Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3

C7.6: General Relativity II

23 February, 2022

Section A: Introductory

Question 1. Null hypersurfaces

Let (M, g) be a $(d+1)$-dimensional Lorentzian manifold and let $f : M \to \mathbb{R}$ be a function such that for $f^{-1}(0) =: \Sigma$ we have $df \neq 0$ on Σ, i.e. Σ is a hypersurface. Let now Σ be a null hypersurface, i.e. we additionally impose the null condition $g^{-1}(df, df)\big|_\Sigma = 0$ on its normal covector field df.

Show/recall that one can locally introduce coordinates $\{y^0, \dots, y^d\}$ such that $x^0 = f$. Let $g_{\mu\nu}$ be the components of g in these coordinates. Show that $\det\left(\{g_{ij}\}_{i,j=1}^d\right)\Big|_\Sigma = 0$.

Section B: Core

Question 2. Point mass

Consider the linearised Einstein equations in the wave gauge

$$\Box\overline{\widetilde{h}}_{\mu\nu} = -16\pi T^{(1)}_{\mu\nu}, \quad \partial^\mu \overline{\widetilde{h}}_{\mu\nu} = 0$$

where we recall that that $\overline{\widetilde{h}}_{\mu\nu} = \widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}$, where η is the Minkowski metric. Now consider a point mass (modelling a spherically symmetric body) of mass εM which is at rest at the coordinate origin $x = y = z = 0$. The corresponding stress-energy tensor is given by $T_{\mu\nu} = \varepsilon T^{(1)}_{\mu\nu} = \varepsilon M \delta^3(\boldsymbol{x}) U_\mu U_\nu$, where $U = \partial_t$ is the four-velocity of the particle. Derive the external gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\widetilde{h}_{\mu\nu}$ under the assumption that it is stationary and compare it to the Schwarzschild metric with mass εM in the region where $\dfrac{\varepsilon M}{\sqrt{x^2 + y^2 + z^2}}$ is small.

Proof. *As no confusion shall arise, we use $h_{\mu\nu}$ to denote the perturbed metric $\widetilde{h}_{\mu\nu}$ in harmonic gauge.*

Since the point mass is at rest, the stress-energy tensor $T^{(1)}_{\mu\nu} = M U_\mu U_\nu \delta^3(\boldsymbol{x})$ satisfies $T^{(1)}_{00} = M\delta^3(\boldsymbol{x})$ and $T^{(1)}_{0i} = T^{(1)}_{jk} = 0$. Since the gravitational field is stationary, $\partial_0 h_{\mu\nu} = 0$ and hence $\Box\overline{h}_{\mu\nu} = \nabla^2\overline{h}_{\mu\nu}$. The linearised Einstein equations become

$$\nabla^2\overline{h}_{00} = -16\pi M\delta^3(\boldsymbol{x}), \qquad \nabla^2\overline{h}_{0i} = 0, \qquad \nabla^2\overline{h}_{jk} = 0, \qquad \partial^i\overline{h}_{i\nu} = 0$$

By imposing the asymptotic boundary condition at infinity, we immediately observe that $\overline{h}_{0i} = 0$ and $\overline{h}_{jk} = 0$. The remaining non-trivial equations are $\nabla^2\overline{h}_{00} = -16\pi M\delta^3(\boldsymbol{x})$. With the boundary condition $\overline{h}_{00}(\boldsymbol{x}) \to 0$ as $\|\boldsymbol{x}\| \to \infty$, the solution is given by

$$\overline{h}_{00}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{-16\pi M\delta^3(\boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|}\,\mathrm{d}^3\boldsymbol{x}' = \frac{4M}{\|\boldsymbol{x}\|}$$

Then

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\overline{h}_{\rho\sigma} = \begin{cases} \dfrac{2M}{\|\boldsymbol{x}\|} & \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

Therefor the full linearised metric is given by

$$\begin{aligned} g = \eta + \varepsilon h &= -\left(1 - \varepsilon\frac{2M}{\|\boldsymbol{x}\|}\right)\mathrm{d}t^2 + \left(1 + \varepsilon\frac{2M}{\|\boldsymbol{x}\|}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \\ &= -\left(1 - \varepsilon\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 + \varepsilon\frac{2M}{r}\right)(\mathrm{d}r^2 + r^2\Omega) \end{aligned}$$

where Ω is the round metric of S^2. Recall that the Schwarzschild metric in the isotropic coordinates is given by

$$g_{\text{Schwarzschild}} = -\left(\frac{1-\varepsilon M/2r}{1+\varepsilon M/2r}\right)^2 \mathrm{d}t^2 + \left(1+\varepsilon\frac{M}{2r}\right)^4 (\mathrm{d}r^2 + r^2\Omega)$$

