2+(\gamma-1)\Phi(r_0)
\end{aligned}$$

The conserved quantity is $a^2+(\gamma-1)\Phi(r)$. (*I don't know any conserved quantity in fluid mechanics related to the speed of sound...*) □

enthalpy

Question 2. Bondi Accretion: go with the flow.

To get some practise working with the equations of GR as well as some insight into relativistic dynamics in a practical problem in astrophysics, consider what is known as (relativistic) Bondi Accretion, the spherical flow of gas into a black hole. (The original Bondi accretion problem was Newtonian accretion onto an ordinary star.) We assume a Schwarzschild metric in the usual spherical coordinates:

$$g_{00}=-\left(1-2GM/rc^2\right),\quad g_{rr}=\left(1-2GM/rc^2\right)^{-1},\quad g_{\theta\theta}=r^2,\quad g_{\phi\phi}=r^2\sin^2\theta$$

a) First, let us assume that particles are neither created or destroyed. So particle number is conserved. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu=nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that particle number conservation implies:

$$J^\mu_{;\mu}=0$$

If nothing depends upon time, show that this integrates to

$$nU^r\left|g'\right|^{1/2}=\text{ constant}$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$, and U^r is...well, you tell me what U^r is.

b) We move on to energy conservation, $T^{t\nu}_{;\nu}=0$. (Refer to §4.6 in the notes.) Show that the only nonvanishing affine connection that we need to use is

$$\Gamma^t_{tr}=\Gamma^t_{rt}=\frac{1}{2}\frac{\partial\ln|g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$\left(P+\rho c^2\right)U^rU_t\left|g'\right|^{1/2}=\text{ constant}$$

where $U_t=g_{t\mu}U^\mu$, and ρ is the total energy density of the fluid in its rest frame, including any thermal energy.

c) We next define

$$\varpi = \mu n$$

where μ is the rest mass per particle and ϖ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2.P and ϖ are assumed to be related by a simple power law relationship,

$$P = K\varpi^\gamma$$

where K is a constant, and γ is called the adiabatic index. This is not an entirely artificial problem: it is valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it!) Show that this implies:

$$\rho = \varpi + \frac{P}{c^2(\gamma - 1)}$$

d) Verify that

$$|g'| = r^4$$

and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + \left(U^r\right)^2\right]^{1/2}$$

(Take care to distinguish U^t and U_t.)

e) With

$$a^2 = \gamma P/\varpi$$

(this is the speed of sound in a nonrelativistic gas), combine our mass and energy conservation equations to show that

$$\left(c^2 + \frac{a^2}{\gamma - 1}\right)^2 \left(c^2 + U^2 - \frac{2GM}{r}\right) = \text{ constant}$$

We have dropped the superscript r on U^r for greater clarity. How does a^2 depend upon ϖ? The other equation we shall use is just that of mass conservation itself. Show that this may be written as

$$4\pi\varpi r^2 U = \dot{m}$$

which defines the net, constant mass accretion rate $\dot{m} < 0$. With a^2 depending entirely on ϖ, and $\varpi = \dot{m}/\left(4\pi r^2 U\right)$, the equation in boldface becomes a single algebraic equation for U as a function of r, and the formal solution to our problem.

f) Three final simple tasks for now:

   i) Show that the constant on the right of the bold equation of problem (2e) is

$$c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

   where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

ii) Show that the Newtonian limit of the equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy (energy plus the work done by pressure) is conserved. This is the original nonrelativistic Bondi 1952 solution for accretion onto a star.

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c^2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c(1 - R_S/r)$$

Like stalled photons, from the point of view of a distant observer, the flow never crosses R_S.

Proof. a) In Special Relativity, we have known that in the Minkowski spacetime $(\mathbb{R}^4, \eta)$, the 4-current is given by $J^\mu = nU^\mu$, and the continuity equation $\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$ upgrades to the covariant form $\partial_\mu J^\mu = 0$. In a general spacetime (M, g), the continuity equation is $\nabla_\mu J^\mu = 0$. In fact, most of the covariant formulae in SR can be generalised to GR by replacing the partial derivative ∂_μ by the covariant derivative ∇_μ.

