1. A
2. A
3. A
4. A
5. B/C
6. NA

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Problem Sheet 2

Conformal Field Theory

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Question 1

Consider the canonical energy-momentum tensor for the free boson in $d > 2$. Find an improvement term which makes it classically traceless without spoiling classical conservation.

[*Hint. Note that the index structure suggests a general ansatz of the form:* $(\alpha\eta^{\mu\nu}\partial_\rho\partial^\rho + \beta\partial^\mu\partial^\nu) f(x)$.]

Proof. The action and the canoncial stress-energy tensor for the free boson are given in the notes:

$$S[\phi] = \int \mathrm{d}^d x \,\mathcal{L} = \frac{1}{2}\int \mathrm{d}^d x \,\partial_\mu\phi \cdot \partial^\mu\phi.$$

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi = -\frac{1}{2}\eta^{\mu\nu}\partial_\rho\phi \cdot \partial_\rho\phi + \partial^\mu\phi \cdot \partial^\nu\phi$$

The trace of the canonical stress-energy tensor is given by

$$(T^\mu{}_\mu)_c = -\frac{1}{2}(d-2)\partial_\rho\phi \cdot \partial^\rho\phi.$$

OK

We seek for an improvement $B^{\mu\nu}$ such that $T^{\mu\nu} + T_c^{\mu\nu} + B^{\mu\nu}$ satisfies the conservation equation $\partial_\mu T^{\mu\nu} = 0$. By the hint we consider $B^{\mu\nu}$ of the form

$$B^{\mu\nu} = (\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)f(\phi).$$

Indeed,

$$\partial_\mu B^{\mu\nu} = (\partial^\nu\partial_\rho\partial^\rho - \partial_\mu\partial^\mu\partial^\nu)f(\phi) = 0.$$

OK

The trace of $B^{\mu\nu}$ is given by

$$B^\mu{}_\mu = (d-1)\partial_\rho\partial^\rho f(\phi).$$

If we take $f(\phi) = \dfrac{d-2}{4(d-1)}\phi^2$, then

$$B^\mu{}_\mu = \frac{d-2}{4}\partial_\rho\partial^\rho(\phi^2) = \frac{d-2}{2}(\partial_\rho\phi\partial^\rho\phi),$$

which implies that $T^\mu{}_\mu = 0$, where

$$T^{\mu\nu} = T_c^{\mu\nu} + \frac{d-2}{4(d-1)}(\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)\phi^2.$$

OK □

Question 2

Consider two dimensional Liouville theory with the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2 e^\phi$$

Write down the canonical energy momentum tensor and verify that it is conserved (on the equations of motion). Add a term such that it is also traceless, without spoiling conservation.

Proof. The canonical stress-energy tensor is given by

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi = -\frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi\partial^\rho\phi + m^2\,\mathrm{e}^\phi) + \partial^\mu\phi\partial^\nu\phi.$$

OK

For the equation of motion, we write down the Euler–Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \implies \partial_\mu \partial^\mu \phi = \frac{1}{2} m^2 \,\mathrm{e}^{\phi}\,.$$

OK

So

$$\begin{aligned} \partial_\mu T_c^{\mu\nu} &= -\partial^\nu \partial_\rho \phi \partial^\rho \phi - \frac{1}{2} m^2 \,\mathrm{e}^{\phi}\, \partial^\nu \phi + \partial_\mu \partial^\mu \phi \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi \\ &= \left(\partial_\mu \partial^\mu \phi - \frac{1}{2} m^2 \,\mathrm{e}^{\phi} \right) \partial^\nu \phi = 0 \end{aligned}$$

OK

The canonical stress-energy tensor indeed satisfies the conservation equation. The trace is given by

$$(T^\mu{}_\mu)_c = -\frac{1}{2}(d-2) \partial_\rho \phi \partial^\rho \phi - \frac{d}{2} m^2 \,\mathrm{e}^{\phi}\,.$$

OK

From the previous question we know that the first term can be eliminated by adding a term $B^{\mu\nu} = \frac{d-2}{4(d-1)}(\eta^{\mu\nu}\partial_\rho \partial^\rho - \partial^\mu \partial^\nu)\phi^2$ to $T_c^{\mu\nu}$. For the second term, we use the equation of motion, and consider

$$C^{\mu\nu} = (\eta^{\mu\nu}\partial_\rho \partial^\rho - \partial^\mu \partial^\nu) f(\phi)$$

with

$$C^\mu{}_\mu = (d-1)\partial_\rho \partial^\rho f(\phi) = d \partial_\rho \partial^\rho \phi.$$

This implies that $f(\phi) = \frac{d}{d-1}\phi$. In summary, $T^\mu{}_\mu = 0$ for

OK, it was enough to consider d=2 as required in the question

$$T^{\mu\nu} = T_c^{\mu\nu} + \frac{1}{d-1}(\eta^{\mu\nu}\partial_\rho \partial^\rho - \partial^\mu \partial^\nu) \left(\frac{d-2}{4}\phi^2 + d\phi \right).$$

□

Question 3

Prove the following property under special conformal transformations

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{\gamma_i^{1/2} \gamma_j^{1/2}}$$

where $\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2$.

