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Problem Sheet 4
A4: Integration

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Question 1

Evaluate $\int_0^1 \left(\int_0^x \mathrm{e}^{-y}\,\mathrm{d}y\right)\mathrm{d}x$ and $\int_0^1 \left(\int_0^{x-x^2} (x+y)\,\mathrm{d}y\right)\mathrm{d}x$

(a) directly;

(b) be reversing the order of integration.

Solution. (a) $\int_0^x \mathrm{e}^{-y}\,\mathrm{d}y = -e^{-y}|_{y=0}^{y=x} = 1 - e^{-x}$ by Fundamental Theorem of Calculus.

Hence $\int_0^1 \left(\int_0^x \mathrm{e}^{-y}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 (1-\mathrm{e}^{-x})\,\mathrm{d}x = x + \mathrm{e}^{-x}|_0^1 = 1 + \mathrm{e}^{-1} - 1 = 1/\mathrm{e}$ again by Fundamental Theorem of Calculus.

$\int_0^{x-x^2} (x+y)\,\mathrm{d}y = \left(xy + \frac{1}{2}y^2\right)_{y=0}^{y=x-x^2} = x(x-x^2) + \frac{1}{2}\left(x-x^2\right)^2 = \frac{1}{2}x^4 - 2x^3 + \frac{3}{2}x^2$ by Fundamental Theorem of Calculus.

Hence $\int_0^1 \left(\int_0^{x-x^2} (x+y)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\frac{1}{2}x^4 - 2x^3 + \frac{3}{2}x^2\right)\,\mathrm{d}x = \left(\frac{1}{10}x^5 - \frac{1}{2}x^4 + \frac{1}{2}x^3\right)_0^1 = \frac{1}{10}$ by Fundamental Theorem of Calculus.

(b) Notice that both integrands are bounded measurable and hence are integrable. By Fubini's Theorem we can reverse the order of integration.

For the first integral, notice that

$$\{(x,y)\in\mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\} = \{(x,y)\in\mathbb{R}^2 : 0 \leqslant y \leqslant 1,\ y \leqslant x \leqslant 1\}$$

Hence

$$\int_0^1 \left(\int_0^x \mathrm{e}^{-y}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\int_y^1 \mathrm{e}^{-y}\,\mathrm{d}x\right)\mathrm{d}y = \int_0^1 (1-y)\mathrm{e}^{-y}\,\mathrm{d}y = y\mathrm{e}^{-y}|_0^1 = 1/\mathrm{e}$$

For the second integral, notice that

$$\{(x,y)\in\mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x-x^2\} = \left\{(x,y)\in\mathbb{R}^2 : 0 \leqslant y \leqslant \frac{1}{4},\ \frac{1}{2} - \sqrt{\frac{1}{4}-y} \leqslant x \leqslant \frac{1}{2} + \sqrt{\frac{1}{4}+y}\right\}$$

Hence

$$\begin{aligned}
\int_0^1 \left(\int_0^{x-x^2} (x+y)\,\mathrm{d}y\right)\mathrm{d}x &= \int_0^{1/4} \left(\int_{1/2-\sqrt{1/4-y}}^{1/2+\sqrt{1/4-y}} (x+y)\,\mathrm{d}x\right)\mathrm{d}y = \int_0^{1/4} \mathrm{d}y\left(xy + \frac{1}{2}x^2\right)_{x=1/2-\sqrt{1/4-y}}^{x=1/2+\sqrt{1/4-y}} \\
&= \int_0^{1/4} (1+2y)\sqrt{\frac{1}{4}-y}\,\mathrm{d}y \\
&= \int_{1/2}^0 \left(1 + 2\left(\frac{1}{4}-t^2\right)\right) t\,\mathrm{d}\left(\frac{1}{4}-t^2\right) \qquad \text{substitute } t = \sqrt{\frac{1}{4}-y} \Longrightarrow y = \frac{1}{4}-t^2 \\
&= \int_0^{1/2} \left(3t^2 - 4t^4\right)\,\mathrm{d}t = \left(t^3 - \frac{4}{5}t^5\right)_0^{1/2} = \frac{1}{8} - \frac{1}{40} = \frac{1}{10}
\end{aligned}$$

□

Question 2

In each of the following cases, is f integrable over the given region?