For a spherically symmetric stationary solution, we can simply put $U^\theta = U^\varphi = 0$. The only non-zero 4-velocity components are U^t and U^r. But U^t does not contribute to the equation because $\partial_t U^t = 0$. Using the covariant divergence formula,

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{|\det g|}} \frac{\partial\left(\sqrt{|\det g|} J^\mu\right)}{\partial x^\mu} = \frac{1}{r^2 \sin\theta} \frac{\partial\left(r^2 \sin\theta\, n U^r\right)}{\partial r} = \frac{1}{r^2} \frac{\partial}{\partial r}(nU^r r^2) = 0$$

which integrates to

$$nU^r r^2 = f(t, \theta, \varphi)$$

for some function f. But $f = \text{const}$ since we are looking for a spherically symmetric stationary solution.

b) The contravariant energy-momentum tensor is given by

$$T^{\mu\nu} = Pg^{\mu\nu} + \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu$$

For this problem, the non-zero components are

$$T^{tt} = -P\left(1 - \frac{R_S}{r}\right)^{-1} + \left(\rho + \frac{P}{c^2}\right)(U^t)^2, \qquad T^{rr} = P\left(1 - \frac{R_S}{r}\right) + \left(\rho + \frac{P}{c^2}\right)(U^r)^2, \qquad T^{tr} = \left(\rho + \frac{P}{c^2}\right) U^t U^r$$

$$T^{\theta\theta} = \frac{P}{r^2}, \qquad T^{\varphi\varphi} = \frac{P}{r^2 \sin^2\theta}$$

We expand $\nabla_\nu T^{t\nu}$ using the Christoffel symbols:

$$\begin{aligned}
\nabla_\nu T^{t\nu} &= \partial_\nu T^{t\nu} + \Gamma^\nu_{\nu\lambda} T^{\lambda t} + \Gamma^t_{\nu\lambda} T^{\nu\lambda} \\
&= \partial_t T^{tt} + \partial_r T^{tr} + \left(\Gamma^t_{tt} + \Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\varphi_{\varphi t}\right) T^{tt} + \left(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\varphi_{\varphi r}\right) T^{tr} \\
&\quad + \left(\Gamma^t_{tt} T^{tt} + \Gamma^t_{rr} T^{rr} + \Gamma^t_{tr} T^{tr} + \Gamma^t_{\theta\theta} T^{\theta\theta} + \Gamma^t_{\varphi\varphi} T^{\varphi\varphi}\right)
\end{aligned}$$

For a diagonal metric g, the Christoffel symbols are given by

$$\Gamma^a_{ab} = \frac{1}{2g_{aa}} \partial_b g_{aa}, \qquad \Gamma^a_{bb} = -\frac{1}{2g_{aa}} \partial_a g_{bb}\ (a \neq b), \qquad \Gamma^a_{bc} = 0\ (a, b, c \text{ dinstinct})$$

We note that all Christoffel symbols appear in the above equation vanish except for Γ^t_{tr}, which is given by

$$\Gamma^t_{tr} = \frac{1}{2g_{tt}} \partial_r g_{tt} = \frac{1}{2} \partial_r \left(\ln|g_{tt}|\right) = -\frac{R_S}{2r(r - R_S)}$$

Use $T^\mu{}_{\nu;\mu} = \frac{1}{\sqrt{g}} \partial_\mu\left(\sqrt{g}\, T^\mu{}_\nu\right) + \Gamma^\lambda_{\mu\nu} T^\mu{}_\lambda$ (2.12 of the notes)

Substituting into the equation above: the r^2 comes from $\sqrt{g}$

$$\partial_r T^{tr} + 2\Gamma^t_{tr} T^{tr} = 0 \implies \frac{\partial T^{tr}}{\partial r} + \frac{T^{tr}}{g_{tt}}\frac{\partial g_{tt}}{\partial r} = 0 \implies \frac{\partial (T^{tr} g_{tt})}{\partial r} = 0$$

Therefore

$$\text{const} = T^{tr} g_{tt} = \left(\rho + \frac{P}{c^2}\right) U^t U^r g_{tt} = \left(\rho + \frac{P}{c^2}\right) U_t U^r$$

(I did not find the r^2 factor in the solution...) □

Question 3. Kinematic and gravitational redshifts.