Proof. For my convenience I use $x := x_i$ and $y := x_j$. This equality is proved by brute force computation.

$$\begin{aligned} x'^\mu - y'^\mu &= \frac{x^\mu - b^\mu x^2}{\gamma_x} - \frac{y^\mu - b^\mu y^2}{\gamma_y} \\ \implies (x'-y')^2 &= \frac{x^2 + b^2 x^4 - 2x^2(b\cdot x)}{\gamma_x^2} + \frac{y^2 + b^2 y^4 - 2y^2(b\cdot y)}{\gamma_y^2} - \frac{2}{\gamma_x \gamma_y}\left(x\cdot y + b^2 x^2 y^2 - (b\cdot x) y^2 - (b \cdot y) x^2\right) \\ &= \frac{x^2}{\gamma_x} + \frac{y^2}{\gamma_y} + \frac{-2x\cdot y - y^2(1 - 2(b\cdot x) + b^2 x^2) - x^2(1 - 2(b\cdot y) + b^2 y^2) + x^2 + y^2}{\gamma_x \gamma_y} \\ &= \frac{x^2}{\gamma_x} + \frac{y^2}{\gamma_y} + \frac{-2x\cdot y - \gamma_x y^2 - \gamma_y x^2 + x^2 + y^2}{\gamma_x \gamma_y} \\ &= \frac{(x-y)^2}{\gamma_x \gamma_y} \end{aligned}$$

OK

By taking the sqaure root we obtain the desired equality.

□

Question 4

Consider the inversion tensor $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$. Show that under inversions

$$I_{\mu\alpha}(x) I^{\alpha\beta}(x-y) I_{\beta\nu}(y) = I_{\mu\nu}\left(x' - y'\right)$$

where $(x')^\mu = x^\mu / x^2$ and $(y')^\mu = y^\mu / y^2$.

Proof. It seems that this question is nothing but a brute force computation. I did not find a neat way to do this... We expand the left-hand side of the equation:

$$\begin{aligned} I_{\mu\alpha}(x) I^{\alpha\beta}(x-y) I_{\beta\nu}(y) &= \left(\eta_{\mu\alpha} - 2x'_\mu x_\alpha\right)\left(\eta^{\alpha\beta} - 2\frac{(x^\alpha - y^\alpha)(x^\beta - y^\beta)}{(x-y)^2}\right)\left(\eta_{\beta\nu} - 2y_\beta y'_\nu\right) \\ &= \eta_{\mu\nu} - 2x'_\mu x_\nu - 2y_\mu y'_\nu + 4x'_\mu y'_\nu x \cdot y \\ &\quad + 4\frac{x^2 - x\cdot y}{(x-y)^2} x'_\mu (x_\nu - y_\nu) - 4\frac{y^2 - x \cdot y}{(x-y)^2} y'_\nu (x_\mu - y_\mu) \\ &\quad - 2\frac{(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x-y)^2} + 8\frac{(x^2 - x\cdot y)(y^2 - x \cdot y)}{(x-y)^2} x'_\mu y'_\nu \end{aligned}$$

There are 8 terms in the expansion. We group them in the following way:

- Coefficients of $x_\mu x_\nu$:
$$-\frac{2}{x^2} + \frac{4(x^2 - x\cdot y)}{x^2(x-y)^2} - \frac{2}{(x-y)^2} = -\frac{2y^2}{x^2(x-y)^2}.$$
- Coefficients of $y_\mu y_\nu$:
$$-\frac{2}{y^2} - \frac{4(y^2 - x\cdot y)}{y^2(x-y)^2} - \frac{2}{(x-y)^2} = -\frac{2x^2}{y^2(x-y)^2}.$$
- Coefficient of $x_\nu y_\mu$:
$$\frac{2}{(x-y)^2}.$$
- Coefficients of $x_\mu y_\nu$:
$$\frac{4x\cdot y}{x^2 y^2} - \frac{4(x^2 - x\cdot y)}{x^2(x-y)^2} - \frac{4(y^2 - x \cdot y)}{y^2(x-y)^2} + \frac{2}{(x-y)^2} + \frac{8(x^2 - x\cdot y)(y^2 - x\cdot y)}{x^2y^2(x-y)^2} = \frac{2}{(x-y)^2}.$$