(i) $f(x,y) = \mathrm{e}^{-xy}$ over $[0,\infty)\times[0,\infty)$;

(ii) $f(x,y) = \mathrm{e}^{-xy}$ over $\{(x,y):\ 0<x<y<x+x^2\}$;

(iii) $f(x,y) = \dfrac{(\sin x)(\sin y)}{x^2+y^2}$ over $\left(-\frac{\pi}{2},\,\frac{\pi}{2}\right)\times\left(-\frac{\pi}{2},\,\frac{\pi}{2}\right)$.

Solution. (i) Suppose that f is integrable. Then by Fubini's Theorem,

$$\int_{[0,\infty)\times[0,\infty)} f(x,y)\,\mathrm{d}(x\times y) = \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-xy}\,\mathrm{d}x\right)\,\mathrm{d}y = \int_0^\infty \frac{1}{y}\,\mathrm{d}y = +\infty$$

which is a contradiction. Hence f is not integrable over $[0,\infty)\times[0,\infty)$.

(ii) Notice that f is non-negative over its domain. We can compute:

$$\int_0^\infty \left(\int_x^{x+x^2} \mathrm{e}^{-xy}\,\mathrm{d}y\right)\,\mathrm{d}x \leqslant \int_0^\infty \left(\int_x^{x+x^2} \mathrm{e}^{-y}\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^\infty \left(\mathrm{e}^{-x} - \mathrm{e}^{-x-x^2}\right)\,\mathrm{d}x \leqslant \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1$$

Hence by Tonelli's Theorem, f is integrable over $\{(x,y):\ 0<x<y<x+x^2\}$.

(iii) By symmetry, f is integrable over $\left(-\frac{\pi}{2},\,\frac{\pi}{2}\right)\times\left(-\frac{\pi}{2},\,\frac{\pi}{2}\right)$ if and only if it is integrable over $\left(0,\,\frac{\pi}{2}\right)\times\left(0,\,\frac{\pi}{2}\right)$. On $\left(0,\,\frac{\pi}{2}\right)\times\left(0,\,\frac{\pi}{2}\right)$, f is non-negative. We can compute the integral by successive single integrals. Since $\sin x$ is concave on $\left(0,\frac{\pi}{2}\right)$, we have $\sin x\leqslant x$ on $\left(0,\,\frac{\pi}{2}\right)$.

$$\begin{aligned}\int_{\left(0,\,\frac{\pi}{2}\right)\times\left(0,\,\frac{\pi}{2}\right)} f(x,y)\,\mathrm{d}(x\times y) &\leqslant \int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{xy}{x^2+y^2}\,\mathrm{d}x\right)\,\mathrm{d}y = \int_0^{\pi/2}\left(\frac{1}{2}y\log(x^2+y^2)\right)_{x=0}^{x=\pi/2}\,\mathrm{d}y\\ &= \int_0^{\pi/2}\frac{1}{2}y\log\left(1+\frac{\pi^2}{4y^2}\right)\,\mathrm{d}y\end{aligned}$$

Notice that $\lim_{y\to 0^+}\frac{1}{2}y\log\left(1+\frac{\pi^2}{4y^2}\right) = 0$ (by Taylor expansion of $\log$ at $y=1$). In particular the integrand is bounded on a bounded domain. Hence it is integrable. By Tonelli's Theorem, f is integrable over $\left(0,\,\frac{\pi}{2}\right)\times\left(0,\,\frac{\pi}{2}\right)$. □

Question 3

(a) Let $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\cos(x\cos\theta)\,\mathrm{d}\theta$. Show that $\displaystyle\int_0^\infty J_0(x)\mathrm{e}^{-ax}\,\mathrm{d}x = \frac{1}{\sqrt{1+a^2}}$ if $a>0$.

(b) Take $b>a>1$. By considering x^{-y} over $(1,\infty)\times(a,b)$, show that $\displaystyle\int_1^\infty \frac{x^{-a}-x^{-b}}{\log x}\,\mathrm{d}x$ exists, and find its value.

Solution. (a) Let $f(x,\theta) = \cos(x\cos\theta)\mathrm{e}^{-ax}$ defined on $(0,\infty)\times(0,\pi/2)$. We observe that on the given domain $|f(x,\theta)|\leqslant \mathrm{e}^{-ax}$, which is integrable. Hence f is integrable and by Fubini's Theorem we can exchange the order of integration:

$$\begin{aligned}\int_0^\infty J_0(x)\mathrm{e}^{-ax}\,\mathrm{d}x &= \frac{2}{\pi}\int_0^\infty\left(\int_0^{\pi/2}\cos(x\cos\theta)\mathrm{e}^{-ax}\,\mathrm{d}\theta\right)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2}\left(\int_0^\infty\cos(x\cos\theta)\mathrm{e}^{-ax}\,\mathrm{d}x\right)\,\mathrm{d}\theta\\ &= -\frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta\,\mathrm{Re}\left(\frac{1}{\mathrm{i}\cos\theta - a}\right) = \frac{2}{\pi}\int_0^{\pi/2}\frac{a}{a^2+\cos^2\theta}\,\mathrm{d}\theta\end{aligned}$$