a) One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R) V^\mu(R)}{p_\mu(E) V^\mu(E)}$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

b) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits

$$\odot - - - - - - - - - \to \text{observer} \gtrdot$$

Show that in t, r, θ, ϕ coordinates for the 0,1,2,3 components,

$$V^\mu(R) = (1,0,0,0), \quad V^\mu(E) = V^0_E(1,0,0,d\phi/cdt), \text{ with } V^0_E = dt/d\tau$$

Then, using $g_{\mu\rho}V^\mu V^\rho = -1$), conclude that

$$V^0_E = \left(1 - 3GM/rc^2\right)^{-1/2}$$

You may use a result from problem (5c) below. (You will prove it later!)

c) Finally, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2}\left(1 + \frac{\Omega p_\phi(E)}{c p_0(E)}\right)^{-1}, \quad \Omega^2 = GM/r^3$$

A result of problem (3) from Problem Set 1 may be useful.

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$\left(1 - 3GM/rc^2\right)^{1/2}$$

Why? From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - 3GM/rc^2\right)^{1/2} / \left(1 \pm \left(rc^2/GM - 2\right)^{-1/2}\right)$$

(Hint: $g^{\nu\rho}p_\nu p_\rho = 0$.) Interpret the ± sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

Question 4. The perihelion advance of Mercury.

a) In the notes we found that the differential equation for $u = 1/r$ for Mercury's orbit could be written as follows. $u =$

$u_N + \delta u$ with the Newtonian solution u_N given by

$$u_N = \left(GM/J^2\right)(1+\epsilon\cos\phi)$$

and the differential equation for δu is

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2 J^4}\left(1+2\epsilon\cos\phi+\epsilon^2\cos^2\phi\right)$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a\left(b+2\epsilon e^{i\phi}+\epsilon^2 e^{2i\phi}/2\right)$$

where $a = 3(GM)^3/\left(c^2 J^4\right)$ and $b = 1+\epsilon^2/2$ To solve this, try a solution of the form

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2 e^{2i\phi}$$

where the A's are constants. Why do we need an additional factor of ϕ in the A_1 term?

b) Show that the solution for $u = u_N + \delta u$ is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi$$

Since a is very small, show that this equivalent to

$$u = ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}[1+\epsilon(\cos\phi(1-\alpha))]$$

where

$$\alpha = aJ^2/GM = 3(GM/Jc)^2$$

c) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\Delta\phi = 6\pi\left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.

Proof. a) We can complexify the equation by considering $\delta u = \mathrm{Re}(\delta u^*)$. Note that $\cos n\varphi = \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}n\varphi}\right)$. Thus

$$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi) = \mathrm{Re}\left(\frac{1}{2}(1+\mathrm{e}^{2\mathrm{i}\varphi})\right)$$

Since the differential operator $\mathrm{d}/\mathrm{d}\varphi$ and Re commutes, we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)\delta u^* = \frac{3(GM)^3}{c^2 J^4}\left(1+2\varepsilon\,\mathrm{e}^{\mathrm{i}\varphi}+\varepsilon^2\frac{1}{2}(1+\mathrm{e}^{2\mathrm{i}\varphi})\right) = \frac{3(GM)^3}{c^2 J^4}\left(1+\frac{\varepsilon^2}{2}+2\varepsilon\,\mathrm{e}^{\mathrm{i}\varphi}+\varepsilon^2\,\mathrm{e}^{2\mathrm{i}\varphi}\right)$$

This is a inhomogeneous second-order linear ODE. We can look for the particular solutions termwise. From the theory of ODE, we know the following fact: Suppose that $\mathscr{L}$ is a linear differential operator with constant coefficients. Let λ be an eigenvalue of $\mathscr{L}$ of multiplicity n ($n=0$ if λ is not an eigenvalue). Then the inhomogeneous problem $\mathscr{L}y = \mathrm{e}^{\lambda x}$ has a particular solution of the form $y = Ax^n\,\mathrm{e}^{\lambda x}$. (The proof is to rewrite $\mathscr{L}$ as a system of first-order linear ODEs, and put the corresponding matrix of $\mathscr{L}$ into a Jordan normal form.)

