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**Problem Sheet 4**

Minimal Models

# Conformal Field Theory

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1. C
2. A
3. NA
4. NA

primary & descendent -> zero norm

**Question 1**

Given a Virasoro primary $|h\rangle$ determine the conditions for $|h\rangle$ to have level three null descendants. Write explicitly the expression for the descendants in each case.

*Proof.* The level-3 descendants are spanned by

$$|s_1\rangle := L_{-1}^3 |h\rangle\,, \qquad |s_2\rangle := L_{-2}L_{-1} |h\rangle\,, \qquad |s_3\rangle := L_{-3} |h\rangle\,.$$

Suppose that $|\chi\rangle = \sum_i a_i |s_i\rangle$ is a null descendant. Then for $\boldsymbol{a} = (a_1, a_2, a_3) \neq 0$,

$$\langle\chi|\chi\rangle = \sum_{i,j} \bar{a}_i a_j \langle s_i|s_j\rangle = \sum_{i,j} \bar{a}_i a_j M_{ij}^{(3)} = 0 \iff \det M^{(3)} = 0.$$

This is the Kac determinant. By (7.11) in the notes, we have

$$(h - h_{1,1})^2 (h - h_{1,2})(h - h_{2,1})(h - h_{1,3})(h - h_{3,1}) = 0.$$

It is now clear that the Verma module contains level-3 null descendants if and only if

$$(h, c) \in \bigcup_{r,s \geqslant 1,\ rs \leqslant 3} \{h - h_{r,s}(c) = 0\}\,.$$

The explicit solutions are known according to (7.12) and (7.13) in the notes:

$$c(m) = 1 - \frac{6}{m(m+1)}, \qquad h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

To compute the combination $\boldsymbol{a}$ is equivalent to computing the eigenvectors of $M^{(3)}$. I don't see there is a clear way to calculate manually. □

**The goal of the problem was computing M(3), the condition det=0 and its null eigenvector. If you still have problems with this after the class let me know.**

**Question 2**

In the lectures we have derived a differential equation for a correlator involving $\phi(z)$

$$\langle \phi(z) \phi_{h_1}(z_1) \cdots \rangle$$

where $\phi(z)$ is a primary field with a level two null descendant.

(a) Verify that the equation is automatically satisfied for two point functions of primary operators.

(b) Consider now a three point function and derive the selection rules stated in the lectures.

*Proof.* (a) The differential equation is (7.22) in the notes:

$$\left( \sum_{i=1}^{n} \left[ \frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right] - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \right) \langle \phi(z) \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) \rangle = 0.$$

Consider the 2-point function of $\phi_h(z)$ and $\phi_{h_1}(z_1)$, where $h = h_1$ is required.

$$\langle \phi_h(z) \phi_h(z_1) \rangle = \frac{C_{12}}{(z - z_1)^{2h}}.$$

Then

$$\begin{aligned}
&\left(\frac{h}{(z-z_1)^2}+\frac{1}{z-z_1}\frac{\partial}{\partial z_1}-\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\langle\phi_h(z)\phi_h(z_1)\rangle \\
&=C_{12}\left(\frac{h}{(z-z_1)^2}+\frac{1}{z-z_1}\frac{\partial}{\partial z_1}-\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\frac{1}{(z-z_1)^{2h}} \\
&=C_{12}\left(\frac{h}{(z-z_1)^{2h+2}}+\frac{2h}{(z-z_1)^{2h+2}}-\frac{3\cdot 2h}{2(z-z_1)^{2h+2}}\right) \\
&=0.
\end{aligned}$$

This shows that (7.22) is automatically satisfied. OK

(b) Most works have been done in the lecture notes. I copy some of them here and add some calculation details. Consider the 3-point function

$$G_{h12}:=\langle\phi(z)\phi_{h_1}(z_1)\,\phi_{h_2}(z_2)\rangle=\frac{C_{h12}}{(z-z_1)^{h+h_1-h_2}\,(z_1-z_2)^{h_1+h_2-h}\,(z-z_2)^{h+h_2-h_1}}.$$

Inserting this into (7.22):

$$0=\left(\frac{h_1}{(z-z_1)^2}+\frac{1}{z-z_1}\frac{\partial}{\partial z_1}+\frac{h_2}{(z-z_2)^2}+\frac{1}{z-z_2}\frac{\partial}{\partial z_2}-\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)G_{h12}$$