Therefore the original expression simplifies to

$$\begin{aligned} &\eta_{\mu\nu} - \frac{2(y^4 x_\mu x_\nu + x^4 y_\mu y_\nu - x^2y^2 x_\mu y_\nu - x^2 y^2 x_\nu y_\mu)}{x^2 y^2 (x - y^2)} \\ &= \eta_{\mu\nu} - \frac{2(y^2 x_\mu - x^2 y_\mu)(y^2 x_\nu - x^2 y_\nu)}{x^2 y^2 (x-y)^2} \\ &= \eta_{\mu\nu} - 2\frac{(x'_\mu - y'_\mu)(x'_\nu - y'_\nu)}{(x'-y')^2} \\ &= I_{\mu\nu}(x' - y'). \end{aligned}$$

OK

We have thus shown that this is equal to the right-hand side of the claimed equality. (*It took me an hour to do this computation...*) □

By Q3, for any conformal transformation, $(x'-y')^2 = \Omega(x)\Omega(y)(x-y)^2$

$$\frac{I_{\mu\nu}(x-y)}{(x-y)^2} = -\frac{1}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\log(x-y)^2 \Rightarrow \frac{I_{\mu\nu}(x'-y')}{(x'-y')^2} \overset{!}{=} -\frac{1}{2}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial y'^\nu}\ln(x-y)^2 = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial y^\beta}{\partial y'^\nu}\frac{I_{\alpha\beta}(x-y)}{(x-y)^2}$$

$$= R^\alpha{}_\mu(x) I_{\alpha\beta}(x-y) R^\beta{}_\nu(y)$$

$(R_{\alpha\mu}(x) = I_{\alpha\mu}(x)$ for inversion$)$

Question 5

Consider the OPE $\phi_1(x)\phi_2(0)$ and the contribution to this OPE from a single primary $\phi_\Delta(0)$ plus all its tower of descendants:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} \left(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \cdots\right)|0\rangle$$

In the lectures it has been shown how to fix α by acting on both sides with K_μ. By following the same idea compute the next orders in the above expansion, quadratic in x.

Proof. The OPE of $\phi_1(x)\phi_2(0)\,|0\rangle$ in the second order is given by

You are missing a term in the ansatz!

$$\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} \left(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \beta x^\mu x^\nu \partial_\mu \partial_\nu \phi_\Delta(0) + \cdots\right)|0\rangle\,. \qquad (*)$$

From the lectures we know that $\alpha = \dfrac{\Delta_1+\Delta_2-\Delta}{2\Delta}$. To compute β, we again let K_μ acts on the expression. The second order term of $K_\mu\phi_1(x)\phi_2(0)\,|0\rangle$ is given by

$$\mathrm{i}\alpha(\Delta_1+\Delta_2-\Delta)\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} x_\mu x^\nu \partial_\nu \phi_\Delta(0)\,|0\rangle\,.$$

The second order term of K_μ acting on equation $(*)$ is given by

$$\begin{aligned} -\beta\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} x^\nu x^\rho K_\mu P_\nu P_\rho \phi_\Delta(0)\,|0\rangle &= -2\mathrm{i}\beta\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} x^\nu x^\rho \left(\eta_{\mu\rho}P_\nu D + \eta_{\mu\nu} D P_\rho\right)\phi_\Delta(0)\,|0\rangle \\ &= -2\mathrm{i}\beta\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} x^\nu x^\rho \left(\eta_{\mu\rho}P_\nu D + \eta_{\mu\nu} P_\rho D + \mathrm{i}\eta_{\mu\nu}P_\rho\right)\phi_\Delta(0)\,|0\rangle \\ &= 2\mathrm{i}\beta\frac{\text{const}}{|x|^{\Delta_1+\Delta_2-\Delta}} (2\Delta+1) x_\mu x^\nu \partial_\nu \phi_\Delta(0)\,|0\rangle \end{aligned}$$

Therefore we must have

$$\beta = \frac{\alpha(\Delta_1+\Delta_2-\Delta)}{2(2\Delta+1)} = \frac{(\Delta_1+\Delta_2-\Delta)^2}{4\Delta(\Delta+1)}.$$

□

Hiding too many steps and final result is wrong

Question 6

Rederive the results above by considering appropriate two and three point functions in a conformal field theory, following the "practical method" described in the lecture notes.