The linear differential operator $\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)$ has an eigenvalue i of multiplicity 1. So the particular solution of the problem $\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)\delta u^* = 2\varepsilon\,\mathrm{e}^{\mathrm{i}\varphi}$ is of the form $\delta u^*(\varphi) = A_1\varphi\,\mathrm{e}^{\mathrm{i}\varphi}$.

Next we determine the coefficients A_0, A_1, A_2.

$$ab = \left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)A_0 = A_0 \implies A_0 = \frac{3(GM)^3}{c^2J^4}\left(1+\frac{\varepsilon^2}{2}\right)$$

$$2\varepsilon a\,\mathrm{e}^{\mathrm{i}\varphi} = \left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)\left(A_1\varphi\,\mathrm{e}^{\mathrm{i}\varphi}\right) = 2\mathrm{i}A_1 \implies A_1 = -\mathrm{i}\varepsilon\frac{3(GM)^3}{c^2J^4}$$

$$\frac{1}{2}\varepsilon^2 a\,\mathrm{e}^{2\mathrm{i}\varphi} = \left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+1\right)\left(A_2\,\mathrm{e}^{2\mathrm{i}\varphi}\right) = -3A_2 \implies A_2 = -\frac{1}{2}\varepsilon^2\frac{(GM)^3}{c^2J^4}$$

The full particular solution is given by

$$\delta u^* = \frac{(GM)^3}{c^2J^4}\left(3\left(1+\frac{\varepsilon^2}{2}\right) - 3\mathrm{i}\varepsilon\varphi\,\mathrm{e}^{\mathrm{i}\varphi} - \frac{1}{2}\varepsilon^2\,\mathrm{e}^{2\mathrm{i}\varphi}\right)$$

The real part is

$$\delta u = \mathrm{Re}(\delta u^*) = \frac{(GM)^3}{c^2J^4}\left(3\left(1+\frac{\varepsilon^2}{2}\right) + 3\varepsilon\varphi\sin\varphi - \frac{1}{2}\varepsilon^2\cos 2\varphi\right)$$

In principle we should also include the general solution to the homogeneous problem, which gives the full general solution

$$\delta u = \left(C_1\cos\varphi + C_2\sin\varphi\right) + \frac{(GM)^3}{c^2J^4}\left(3\left(1+\frac{\varepsilon^2}{2}\right) + 3\varepsilon\varphi(\cos\varphi+\sin\varphi) - \frac{1}{2}\varepsilon^2\cos 2\varphi\right)$$

But the constants C_1 and C_2 can be absorbed into the Newtonian solution u_N and can be discarded here.

b) We have

$$\begin{aligned} u = u_N + \delta u &= \frac{GM}{J^2}(1+\varepsilon\cos\varphi) + \frac{(GM)^3}{c^2J^4}\left(3\left(1+\frac{\varepsilon^2}{2}\right) + 3\varepsilon\varphi\sin\varphi - \frac{1}{2}\varepsilon^2\cos 2\varphi\right) \\ &= a\left(1+\frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{6}\cos 2\varphi\right) + \frac{GM}{J^2}\left(1+\varepsilon\left(\cos\varphi + \alpha\varphi\sin\varphi\right)\right) \end{aligned}$$

Since $\alpha\varphi \ll 1$, to the order $\mathrm{O}(\alpha\varphi)$ we have

$$\cos\varphi + \alpha\varphi\sin\varphi = \cos\varphi\cos\alpha\varphi + \sin\alpha\varphi\sin\varphi = \cos(\varphi(1-\alpha))$$

Hence the solution is approximately

$$u = a\left(1+\frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{6}\cos 2\varphi\right) + \frac{GM}{J^2}\left(1+\varepsilon\cos(\varphi(1-\alpha))\right)$$

c) The period of the solution is

$$\frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha) = 2\pi + 6\pi\left(\frac{GM}{Jc}\right)^2$$

Hence the precession angle per period is

$$\Delta\varphi = 6\pi\left(\frac{GM}{Jc}\right)^2$$

□

Question 5. Black hole orbits.

