Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 2

ASO: Special Relativity

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Question 1. Lorentz transformations and velocity.

Let O and O' be two non-accelerating observers whose inertial coordinate systems are related by a proper orthochronous Lorentz transformation

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = L \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}.$$

Show that the Lorentz transformation matrix L must be of the form

$$\begin{pmatrix} \gamma & -\gamma v_1'/c & -\gamma v_2'/c & -\gamma v_3'/c \\ \gamma v_1/c & * & * & * \\ \gamma v_2/c & * & * & * \\ \gamma v_3/c & * & * & * \end{pmatrix}$$

where $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity of the observer O' in frame O, $\boldsymbol{v}' = (v_1', v_2', v_3')$ is the velocity of the observer O in frame O', and $\gamma = \gamma(v) = \gamma(v')$.

Proof. Suppose that L is a proper orthochronous Lorentz transformation. We write it in the block matrix form

$$L = \begin{pmatrix} \gamma & \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b} & M \end{pmatrix}$$

where $\gamma > 0$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, and $M \in M_{3\times 3}(\mathbb{R})$.

The Minkowski space has metric tensor $g = \mathrm{diag}(1, -1, -1, -1)$. Since $L^{\mathrm{T}} g L = g$, we have:

$$\begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & -I_3 \end{pmatrix} = \begin{pmatrix} \gamma & \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{a} & M^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & -I_3 \end{pmatrix} \begin{pmatrix} \gamma & \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b} & M \end{pmatrix} = \begin{pmatrix} \gamma^2 - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & \gamma \boldsymbol{a}^{\mathrm{T}} - \boldsymbol{b}^{\mathrm{T}} M \\ \gamma \boldsymbol{a} - M^{\mathrm{T}} \boldsymbol{b} & \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} - M^{\mathrm{T}} M \end{pmatrix}$$

Then we have $\gamma^2 - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} = 1$. So $\|\boldsymbol{b}\| = \sqrt{\gamma^2 - 1}$. Since $\gamma > 0$, there exists $0 < v < c$ such that $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$. Then $\|\boldsymbol{b}\| = \dfrac{v/c}{\sqrt{1 - v^2/c^2}} = \dfrac{\gamma v}{c}$. We write $\boldsymbol{b} = \left(\dfrac{\gamma v_1}{c}, \dfrac{\gamma v_2}{c}, \dfrac{\gamma v_3}{c}\right)$ so that $v^2 = v_1^2 + v_2^2 + v_3^2$.

Note that $g = g^{-1}$. We have:

$$L^{\mathrm{T}} g L = g \implies L^{-1} g^{-1} (L^{-1})^{\mathrm{T}} = g^{-1} \implies L^{-1} g (L^{-1})^{\mathrm{T}} = g \implies g = L g L^{\mathrm{T}}$$

We in fact have a symmetry in $\boldsymbol{a}$ and $\boldsymbol{b}$, which implies that $\|a\| = \sqrt{\gamma^2 - 1} = \dfrac{\gamma v}{c}$. Then we can write $\boldsymbol{a} = \left(-\dfrac{\gamma v_1'}{c}, -\dfrac{\gamma v_2'}{c}, -\dfrac{\gamma v_3'}{c}\right)$ so that $v^2 = (v_1')^2 + (v_2')^2 + (v_3')^2$. In conclusion, L must be in the form

$$\begin{pmatrix} \gamma & -\gamma v_1'/c & -\gamma v_2'/c & -\gamma v_3'/c \\ \gamma v_1/c & * & * & * \\ \gamma v_2/c & * & * & * \\ \gamma v_3/c & * & * & * \end{pmatrix}$$

where $v^2 = v_1^2 + v_2^2 + v_3^2 = (v_1')^2 + (v_2')^2 + (v_3')^2$ and $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$. $\square$

Question 2. Lorentz matrices.

Which of the following matrices represent Lorentz transformations? Which are proper? Which are orthochronous?

$$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix} -2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Solution. We just need to compute $L^{\mathrm{T}} g L$ for each matrix L.