When $\varepsilon M/r \ll 1$, we can approximate the second term to obtain

$$g_{\text{Schwarzschild}} \simeq -\left(1-\varepsilon\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1+\varepsilon\frac{2M}{r}\right)(\mathrm{d}r^2 + r^2\Omega)$$

which agrees with the result for point mass above. □

Question 3. Quadrupole curvature

According to the Quadrupole formula the asymptotically leading order terms of the metric corrections $\widetilde{h}_{\mu\nu}$ of a radiating system in wave gauge take the form

$$\begin{aligned}
\overline{\widetilde{h}}_{00}(t,\boldsymbol{x}) &\simeq \frac{4M}{r} + \frac{2x_i x_k}{r^3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ik}(t-r),\\
\overline{\widetilde{h}}_{0i}(t,\boldsymbol{x}) &\simeq -\frac{2x^k}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ik}(t-r),\\
\overline{\widetilde{h}}_{ij}(t,\boldsymbol{x}) &\simeq \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ij}(t-r).
\end{aligned}$$

Recall that $\overline{h}_{\mu\nu} = \widetilde{h}_{\mu\nu} - \eta_{\mu\nu}\widetilde{h}/2$, where η is the Minkowski metric. Compute the metric corrections $\widetilde{h}_{\mu\nu}$ and show that the curvature component $R^i{}_{00j}$ of $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\widetilde{h}_{\mu\nu}$ to leading order in $0 < \varepsilon \ll 1$ and to leading order in $\frac{1}{r}$ is given by

$$R^i{}_{00j}(t,\boldsymbol{x}) \simeq \frac{\varepsilon}{r}\left[\Pi^m{}_i\Pi^n{}_j - \frac{1}{2}\Pi^{mn}\Pi_{ij}\right]\frac{\mathrm{d}^4}{\mathrm{d}t^4}Q_{mn}(t-r)$$

where $\Pi^{mn} = \delta^{mn} - x^m x^n/r^2$. What is the interpretation of $\Pi^m{}_n$?

Proof. As no confusion shall arise, we use $h_{\mu\nu}$ to denote the perturbed metric $\widetilde{h}_{\mu\nu}$ in harmonic gauge.

The trace of $\overline{h}$ is given by

$$\begin{aligned}
\operatorname{tr}\overline{h} = -\overline{h}_{00} + \overline{h}_{ii} &= -\frac{4M}{r} - \frac{2x^i x^j}{r^3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ij}(t-r) + \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ii}(t-r)\\
&= -\frac{4M}{r} + \frac{2(r^2\delta^{ij} - x^i x^j)}{r^3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ij}(t-r)
\end{aligned}$$

Then

$$\begin{aligned}
h_{00} &= \overline{h}_{00} + \frac{1}{2}\operatorname{tr}\overline{h} = \frac{1}{2}(\overline{h}_{00} + \overline{h}_{ii}) = \frac{2M}{r} + \frac{r^2\delta^{ij} + x^i x^j}{r^3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ij}(t-r)\\
\text{(no sum over } k\text{)}\quad h_{kk} &= \overline{h}_{kk} - \frac{1}{2}\operatorname{tr}\overline{h} = \frac{2M}{r}\left(-\frac{r^2\delta^{ij} - x^i x^j}{r^3} + \frac{2\delta^i_k\delta^j_k}{r}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q_{ij}(t-r)\\
h_{\mu\nu} &= \overline{h}_{\mu\nu} \qquad (\mu \neq \nu)
\end{aligned}$$

From the notes we have the following formula for the Riemann curvature:

$$R^\mu{}_{\kappa\rho\nu} = \varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_\rho\partial_\kappa h_{\nu\sigma} - \partial_\rho\partial_\sigma h_{\nu\kappa} - \partial_\nu\partial_\kappa h_{\rho\sigma} + \partial_\nu\partial_\sigma h_{\rho\kappa}\right) + \mathcal{O}\left(\varepsilon^2\right)$$

So, to the leading order of ε we have

$$R^i{}_{00j} \simeq \frac{1}{2}\varepsilon\left(\partial_0^2 h_{ij} + \partial_i\partial_j h_{00} - \partial_0\partial_j h_{0i} - \partial_0\partial_i h_{0j}\right)$$