a) In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathscr{E}$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathscr{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathscr{E}_S$$

where r is the standard radial coordinate, l_S and $\mathscr{E}_S$ are constants, $\Phi_S(r)$ is an effective potential function, and $v_S = dr/d\tau$. Determine l_S and $\mathscr{E}_S$ in terms of the fundamental angular momentum and energy constants J and E from lecture (or the notes). Express $\Phi_S(r)$ in terms of $l_S, \mathscr{E}_S$, the speed of light c, GM and r. The form of $l_S, \mathscr{E}_S$, and Φ_S should be chosen to go over to their Newtonian counterparts in the limit $E \to c^2, c \to \infty, E - c^2 \to$ finite.

b) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for Φ_S, and what does it mean physically if it does not exist?

c) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3}$$

exactly the Newtonian form. Here $\Omega \equiv d\phi/dt$ at the coordinate location r, where dt is the proper time interval at infinity. Derive expressions for E and J in terms of GM, c^2 and r.

d) Below what value of r does Φ_S not have any local extrema? (Answer: $6GM/c^2$.)

Proof. a) We assume that the particle is massive, so that its 4-velocity is always timelike. First we starts from the Schwarzschild metric:

$$g = -\left(1 - \frac{R_S}{r}\right)c^2 \mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

The corresponding Lagrangian is given by

$$\mathscr{L} = -\left(1 - \frac{R_S}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right)$$

where the dot denotes the derivative with respect to the proper time τ. We observe that t and φ are ignorable coordinates. We have

$$\frac{\partial \mathscr{L}}{\partial \dot{t}} = -2\left(1 - \frac{R_S}{r}\right)c^2 \dot{t} = \text{const}, \qquad \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} = 2r^2 \sin^2\theta \dot{\varphi} = \text{const}$$

We can use the SO(3) symmetry of the manifold to fix the orbits on the plane $\theta = \pi/2$. Then $\dot{\theta} = 0$. We set the constants $J := r^2\dot{\varphi}$ and $E := \left(1 - \frac{R_S}{r}\right)c^2 \dot{t}$, which are the angular momentum and energy per unit mass.

Note that by the definition of proper time, $c\mathrm{d}\tau = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$. Therefore $\mathscr{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$. This gives

$$\mathscr{L} = -\left(1 - \frac{R_S}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \dot{r}^2 + r^2\dot{\varphi}^2 = -\left(1 - \frac{R_S}{r}\right)^{-1} \frac{E^2}{c^2} + \left(1 - \frac{R_S}{r}\right)^{-1} \dot{r}^2 + \frac{J^2}{r^2} = -c^2$$

Rearrangine the expression:

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \left(\frac{GM}{r} + \frac{GMJ^2}{c^2 r^3}\right) = \frac{E^2 - c^4}{2c^2}$$

This is the radial orbit equation we want.

(*The way that the notes introduces the constants J and E is unsatistory from my perspective. We don't need any mysterous parameter p. We can simply choose the* ***affine parameter****, which is a parameter such that the tangent vector of the geodesic is parallel transported along the geodesic.*)

$$v_S = \frac{\mathrm{d}r}{\mathrm{d}\tau}, \qquad \ell_S = J, \qquad \Phi_S(r) = -\frac{GM}{r} - \frac{GMJ^2}{c^2 r^3} = -\frac{GM}{r} - \frac{GM\ell_S^2}{c^2 r^3}, \qquad \mathscr{E}_S = \frac{E^2 - c^4}{2c^2}$$

In the gravitational potential $\Phi_S(r)$, we can clearly find the relativistic correction term $-\frac{GMJ^2}{c^2 r^3}$.

It seems that you have lost some factors of E at some point

b) The effective potential is given by

$$V_{\text{eff}}(r) = \frac{J^2}{2r^2} - \frac{GM}{r} - \frac{GMJ^2}{c^2 r^3}$$

I borrow the following figure from the Part C GR1 notes. In the figure $\Omega = J$ and $G = c = 1$.