Note that

$$\begin{aligned}
\frac{\partial}{\partial z_1}G_{h12}&=\frac{C_{h12}}{(z_1-z_2)^{h_1+h_2-h}\,(z-z_2)^{h+h_2-h_1}}\frac{\partial}{\partial z_1}\frac{1}{(z-z_1)^{h+h_1-h_2}} \\
&\quad+\frac{C_{h12}}{(z-z_1)^{h+h_1-h_2}\,(z-z_2)^{h+h_2-h_1}}\frac{\partial}{\partial z_1}\frac{1}{(z_1-z_2)^{h_1+h_2-h}} \\
&=G_{h12}\left(\frac{h+h_1-h_2}{z-z_1}-\frac{h_1+h_2-h}{z_1-z_2}\right),
\end{aligned}$$

and, symmetrically,

$$\frac{\partial}{\partial z_2}G_{h12}=G_{h12}\left(\frac{h+h_2-h_1}{z-z_2}+\frac{h_1+h_2-h}{z_1-z_2}\right).$$

For the derivative with respect to $z$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial z^2}G_{h12}&=\frac{C_{h12}}{(z_1-z_2)^{h_1+h_2-h}}\left(\frac{1}{(z-z_1)^{h+h_1-h_2}}\frac{\partial^2}{\partial z^2}\frac{1}{(z-z_2)^{h+h_2-h_1}}\right. \\
&\quad\left.+\frac{1}{(z-z_2)^{h+h_2-h_1}}\frac{\partial^2}{\partial z^2}\frac{1}{(z-z_1)^{h+h_1-h_2}}+2\frac{\partial}{\partial z}\frac{1}{(z-z_1)^{h+h_1-h_2}}\frac{\partial}{\partial z}\frac{1}{(z-z_2)^{h+h_2-h_1}}\right) \\
&=G_{h12}\left(\frac{(h+h_1-h_2)(h+h_1-h_2+1)}{(z-z_1)^2}+\frac{(h+h_2-h_1)(h+h_2-h_1+1)}{(z-z_2)^2}+\frac{2(h+h_1-h_2)(h+h_2-h_1)}{(z-z_1)(z-z_2)}\right).
\end{aligned}$$

Substituting these expression back, we obtain an equation of the form

$$G_{h12}\left(\frac{a_1}{(z-z_1)^2}+\frac{a_2}{(z-z_2)^2}+\frac{a_3}{(z-z_1)(z_1-z_2)}+\frac{a_4}{(z-z_2)(z_1-z_2)}+\frac{a_5}{(z-z_1)(z-z_2)}\right)=0,$$

where:

$$a_1=h_1+(h+h_1-h_2)-\frac{3}{2(2h+1)}(h+h_1-h_2)(h+h_1-h_2+1)$$

$$
\begin{aligned}
a_2 &= h_2 + (h + h_2 - h_1) - \frac{3}{2(2h+1)}(h + h_2 - h_1)(h + h_2 - h_1 + 1) \\
a_3 &= h - h_1 - h_2 \\
a_4 &= h_1 + h_2 - h \\
a_5 &= -\frac{3}{2h+1}(h + h_1 - h_2)(h + h_2 - h_1).
\end{aligned}
$$

These coefficients are not independent. We may first clear the denominators. Note that $a_3 + a_4 = 0$. Hence the left-hand side is given by

$$
\begin{aligned}
&G_{h12}\left(\frac{a_1}{(z-z_1)^2} + \frac{a_2}{(z-z_2)^2} + \frac{a_3 + a_5}{(z-z_1)(z-z_2)}\right) \\
&= \frac{G_{h12}}{(z-z_1)^2(z-z_2)^2}\left(a_1(z-z_2)^2 + a_2(z-z_1)^2 + (a_3+a_5)(z-z_1)(z-z_2)\right) = 0.
\end{aligned}
$$

By comparing coefficients, we note that the constraints are

$$a_1 = a_2 = a_3 + a_5 = 0.$$

The constraint (7.24) in the notes is exactly the equation $a_2 = 0$. Furthermore, we need to show that $a_1 = 0$ and $a_3 + a_5 = 0$ do not impose more constraints on $h_2$ (otherwise we may obtain no solutions for $h_2$). By a brute force computation we note that all three equations lead to

$$h^2 - 3(h_1^2 + h_2^2) + 2h(h_1 + h_2) + 6h_1h_2 - h + h_1 + h_2 = 0.$$

This concludes the proof. OK □

## Question 3

Consider the critical Ising model introduced in the lectures, and the four point correlator of four identical operators $\epsilon(z, \overline{z})$, of conformal dimension $h = \overline{h} = 1/2$.

(a) Explain why conformal symmetry implies:

$$\langle \epsilon(z_1, \overline{z}_1) \cdots \epsilon(z_4, \overline{z}_4) \rangle = \frac{g(\eta, \overline{\eta})}{|z_{12}|^{4h} |z_{34}|^{4h}}$$

(b) Given that $\epsilon(z, \overline{z})$ admits a level two null descendant, write down a differential equation for $g(\eta, \overline{\eta})$. [You may focus in the holomorphic dependence only.]