Note that $h_{\mu\nu} = \mathcal{O}(r^{-1})$ and $\partial_i h_{\mu\nu} = \mathcal{O}(r^{-2})$. So to the order of $\mathcal{O}(r^{-1})$ we simply have

$$\begin{aligned} R^i{}_{00j} &\simeq \frac{1}{2}\varepsilon\partial_0^2 h_{ij} \\ &= \frac{\varepsilon}{r}\left(\delta_i^m\delta_j^n - \frac{1}{2}\delta_{ij}\left(\delta^{mn} - \frac{x^m x^n}{r^2}\right)\right)\frac{\mathrm{d}^4}{\mathrm{d}t^4}Q_{mn}(t-r) \\ &= \frac{\varepsilon}{r}\left(\delta_i^m\delta_j^n - \frac{1}{2}\delta_{ij}\Pi^{mn}\right)\frac{\mathrm{d}^4}{\mathrm{d}t^4}Q_{mn}(t-r) \end{aligned}$$

□

The approximation above is incorrect. For an expression of the form $f(x_i, x_j, r)\ddot{Q}_{ij}(t-r)$, the spatial derivative to the leading order is given by

$$\partial_i f(x_i, x_j, r)\ddot{Q}_{ij}(t-r) \sim f(x_i, x_j, r)\partial_i\ddot{Q}_{ij}(t-r) = -\frac{x_i}{r}f(x_i, x_j, r)\ddot{Q}_{ij}(t-r)$$

So each spatial derivative ∂_i acts on $\ddot{Q}_{ij}(t-r)$ as $-\frac{x_i}{r}\frac{\mathrm{d}}{\mathrm{d}t}$ and each time derivative ∂_t acts as $\frac{\mathrm{d}}{\mathrm{d}t}$. Then we have

$$R^i{}_{00j} \sim \frac{\varepsilon}{2}\left(\partial_0^2 h_{ij} + \frac{x_i x_j}{r^2}\partial_0^2 h_{ij} + \frac{x_j}{r}\partial_0^2 h_{0i} + \frac{x_i}{r}\partial_0^2 h_{0j}\right)$$

Expanding the expression shall produce the correct form of the Riemann curvature.

$\Pi(\boldsymbol{x}) : \mathrm{T}_{\boldsymbol{x}}\mathbb{R}^3 \to \mathrm{T}_{\boldsymbol{x}}\partial B(0, \boldsymbol{x})$ *is the push-forward of the orthogonal projection* $\pi : \mathbb{R}^3 \to \partial B(0, \boldsymbol{x})$.

Question 4. Gravitational radiation from a binary

Consider two stars, each of mass m and modelled as point particles, moving in a circular *Newtonian* orbit of radius R in the (x, y)-plane centred at the origin.

(a) Show that their trajectories may be taken to be $\gamma_\pm(t) = (t, \pm R\cos(\omega t), \pm R\sin(\omega t), 0)$, where $\omega^2 = \frac{m}{4R^3}$.

(b) Consider the corresponding stress-energy tensor

$$T^{\mu\nu} = \sum_{a=\pm} m \int \dot{\gamma}_a^\mu \dot{\gamma}_a^\nu \delta^4\left(x - \gamma_a(\tau)\right) \mathrm{d}\tau$$

where τ is proper time of the particles and $\dot{\gamma}_a^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma_a^\mu$. Compute the stress-energy tensor and the quadrupole moment in the slow-motion approximation.

(c) Recall that our derivation of the quadrupole formula required the system to be non-self-gravitating, which is violated in this scenario. However, assume now that it still serves as a good approximation even in the weakly-self-gravitating case if all other assumptions made in its derivation are met. What restrictions does this impose on the parameters m and R?

Consider the stars to be each of one solar mass $\approx 2 \cdot 10^{30}$ kg. Give an order of magnitude estimate on R for which we might expect the quadrupole formula to be a good approximation.

(d) Compute the metric corrections $\bar{\bar{h}}_{\mu\nu}$ according to the quadrupole formula.