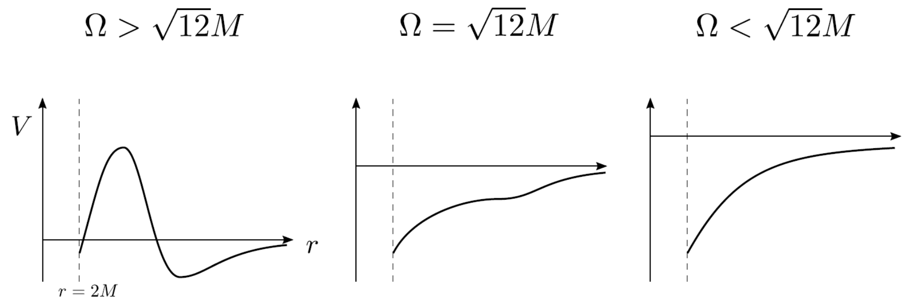

Figure 6.2: The effective potential in a Schwarzschild spacetime, for various values of the conserved angular momentum Ω.

For the Newtonian effective potential, at a local extermum we have

$$V'_{\text{eff}}(r) = \frac{GM}{r^2} - \frac{J^2}{r^3} = 0 \implies r = \frac{J^2}{GM}$$

This is a local minimum because

$$V''_{\text{eff}}\left(\frac{J^2}{GM}\right) = \left(-\frac{2GM}{r^3} + \frac{3J^2}{r^4}\right)_{r=J^2/GM} = GM\left(\frac{GM}{J^2}\right)^3 > 0$$

For the Schwarzschild effective potential, at a local extermum we have

$$V'_{\text{eff}}(r) = \frac{GM}{r^2} - \frac{J^2}{r^3} + \frac{3GMJ^2}{c^2 r^4} = 0 \implies r^2 - \frac{J^2}{GM}r + \frac{3J^2}{c^2} = 0$$

The discriminant of the quadratic equation is given by

$$\Delta = \left(\frac{J^2}{GM}\right)^2 - \frac{12J^2}{c^2}, \qquad \Delta > 0 \iff J > \sqrt{12}\frac{GM}{c}$$

When $J < \sqrt{12}GM/c$, the effective potential does not have any extremum, so there are no bound orbits. Physically, particles with angular momentum smaller than $\sqrt{12}GM/c$ will either fall onto the event horizon $r = R_S$ or escape to $r \to \infty$ eventually.

c) Along a circular orbit, we have $\dot{r} = 0$ and $\ddot{r} = 0$. The Lagrangian simplifies to

$$\mathscr{L} = -\left(1 - \frac{R_S}{r}\right)c^2\dot{t}^2 + r^2\dot{\varphi}^2 = -c^2$$

The Euler-Lagrange equation for r is

$$\frac{\partial \mathscr{L}}{\partial r} = -\frac{R_S}{r^2}c^2\dot{t}^2 + 2r\dot{\varphi}^2 = 0$$

Therefore

$$\Omega^2 = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \frac{\dot{\varphi}^2}{\dot{t}^2} = \frac{R_S c^2}{2r^3} = \frac{GM}{r^3}$$

which is consistent with the classical Kepler third law.

Along a circular orbit, we must have $V'_{\text{eff}}(r) = 0$. Solving J in terms of r,

$$r^2 - \frac{J^2}{GM}r + \frac{3J^2}{c^2} = 0 \implies J = r\left(\frac{r}{GM} - \frac{3}{c^2}\right)^{-1/2}$$

Still something is wrong because of the factors of E

We substitute J into the radial equation:

$$\frac{J^2}{2r^2} - \left(\frac{GM}{r} + \frac{GMJ^2}{c^2r^3}\right) = \frac{E^2 - c^4}{2c^2} \implies E = c\sqrt{c^2 - \frac{GM}{r}\frac{3rc^2 - 8GM}{rc^2 - 3GM}} = c^2\sqrt{\frac{2(r - R_S)(r - 2R_S)}{2r^2 - 3R_S r}}$$

d) $\Phi_S(r)$ *is monotonic and never has a local extremum. I assume that the question is asking about* V_{eff}.

$$V'_{\text{eff}}(r) = 0 \implies r_\pm = \frac{J^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{c^2J^2}}\right)$$

From the sketch of $V_{\text{eff}}(r)$ it is easy to see that r_- is an unstable orbit and r_+ is a stable orbit. The minimum value of r_+ is achieved when $J^2 = \sqrt{12}GM/c$, where

$$(r_+)_{\text{min}} = \frac{J^2}{2GM} = \frac{6GM}{c^2}$$

There are no stable circular orbits with $r < 6GM/c^2$. □