Change of variable: $u = \tan\theta \Longrightarrow \mathrm{d}u = \dfrac{1}{\cos^2\theta}\,\mathrm{d}\theta = \dfrac{1}{1+u^2}\,\mathrm{d}\theta$. Hence

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi/2}\frac{a}{a^2+\cos^2\theta}\,\mathrm{d}\theta &= \frac{2a}{\pi}\int_0^{\infty}\frac{1}{a^2+\dfrac{1}{1+u^2}}(1+u^2)\,\mathrm{d}u = \frac{2a}{\pi}\int_0^{\infty}\frac{1}{(1+a^2)+a^2u^2}\,\mathrm{d}u\\ &= \frac{2}{\pi\sqrt{1+a^2}}\int_0^{\infty}\frac{1}{1+\left(\dfrac{a}{\sqrt{1+a^2}}u\right)^2}\,\mathrm{d}\left(\frac{a}{\sqrt{1+a^2}}u\right) = \frac{2}{\pi\sqrt{1+a^2}}\cdot\frac{\pi}{2}\\ &= \frac{1}{\sqrt{1+a^2}}\end{aligned}$$

(b) Notice that

$$\frac{x^{-a}-x^{-b}}{\log x} = \int_a^b x^{-y}\,\mathrm{d}y$$

for all $x > 1$. Notice that x^{-y} is non-negative. By Tonelli's Theorem the double integral exists. By Fubini's Theorem we can exchange the order of intgeration. We have:

$$\int_1^{\infty}\frac{x^{-a}-x^{-b}}{\log x}\,\mathrm{d}x = \int_1^{\infty}\left(\int_a^b x^{-y}\,\mathrm{d}y\right)\,\mathrm{d}x = \int_a^b\left(\int_1^{\infty}x^{-y}\,\mathrm{d}x\right)\,\mathrm{d}y$$

provided that the rightmost integral exists.

$$\int_1^{\infty}x^{-y}\,\mathrm{d}x = \left(\frac{1}{1-y}x^{1-y}\right)_{x=1}^{x=\infty} = \frac{1}{y-1}$$

by Fundamental Theorem of Calculus and Monotone Convergence Theorem.

$$\int_a^b\frac{1}{y-1}\,\mathrm{d}y = \log(y-1)_a^b = \log\left(\frac{b-1}{a-1}\right)$$

In conclusion $\displaystyle\int_1^{\infty}\frac{x^{-a}-x^{-b}}{\log x}\,\mathrm{d}x$ exists and equals to $\log\left(\dfrac{b-1}{a-1}\right)$. □

Question 4

Let $f \in \mathcal{L}^1(\mathbb{R})$ be non-negative with $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$, and let $F(x) = \displaystyle\int_{-\infty}^{x} f(y)\,\mathrm{d}y$, Assume that $xf(x) \in \mathcal{L}^1(\mathbb{R})$. Use Fubini's Theorem to prove that

$$\int_0^{+\infty}(1-F(x))\,\mathrm{d}x = \int_0^{+\infty}xf(x)\,\mathrm{d}x, \qquad \int_{-\infty}^{0}F(x)\,\mathrm{d}x = -\int_{-\infty}^{0}xf(x)\,\mathrm{d}x.$$

Now let g be a bounded measurable function, and let

$$G(y) = \int_{\{g(x)\leqslant y\}} f(x)\,\mathrm{d}x$$

Prove that

$$\int_0^{+\infty}(1-G(y)-G(-y))\,\mathrm{d}y = \int_{-\infty}^{+\infty}f(x)g(x)\,\mathrm{d}x$$

Proof. For the first one,

$$1-F(x) = \int_{\mathbb{R}} f(y)\,\mathrm{d}y - \int_{\infty}^{x} f(y)\,\mathrm{d}y = \int_x^{+\infty} f(y)\,\mathrm{d}y$$

Notice that

$$\int_0^\infty \left(\int_0^x f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^\infty xf(x)\,\mathrm{d}x < +\infty$$

as $xf(x) \in \mathcal{L}^1(\mathbb{R})$. Since $xf(x)$ is non-negative, by Tonelli's Theorem the double integral exists and by Fubini's Theorem we can exchange the order of integration.