Proof. (a) We can solve the trajectories purely in Newtonian mechanics. We work in the centre-of-mass frame and use the polar coordinates. The Lagrangian of one of the star is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m^2}{r}$$

Subsituting into the Euler-Lagrange equations we obtain the equations of motion:

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{m^2}{(2r)^2}, \qquad r^2\dot{\theta} = \text{const}$$

In a circular orbit $r = R = \text{const}$. So $\omega^2 = \dot{\theta}^2 = \frac{m}{4R^3}$. Then the orbit is given by

$$\gamma_+(t) = \left(t, R\cos\left(\sqrt{\frac{m}{4R^3}}t\right), R\sin\left(\sqrt{\frac{m}{4R^3}}t\right), 0\right)$$

As the centre of mass is at rest, $\gamma_+(t)^i + \gamma_-(t)^i = 0$. Then

$$\gamma_-(t) = \left(t, -R\cos\left(\sqrt{\frac{m}{4R^3}}t\right), -R\sin\left(\sqrt{\frac{m}{4R^3}}t\right), 0\right)$$

(b) We have $\mathrm{d}\gamma_\pm/\mathrm{d}t = (1, \mp R\omega\sin(\omega t), \pm R\omega\cos(\omega t), 0)$ and $\eta(\mathrm{d}\gamma_\pm/\mathrm{d}t, \mathrm{d}\gamma_\pm/\mathrm{d}t) = -1 + R^2\omega^2$. Then

$$\dot{\gamma}_\pm(t) = \frac{1}{\sqrt{1 - m/4R}}(1, \mp R\omega\sin(\omega t), \pm R\omega\cos(\omega t), 0)$$

where $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - m/4R}}$ The non-zero stress-energy tensor components is given by

$$T^{00} = \sum_s \frac{m}{\sqrt{1 - m/4R}} \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t \qquad T^{01} = -\sum_s \frac{sR\omega m}{\sqrt{1 - m/4R}} \sin(\omega t) \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t$$

$$T^{02} = \sum_s \frac{sR\omega m}{\sqrt{1 - m/4R}} \cos(\omega t) \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t \quad T^{12} = -\sum_s \frac{R^2\omega^2 m}{\sqrt{1 - m/4R}} \sin(\omega t)\cos(\omega t) \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t$$

$$T^{11} = \sum_s \frac{R^2\omega^2 m}{\sqrt{1 - m/4R}} \sin^2(\omega t) \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t \quad T^{22} = \sum_s \frac{R^2\omega^2 m}{\sqrt{1 - m/4R}} \cos^2(\omega t) \int_{\mathbb{R}} \delta^4(x - \gamma_s(t))\,\mathrm{d}t$$

In the slow-motion approximation we just neglect the terms with $R\omega \ll 1$ (and $\mathrm{d}/\mathrm{d}\tau \sim \mathrm{d}/\mathrm{d}t$), so the only non-zero commponent of the stress-energy tensor is:

$$T^{00}(t, \boldsymbol{x}) = m\sum_a \int_{\mathbb{R}} \delta^4(x - \gamma_a(t'))\,\mathrm{d}t'$$

The quadrupole moment is given by

$$\begin{aligned} Q_{ij}(t) &= \int_{\mathbb{R}^3} T_{00}(x')x'_i x'_j \,\mathrm{d}^3\boldsymbol{x}' \\ &= \sum_s \frac{m}{\sqrt{1 - m/4R}} \int_{\mathbb{R}^3}\int_{\mathbb{R}} x'_i x'_j \delta^4(x - \gamma_s(t'))\,\mathrm{d}t'\,\mathrm{d}^3\boldsymbol{x}' \\ &= \sum_s \frac{m}{\sqrt{1 - m/4R}} \gamma_s(t)_i \cdot \gamma_s(t)_j \end{aligned}$$

The non-zero components are given by

$$Q_{11}(t) = \frac{2mR^2}{\sqrt{1-m/4R}}\cos^2(\omega t), \quad Q_{12}(t) = \frac{2mR^2}{\sqrt{1-m/4R}}\cos(\omega t)\sin(\omega t), \quad Q_{22}(t) = \frac{2mR^2}{\sqrt{1-m/4R}}\sin^2(\omega t)$$

Just neglect the term with $m/4R$.

(c) The quadrupole formula requires that $R\omega = \sqrt{m/4R} \ll 1$. For $m \sim 2 \cdot 10^{30}$ kg, if we require that $\sqrt{m/4R} \sim 10^{-2}$, then $R \sim 10^{34}$ kg $\sim 10^7$ m. This is an extremely small length scale, so we expect that the approximation is good for most of the time.

(d) Just plug the expressions into the quadrupole formula... □

Question 5. Kerr spacetime

Let $M = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$ with the standard $\{t, r, \theta, \varphi\}$ coordinates, where $r_+ = M + \sqrt{M^2 - a^2}$, $M > 0$, and $0 < a < M$. We define the Kerr metric g on M by

$$\begin{aligned} g = &-\left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 - \frac{2Mra\sin^2\theta}{\rho^2}(\mathrm{d}t \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}t) + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 \\ &+ \rho^2\,\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2 \end{aligned}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Show that the vector field ∂_t is a Killing vector field and that it is timelike for $r > M + \sqrt{M^2 - a^2\cos^2\theta}$. Also show that ∂_t is not hypersurface orthogonal. Thus (M, g) is stationary but not static.