For the first one,

$$L^{\mathrm{T}} g L = \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is a Lorentz transformation. Since $L_0^0 > 0$ and $\det L = 1$, it is proper orthochronous.

For the second one,

$$L^{\mathrm{T}} g L = \frac{1}{2}\begin{pmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is a Lorentz transformation. Since $L_0^0 > 0$ and $\det L = -1$, it is improper orthochronous.

For the third one,

$$L^{\mathrm{T}} g L = \frac{1}{2}\begin{pmatrix} -2 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is a Lorentz transformation. Since $L_0^0 < 0$ and $\det L = -1$, it is improper antichronous.

For the fourth one,

$$L^{\mathrm{T}} g L = \frac{1}{2}\begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is a Lorentz transformation. Since $L_0^0 < 0$ and $\det L = 1$, it is proper antichronous. □

Question 3. Geometry of four-vectors.

Show that

(i) If V is a future-pointing timelike four-vector, then there exists an inertial coordinate system in which it has components $(T, 0, 0, 0)$, where $T = \sqrt{g(V, V)}$.

(ii) If V a future-pointing null four-vector, then there exists an inertial coordinate system in which V has components $(1, 1, 0, 0)$.

(iii) The sum of two future-pointing timelike four-vectors is future-pointing timelike.

(iv) The sum of two future-pointing null four-vectors is future-pointing and eithr timelike or null. Under what condition is the sum null?

(v) Every four-vector pseudo-orthogonal to a timelike vector is spacelike.

Proof. (i) This is Proposition 5 in the notes. Let $V = (ct, \boldsymbol{r})$, and $\boldsymbol{r} = r\boldsymbol{e}_r$. Suppose that $\boldsymbol{e}_r, \boldsymbol{a}, \boldsymbol{b}$ form a orthonormal basis of the Euclidean space $\mathbb{R}^3$. Then we have the following 4-vectors:

$$\frac{1}{\sqrt{c^2t^2 - r^2}}(ct, r\boldsymbol{e}_r), \quad \frac{1}{\sqrt{c^2t^2 - r^2}}(r, ct\boldsymbol{e}_r), \quad (0, \boldsymbol{a}), \quad (0, \boldsymbol{b})$$

which form a pseudo-orthonormal basis of the Minkowshi space. Hence it defines a Lorentz transformation and an inertial coordinate system in which $V = (\sqrt{c^2t^2 - r^2}, 0, 0, 0) = (\sqrt{g(V, V)}, 0, 0, 0)$.

(ii) Let $V = (ct, \boldsymbol{r})$ is null, and $\boldsymbol{r} = r\boldsymbol{e}_r$. Suppose that $\boldsymbol{e}_r, \boldsymbol{a}, \boldsymbol{b}$ form a orthonormal basis of the Euclidean space $\mathbb{R}^3$. Then

$$(ct, 0), \quad (0, r\boldsymbol{e}_r), \quad (0, \boldsymbol{a}), \quad (0, \boldsymbol{b})$$

form a pseudo-orthonormal basis. It defines an inertial coordinate system in which $V = (1, 1, 0, 0)$.

(iii) Suppose that X and Y are time-like 4-vectors. By part (i) there exists an inertial coordinate system such that $X = (X^0, 0, 0, 0)$. In this frame we apply a rotation in the spatial coordinates such that Y takes the form $(Y^0, Y^1, 0, 0)$. We have:

$$g(X + Y, X + Y) = (X^0 + Y^0)^2 - (Y^1)^2 > (Y^0)^2 - (Y^1)^2 > 0$$

Hence $X + Y$ is also time-like.

(iv) *We use the convention that repeated greek indices range from 0 to 3, and that repeated latin indices range from 1 to 3.*

Suppose that X and Y are null 4-vectors. By part (ii) there exists an inertial coordinate system such that $X = (1, 1, 0, 0)$. We have:

$$g(X + Y, X + Y) = g(X, X) + g(Y, Y) + 2g(X, Y) = 2g(X, Y) = 2(Y^0 - Y^1)$$

Since Y is null, we have $Y^0 = \sqrt{Y_a Y^a} \geqslant |Y^1|$. Therefore $g(X + Y, X + Y) \geqslant 0$. $X + Y$ is either time-like or null.