(c) Write down the full correlator as a linear combination of solutions to the equation above [reintroducing the anti-holomorphic dependence].

(d) What form do the crossing relations take? Find the most general expression for $g(\eta, \overline{\eta})$ consistent with the crossing relations and with part (c). Can you think how to fix $g(\eta, \overline{\eta})$ completely?

## Question 4

Write down the schematic decomposition of the result of problem 3 in terms of Virasoro conformal blocks. Verify that the small $z, \overline{z}$ behaviour for the identity conformal block of problem 3, and the one computed in the lecture notes, agree with the small $z, \overline{z}$ behaviour you obtained in problem six of the previous sheet.

Q3.

$m=3$:

| Field | $(h,\bar h)$ |
|---|---|
| id. | $(0,0)$ |
| $\sigma$ | $(\frac{1}{16},\frac{1}{16})$ |
| $\varepsilon$ | $(\frac{1}{2},\frac{1}{2})$ |

$\sigma\times\sigma = id + \varepsilon$

$\varepsilon\times\varepsilon = id$

$$\langle\varphi_1(z_1)\cdots\varphi_4(z_4)\rangle = g(\eta)\prod_{i<j} z_{ij}^{h/3-h_i-h_j}\ ,\ h := \Sigma_i h_i$$

$$\langle\varepsilon_1(z_1)\cdots\varepsilon_4(z_4)\rangle = \underbrace{\frac{1}{z_{12}^{2\Delta}z_{34}^{2\Delta}}}_{f(z_i)}\tilde g(\eta)\ ,\ \Delta=\frac{1}{2}.$$

$$\overbrace{\left(\mathcal{L}_{-2}-\frac{3}{2(h+1)}\mathcal{L}_{1}\right)}^{K_2} f(z_i)\tilde g(\eta) = 0$$

$$\left[f(z_i)^{-1}K_2 f(z_i)\tilde g(\eta)\right]_{(z_1,z_2,z_3,z_4)=(0,\eta,1,\infty)} = 0$$

$$\Rightarrow \tilde g''(\eta) - \frac{2(\eta+1)}{3\eta(1-\eta)}\tilde g'(\eta) - \frac{2}{3(1-\eta)^2}g(\eta) = 0 .$$

Frobenius method near the pole $\eta=1$.

$\Rightarrow$ conformal blocks.

$$\left(\tilde g(\eta) = \alpha\underbrace{\frac{\eta^2-\eta+1}{1-\eta}}_{g_\alpha} + \beta\overbrace{\frac{\eta^{5/6}}{(1-\eta)^{1/6}}Q^{[\frac{1}{3},\frac{5}{3}]}(2\eta-1)}^{g_\beta}\right)$$

$\uparrow$ Legendre Q-function

1) Crossing $1\leftrightarrow 2$ : $\tilde g(\frac{\eta}{\eta-1}) = \tilde g(\eta)$

$1\leftrightarrow 3$ : $(1-\eta)^{2h}\tilde g(\eta) = \eta^{2h}\tilde g(1-\eta)$

(from physical symmetry, not from the ODE)

$\Rightarrow \beta = 0$ !

2) Monodromy : $\eta\mapsto e^{2\pi i}\eta$ . $g_\beta$ has non-trivial monodromy.

3) OPE : $\varepsilon\times\varepsilon = id$. Integer powers $\Rightarrow \beta=0$ .

$\hookrightarrow$ can also fix $\alpha$ !

$$\bar g(\eta,\bar\eta) = \tilde\alpha\left|\frac{1-\eta+\eta^2}{1-\eta}\right|^2 \sim (1+O(\eta^2)+O(\bar\eta^2))$$

$$= \underbrace{C_{\varepsilon\varepsilon id}^2}_{=1} \underbrace{F(0|\eta)}_{(1+O(\eta))}\underbrace{F(0|\bar\eta)}_{(1+O(\bar\eta))} \Rightarrow \alpha = 1$$

$$F(0|\eta) = \frac{1-\eta+\eta^2}{1-\eta} = 1+\eta^2 \sum_{k=0}^{\infty} \eta^k \quad \Rightarrow h = \frac{1}{2}, \quad c = \frac{1}{2}.$$

Alternative way: look at level 3 null descendent.

$\Rightarrow$ Get 3rd order ODE of $\tilde{g}(\eta)$

$\Rightarrow$ Combine with 2nd order ODE to get:

$$\tilde{g}'(\eta) = \frac{\eta(\eta-2)}{(\eta-1)(1-\eta+\eta^2)} \tilde{g}(\eta)$$

$$[f^{-1} K_3 f(z_i) \tilde{g}(\eta)] = 0$$