Proof. Suppose that the OPE of $\phi_1(x)\phi_2(0)$ takes the form

$$\phi_1(x)\phi_2(0) = C_\Delta(x,\partial)\phi_\Delta(0).$$

From the notes, the 3-point correlation function is given by

$$\begin{aligned} \langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle &= C_{12\Delta}C_\Delta(x,\partial)\,\langle\phi_\Delta(y)\phi_\Delta(z)\rangle\,\big|_{y=0} \\ &= \frac{C_{12\Delta}}{|x|^{\Delta_1+\Delta_2-\Delta}\cdot|z|^{\Delta_2+\Delta-\Delta_1}\cdot|x-z|^{\Delta_1+\Delta-\Delta_2}}. \end{aligned}$$

The 2-point correlation function is given by

$$\langle\phi_\Delta(y)\phi_\Delta(z)\rangle = \frac{1}{|y-z|^{2\Delta}}.$$

Incomplete

□

Operator Product Expansion

- They converge in CFT but in general QFT !
- State-operator correspondence
- $\varphi_1(x)\varphi_2(0) = \sum_{\mathcal{O}} \sum_{m=0}^{\infty} \frac{C_{12\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta}} \alpha_m(x,\partial_y)\mathcal{O}(y)\Big|_{y=0}$

$\mathcal{O}$ & $C_{12\mathcal{O}}$ comes from specific CFTs

Q5.

$\alpha_0 = 1$; $\alpha_1 = \alpha x^\mu \partial_\mu$ $\quad \alpha_2 = \beta x^\mu x^\nu \partial_\mu \partial_\nu + \gamma x^2 \partial_\mu \partial^\mu$

↑ spin 1 rep. of rotation

$$K_\mu \varphi_1(x)\varphi_2(0)|0\rangle = [K_\mu, \varphi_1(x)]\varphi_2(0)|0\rangle$$
$$= \underbrace{(2i\Delta x_\mu \varphi_1(x) + 2i x_\mu x^\nu \partial_\nu \varphi_1(x) - i x^2 \partial_\mu \varphi_1(x))}_{=:\eta_\mu \varphi_1(x)} \varphi_2(0)|0\rangle$$

$[K_\mu, K_\nu] = 0$

$$\Rightarrow K_\mu K_\nu \varphi_1(x)\varphi_2(0)|0\rangle = \eta_\mu \eta_\nu \varphi_1(x)\varphi_2(0)|0\rangle$$
$$= -\frac{\Delta_1+\Delta_2-\Delta}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(-x^2\eta_{\mu\nu} + x_\mu x_\nu(2+\Delta_1-\Delta_2+\Delta) + \cdots\right)$$

$$K_\mu K_\nu\left(\frac{C_{12\mathcal{O}}}{|x|^{\cdots}}(1+\alpha x^\mu P_\mu + \beta x^\mu x^\nu P_\mu P_\nu + \cdots)|0\rangle\right)$$
$$= -\frac{C_{12\mathcal{O}}}{|x|^{\cdots}}\left(8\Delta(\Delta+1)\beta x_\mu x_\nu + (-4\beta\Delta + 8\Delta(\Delta+1) - 4\gamma\Delta\alpha)x^2\eta_{\mu\nu} + \cdots\right)$$

$$\Rightarrow \beta = \frac{(\Delta_1-\Delta_2+\Delta)(2+\Delta_1-\Delta_2+\Delta)}{8\Delta(\Delta+1)}, \quad \gamma = \cdots\cdots$$

Q6.

$$G_3(x,z) := \langle \varphi_1(x)\varphi_2(0)\varphi_3(z)\rangle = \sum_{\Delta'} C_{12\Delta'}\alpha_{\Delta'}(x,\partial)\langle \varphi_{\Delta'}(y)\varphi_\Delta(z)\rangle\Big|_{y=0}$$
$$= \sum_{\Delta} C_{12\Delta}\alpha_\Delta(x,\partial)\frac{1}{|y-z|^{2\Delta}}\Big|_{y=0}$$

Expand 2-pt function :

$$\frac{1}{|y-z|^{2\Delta}} = \frac{1}{|z|^{2\Delta}}\left(1 + 2\Delta\frac{y\cdot z}{|z|^2} - \Delta\cdot\frac{|y|^2}{|z|^2} + \cdots\right)$$

$0\,\partial$: $\frac{1}{|z|^b}$, $(b = \Delta_2+\Delta-\Delta_1)$, $\quad 1\,\partial$: $\frac{x_\mu z^\mu}{|z|^{b+2}}$, $\quad 2\,\partial$: $\cdots\cdots$

USE MATHEMATICA !