$X + Y$ is null if $Y = (Y^0, Y^1, 0, 0)$ in this inertial coordinate system and $Y^1 > 0$.

(v) **This statement is true for non-zero 4-vectors.**

Suppose that X is time-like and Y is a non-zero 4-vector such that $g(X, Y) = 0$. By part (i) there exists an inertial coordinate system such that $X = (X^0, 0, 0, 0)$. Then

$$g(X, Y) = X_0 Y^0 = 0 \implies Y^0 = 0.$$

Since $Y_a Y^a > 0$, $g(Y, Y) = 0 - Y_a Y^a < 0$. Hence Y is space-like. □

Question 4. A time-like inequality.

Let X and Y be future-pointing, timelike four-vectors, and let $Z = X + Y$. Show that

$$\sqrt{g(Z, Z)} \geqslant \sqrt{g(X, X)} + \sqrt{g(Y, Y)}.$$

When does equality hold? What is the analogous statement in Euclidean geometry?

Now consider two space-time events A and B separated by displacement vector Z, which is future-pointing timelike. One observer travels from A to B in a straight line at constant speed. A second observer travels from A to C with displacement vector X from A in a straight line at constant speed, and then travels from C to B with displacement vector Y from C in a straight line at constant speed, Whose journey from A to B takes longer?

Proof. This is Proposition 8 in the notes.

We continue from Question 3 part (iii). We have shown that Z is time-like. In the inertial coordinate system,

$$\sqrt{g(Z,Z)} = \sqrt{(X^0+Y^0)^2-(Y^1)^2}, \qquad \sqrt{g(X,X)}+\sqrt{g(Y,Y)} = X^0+\sqrt{(Y^0)^2-(Y^1)^2}$$

We have:

$$\begin{aligned} g(Z,Z) &= (X^0+Y^0)^2-(Y^1)^2 = (X^0)^2+(Y^0)^2-(Y^1)^2+2X^0Y^0 \\ &\geqslant (X^0)^2+(Y^0)^2-(Y^1)^2+2X^0\sqrt{(Y^0)^2-(Y^1)^2} \\ &= \left(X^0+\sqrt{(Y^0)^2-(Y^1)^2}\right)^2 \\ &= \left(\sqrt{g(X,X)}+\sqrt{g(Y,Y)}\right)^2 \end{aligned}$$

The equality holds if $Y^0 = \sqrt{(Y^0)^2-(Y^1)^2}$, that is, if $Y^1 = 0$ in the inertial coordinate system.

The inverse inequality is the triangular inequality in a Euclidean space.

The journey $A \to B$ takes longer (in terms of the proper time of the travelers) than $A \to B \to C$. In fact, the proper time between two events is the "length" of the worldline of the traveler. By the inequality we just proved, we deduce that the stright lines are the "longest" curves between two time-like events (which means that stright lines are time-like geodesics in the Minkowski space). □

Question 5. Particle Physics.

A particle of rest mass M and total energy E collides with a particle of rest mass m at rest. Show that the sum E' of the total energies of the two particles in the frame in which their center of mass is at rest is given by

$$E'^2 = (M^2+m^2)c^4 + 2Emc^2.$$

Proof. In the lab frame, M has 4-momentum $P = (E/c, \boldsymbol{p})$ and m has 4-momentum $Q = (mc, \mathbf{0})$. Their total 4-momentum is $P+Q = \left(\frac{E}{c}+mc, \boldsymbol{p}\right)$. In the center of mass frame, the system has zero total 3-momentum. By conservation of 4-momenta, we have:

$$\left\|\left(\frac{E}{c}+mc,\ \boldsymbol{p}\right)\right\| = \left\|\left(\frac{E'}{c},\ \mathbf{0}\right)\right\|$$

Hence

$$(E+mc^2)^2 - p^2c^2 = E'^2$$

But $p^2c^2 = E^2 - M^2c^4$, we conclude that

$$E'^2 = (M^2+m^2)c^4 + 2Emc^2.$$

□

Question 6. Photon scattering.