Proof. Note that $\partial_t g_{\mu\nu} = 0$. Then

$$\begin{aligned} (\mathcal{L}_{\partial_t} g)_{\mu\nu} &= (\mathcal{L}_{\partial_t} g)(\partial_\mu, \partial_\nu) = (\mathcal{L}_{\partial_t} g(\partial_\mu, \partial_\nu)) - g(\mathcal{L}_{\partial_t}\partial_\mu, \partial_\nu) - g(\partial_\mu, \mathcal{L}_{\partial_t}\partial_\nu) \\ &= \partial_t g_{\mu\nu} - g([\partial_t, \partial_\mu], \partial_\nu) - g(\partial_\mu, [\partial_t, \partial_\nu]) \\ &= 0 \end{aligned}$$

Hence ∂_t is a Killing vector field. Next,

$$\begin{aligned} g(\partial_t, \partial_t) = -\left(1 - \frac{2Mr}{\rho^2}\right) = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right) < 0 &\iff 2Mr < r^2 + a^2\cos^2\theta \\ &\iff r > M + \sqrt{M^2 - a^2\cos^2\theta} \end{aligned}$$

So ∂_t is timelike for $r > M + \sqrt{M^2 - a^2\cos^2\theta}$. To show that ∂_t is not hypersurface orthogonal, by Proposition 1.34 it suffices to show that $\partial_t^\flat \wedge \mathrm{d}\partial_t^\flat \neq 0$. We have

$$(\partial_t^\flat)_\mu = (\partial_t)^\nu g_{\mu\nu} = g_{\mu 0}$$

So

$$\partial_t^\flat = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t - \frac{2Mra\sin^2\theta}{\rho^2}\mathrm{d}\varphi$$

Expanding in the coordinates, we have

$$\partial_t^\flat \wedge \mathrm{d}\partial_t^\flat = (g_{tt}\partial_r g_{t\varphi} - g_{t\varphi}\partial_r g_{tt})\,\mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\varphi + (g_{tt}\partial_\theta g_{t\varphi} - g_{t\varphi}\partial_\theta g_{tt})\,\mathrm{d}t \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi$$

By direct computation we can show that this is nonzero. So ∂_t is not hypersurface orthogonal. We conclude that the Kerr spacetime is stationary but not static. □

Question 6. Electromagnetic radiation

This question recalls the derivation of electromagnetic radiation and thus makes explicit the differences and similarities between the gravitational and electromagnetic cases. Let (M, g) be the $(3+1)$-dimensional Minkowski spacetime with canonical coordinates x^μ. Maxwell's equation read $\mathrm{d}F = 0$ and $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$, where J is the source 4 -vector and F is the Faraday tensor, an antisymmetric 2-form.

(a) Show that J satisfies the conservation law $\partial_\mu J^\mu = 0$ by virtue of the Maxwell equations.

(b) We now consider a source J that is compactly supported in space. Show that the charge $Q(t) := \displaystyle\int_{\mathbb{R}^3} J^0(t, \boldsymbol{x})\,\mathrm{d}^3x$ is independent of time.

(c) Consider now the Maxwell equations in the Lorenz gauge

$$\Box\widetilde{A}_\mu = 4\pi J_\mu, \quad \partial^\mu\widetilde{A}_\mu = 0$$

where $\widetilde{A}$ is a 1-form such that $\mathrm{d}\widetilde{A} = F$. Write the solution specified by the boundary condition of no incoming radiation at $t \to -\infty$ and provide an explicit verification of the Lorentz gauge constraint.

(d) Follow the approach taken in the gravitational case in the lectures to derive the following expression for the far field (i.e. for large $r = |\boldsymbol{x}|$)

$$\widetilde{A}_0(t, \boldsymbol{x}) \simeq \frac{Q}{r} + \frac{x_i}{r^2}\frac{\mathrm{d}}{\mathrm{d}t}D_i(t-r), \quad \widetilde{A}_i(t, \boldsymbol{x}) \simeq -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}D_i(t-r), \tag{1}$$

where $D_i(t) := \int_{\mathbb{R}^3} J^0(t, \boldsymbol{x})x_i\,\mathrm{d}^3x$ is the electric dipole moment. Spell out the assumptions under which the above approximation is valid.