Suppose that two photons of energies E_1 and E_2 travel towards one another along the x-axis in a fixed ICS. Show that there can be no interaction in which the outcome is a single photon. Argue that the same conclusion holds when the two photons don't necessarily collide head on, but instead collide at a general angle.

Proof. Suppose that the two photons have 4-momenta

$$P = \frac{E_1}{c}(1, \boldsymbol{e}_1) \qquad Q = \frac{E_2}{c}(1, \boldsymbol{e}_2)$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two unit vectors in $\mathbb{R}^3$. Suppose that after interaction they become one photon. Then the generated photon has 4-momentum

$$P + Q = \left(\frac{E_1 + E_2}{c}, \frac{E_1}{c}\boldsymbol{e}_1 + \frac{E_2}{c}\boldsymbol{e}_2\right)$$

Then $E_1 + E_2 = \|E_1\boldsymbol{e}_1 + E_2\boldsymbol{e}_2\|$. This is only possible if $\boldsymbol{e}_1 = \boldsymbol{e}_2$. But then the two photons travel in the same direction before they interact, which is impossible. The outcome of collision of two photons cannot be a single photon. □

Question 7. Four-acceleration.

A particle travels along a straight line in space relative to a given ICS at a not-necessarily-constant speed. Show that

$$g(A, A) = -c^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2,$$

where A is the four-acceleration, s measures proper time, and ϕ is the (instantaneous) rapidity.

Proof. We apply a rotation in the spatial coordinates such that the x-axis aligns with the trajectory of the particle, hence reducing the problem into 2-dimensional case. The 4-velocity of the particle:

$$V = \frac{\mathrm{d}X}{\mathrm{d}s} = \gamma(c, v, 0, 0)$$

where $v := \frac{\mathrm{d}x}{\mathrm{d}t}$ is the velocity of the particle measured in the given inertial coordinate system, and s is the parameter measuring proper time.

The 4-acceleration:

$$A = \frac{\mathrm{d}V}{\mathrm{d}s} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}(c, v, 0, 0) + \gamma\left(0, \frac{\mathrm{d}v}{\mathrm{d}s}, 0, 0\right) = \left(c\frac{\mathrm{d}\gamma}{\mathrm{d}s}, \frac{\mathrm{d}}{\mathrm{d}s}(\gamma v), 0, 0\right)$$

If $\phi = \phi(s)$ is the rapidity, then we have $v(s) = c\tanh\phi(s)$, $\gamma(s) = \cosh\phi(s)$, and $\gamma v = c\sinh\phi(s)$. Substitute them into the 4-acceleration:

$$A = c\frac{\mathrm{d}\phi}{\mathrm{d}s}(\sinh\phi, \cosh\phi, 0, 0)$$

Hence $g(A, A) = c^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2(\sinh^2\phi - \cosh^2\phi) = -c^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2$. □

Question 8. Constant acceleration motion.

Two rockets accelerating along the x-axis in opposite directions with constant acceleration a have world-lines whose coordinates in a fixed ICS are given by

$$x = -\frac{c^2\cosh(as/c)}{a} \qquad t = \frac{c\sinh(as/c)}{a},$$

and

$$x = \frac{c^2\cosh(as/c)}{a} \qquad t = \frac{c\sinh(as/c)}{a},$$

respectively.

Draw a space-time diagram showing the two world-lines. Show that the parameter s measures proper time along the world-lines.

Let $Z(s)$ denote the displacement four-vector from the event A at proper time $-s$ on the first world-line to the event B at proper time s on the second world-line. Show that

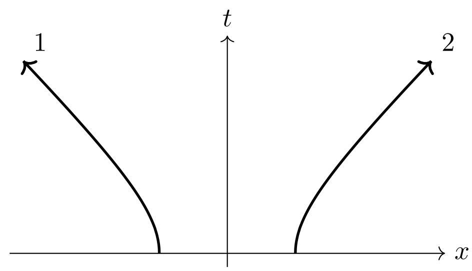
(i) $g(Z, Z)$ is independent of s.

(ii) Z is always pseudo-orthogonal to the four-velocity of the first rocket at A and to the four-velocity of the second rocket at B.

Deduce that observers in the two rockets reckon that A and B are simultaneous for every choice of s, and that they both think that the distance between A and B is independent of s. Thus the two rockets are always the same distance apart, according to the observers. Discuss this apparent absurdity.

Draw a picture of the Euclidean analogue of this situation.

Proof. By the identity $\cosh^2(as/c) - \sinh^2(as/c) = 1$, we have $x^2 - c^2t^2 = c^4a^2$ for both world lines. Th equation is a hyperbola in the space-time diagram, as shown below.



The arc length element:

$$\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 = c^2\cosh^2\left(\frac{as}{c}\right)\mathrm{d}s^2 - c^2\sin^2\left(\frac{as}{c}\right)\mathrm{d}s^2 = c^2\mathrm{d}s^2$$

For two events on the world-line, the proper time between them

$$\tau = \frac{1}{c}\int_X^Y \mathrm{d}\tau = \Delta s.$$

We see that s indeed measures the proper time.

At proper time $-s$, the first rocket has space-time coordinates $A = \left(-\frac{c^2\sinh(as/c)}{a}, -\frac{c^2\cosh(as/c)}{a}, 0, 0\right)$. At proper time s, the second rocket has space-time coordinates $B = \left(\frac{c^2\sinh(as/c)}{a}, \frac{c^2\cosh(as/c)}{a}, 0, 0\right)$. Hence

$$Z = B - A = \left(2\frac{c^2\sinh(as/c)}{a}, 2\frac{c^2\cosh(as/c)}{a}, 0, 0\right)$$

We have

$$g(Z, Z) = \left(2\frac{c^2\sinh(as/c)}{a}\right)^2 - \left(2\frac{c^2\cosh(as/c)}{a}\right)^2 = -\frac{4c^4}{a^2}$$

is independent of s.

The 4-velocity of the second rocket at B:

$$V = \frac{\mathrm{d}X}{\mathrm{d}s} = \left(\frac{c\cosh(as/c)}{a}, \frac{c\sinh(as/c)}{a}, 0, 0\right)$$

We have

$$g(Z, V) = 2\frac{c^2\sinh(as/c)}{a}\cdot\frac{c\cosh(as/c)}{a} - 2\frac{c\cosh(as/c)}{a}\cdot\frac{c\sinh(as/c)}{a} = 0$$

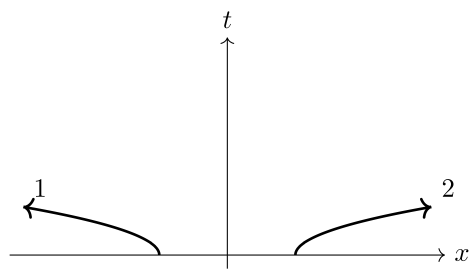
Therefore Z is pseudo-orthogonal to the 4-vector of the second rocket at B. By symmetry, Z is pseudo-orthogonal to the 4-vector of the second rocket at A.

The fact that 4-vector is pseudo-orthogonal to $B - A$ implies that A and B lie in the surface of simultaneity in the frame of either of the rockets.

(*However, using Lorentz transformation to determine the measurement in the frame of the rocket is questionable, as the rocket itself is accelerating. In this case we have to prove that the space-time remains flat in the presence of acceleration. This is obvious in general relativity but is probably beyond the scope of special relativity.*)

The observers in the rockets think that two rockets are always the same distance apart, even though they are moving in opposite direction as measured in the lab frame. This is not unexpected because the acceleration of rockets lead to the effect of length contraction, which cancels the increase of distance between the rockets.

In the non-relativistic limit (Euclidean case), the trajectories of the rockets are parabolae in the space-time diagram, as shown below:



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