Proof. (a) Since F is anti-symmetric, we have

$$\partial_\mu J^\mu = \frac{1}{4\pi}\partial_\nu\partial_\mu F^{\mu\nu} = 0$$

So J is a conserved current.

(b) Since J is compactly supported, for sufficiently large $R > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Q(t) &= \int_{\mathbb{R}^3}\frac{\partial J^0}{\partial t}\,\mathrm{d}^3\boldsymbol{x} = -\int_{\mathbb{R}^3}\frac{\partial J^i}{\partial x^i}\,\mathrm{d}^3\boldsymbol{x} = -\int_{B(0,R)}\frac{\partial J^i}{\partial x^i}\,\mathrm{d}^3\boldsymbol{x} \\ &= -\oint_{\partial B(0,R)}\varepsilon_{ijk}J^i\mathrm{d}x^j\wedge\mathrm{d}x^k = 0\end{aligned}$$

(c) (*Again there is no need to put tilde on the 4-potential.*)

The retarded Green's function G such that $\Box G(x, x') = \delta(x - x')$ is given by

$$G(x, x') = \frac{1}{4\pi\|\boldsymbol{x} - \boldsymbol{x}'\|}\delta\big((t - t') - \|\boldsymbol{x} - \boldsymbol{x}'\|\big)$$

Then the solution to the sourced wave equation is given by

$$A_\mu(x) = \int_{M^4}\frac{J_\mu(x')}{\|\boldsymbol{x} - \boldsymbol{x}'\|}\delta\big((t - t') - \|\boldsymbol{x} - \boldsymbol{x}'\|\big)\,\mathrm{d}^4x' = \int_{M^4}\frac{J_\mu(t - \|\boldsymbol{x} - \boldsymbol{x}'\|, \boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|}\,\mathrm{d}^3\boldsymbol{x}$$

The derivative of the 4-potential is given by

$$\begin{aligned}\partial^\mu A_\mu &= \int_{M^4} 4\pi \left(\frac{\partial J_\mu(x')}{\partial (x')_\mu} G(x,x') + J_\mu(x') \frac{\partial G(x,x')}{\partial (x')_\mu} \right) \mathrm{d}^4 x' \\ &= \int_{M^4} 4\pi J_\mu(x') \frac{\partial G(x,x')}{\partial (x')_\mu} \mathrm{d}^4 x' \\ &= 0\end{aligned}$$

where we have used the conservation equation and the fact the Green's function $G(x,x') = \widetilde{G}(x-x') = \widetilde{G}(x'-x)$ only depends on $\|\boldsymbol{x} - \boldsymbol{x}'\|$ [1]. Hence A indeed satisfies the condition for the Lorenz gauge. [2]

(d)

□

Remark. A rough correspondence between electromagnetism and general relativity:

	EM	GR
Potential	4-potential A	metric g
Gauge freedom	$A \mapsto A + \mathrm{d}\xi$	diffeomorphisms
Observable	Faraday tensor $F = \mathrm{d}A$	Riemann curvature $R(X,Y)Z$
Derivative	1st	2nd

Section C: Optional

Question 7. Electromagnetic radiation

This is a continuation of problem 6 above, i.e. we consider electromagnetic radiation in the electric dipole approximation (1).

(a) Compute to the leading order the electric and magnetic field in the far region and show that they (the leading orders) are orthogonal.

(b) Consider a point particle with charge e. The associated current 4-vector is given by $J^\mu(x) = e \int \delta^4(x - \gamma(\tau))\dot{\gamma}^\mu(\tau)\,\mathrm{d}\tau$, where τ is the proper time of the particle and $\dot{\gamma}^\mu = \mathrm{d}\gamma^\mu/\mathrm{d}\tau$. Show that $\partial_\mu J^\mu = 0$ in the sense of distributions.

(c) Show that there is in fact no need to restrict oneself to the proper time, i.e. that $J^\mu(x) = e \int \delta^4(x - \gamma(s))\dot{\gamma}^\mu(s)\,\mathrm{d}s$, where s is an arbitrary curve parameter. Now consider the point particle to oscillate as

$$t \overset{\gamma}{\mapsto} (t, 0, 0, L\sin(\omega t)),$$

and compute its *electric dipole moment* $D_i(t)$.

[1] As suggested by Question 7, these expressions hold in the sense of distribution.

[2] It is **Lorenz** gauge, not Lorentz gauge! These are two different people!