$$\int_0^\infty xf(x)\,\mathrm{d}x = \int_0^\infty \left(\int_0^x f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^\infty (1-F(x))\,\mathrm{d}x$$

For the second one, notice that

$$\int_{-\infty}^0 \left(\int_x^0 f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = -\int_{-\infty}^0 xf(x)\,\mathrm{d}x$$

exists as $xf(x) \in \mathcal{L}^1(\mathbb{R})$. For $x < 0$, $xf(x)$ is non-positive, so we can apply Tonelli's Theorem to argue that the double integral exists. Next, by Fubini's Theorem we can exchange the order of integration.

$$-\int_{-\infty}^0 xf(x)\,\mathrm{d}x = \int_{-\infty}^0 \left(\int_x^0 f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_{-\infty}^0 \left(\int_{-\infty}^x f(y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_{-\infty}^0 F(x)\,\mathrm{d}x$$

The part for $G(y)$ is simply a generalization of the previous parts, with x replaced by $g(x)$:

$$1 - G(y) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x - \int_{\{g(x)\leqslant y\}} f(x)\,\mathrm{d}x = \int_{\{g(x)>y\}} f(x)\,\mathrm{d}x$$

Then

$$\int_0^\infty (1-G(y))\,\mathrm{d}y = \int_0^\infty \left(\int_{\{g(x)>y\}} f(x)\,\mathrm{d}x\right)\,\mathrm{d}y = \int_{-\infty}^{+\infty} \left(\int_0^{\max\{0,g(x)\}} f(x)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_{\{g(x)>0\}} f(x)g(x)\,\mathrm{d}x$$

$$\int_0^\infty -G(-y)\,\mathrm{d}y = -\int_0^\infty \left(\int_{\{g(x)\leqslant -y\}} f(x)\,\mathrm{d}x\right)\,\mathrm{d}y = \int_{-\infty}^{+\infty} \left(\int_{\min\{0,g(x)\}}^0 f(x)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_{\{g(x)<0\}} f(x)g(x)\,\mathrm{d}x$$

In conclusion,

$$\int_0^\infty (1-G(y)-G(-y))\,\mathrm{d}y = \int_{-\infty}^{+\infty} f(x)g(x)\,\mathrm{d}x$$

□

Question 5

(a) Let $\alpha > 1$ and $f(x,y) = (x^2+y^2)^{-\alpha}$ and $g(x,y) = (1+x^2+y^2)^{-\alpha}$. Show that f is integrable over $[1,+\infty)\times[0,+\infty)$. Deduce that f is integrable over $[0,1]\times[1,+\infty)$ and that g is integrable over $\mathbb{R}^2$.

(b) Use polar coordinates to show that g is integrable over $\mathbb{R}^2$.

Proof. (a) Change of variable: $x = x,\ y = ux$. The Jacobian determinant: $\dfrac{\partial(x,y)}{\partial(x,u)} = x$. Hence:

$$\int_{[1,+\infty)\times[0,+\infty)} \frac{1}{(x^2+y^2)^\alpha}\,\mathrm{d}(x\times y) = \int_{[1,+\infty)\times[0,+\infty)} \frac{x}{x^{2\alpha}(1+u^2)^\alpha}\,\mathrm{d}(x\times u) = \int_1^\infty \frac{1}{x^{2\alpha-1}}\,\mathrm{d}x \int_0^\infty \frac{1}{(1+u^2)^\alpha}\,\mathrm{d}u$$

By Fundamental Theorem of Calculus and Monotone Convergence Theorem we know that

$$\int_1^\infty \frac{1}{x^{2\alpha-1}}\,\mathrm{d}x = \frac{1}{2-2\alpha}x^{2-2\alpha}\Big|_1^\infty = \frac{1}{2(\alpha-1)}$$

In addition, we have

$$\int_0^\infty \frac{1}{(1+u^2)^\alpha}\,\mathrm{d}u \leqslant \int_0^\infty \frac{1}{(1+u^2)}\,\mathrm{d}u = \arctan u|_0^\infty = \frac{\pi}{2}$$

Hence the successive single integrals exist. Since the integrand f is non-negative, by Tonelli's Theorem, f is integrable over $[1,\infty)\times[0,\infty)$. By symmetry of f in the two variables, f is integrable over $[0,\infty)\times[1,\infty)$ and hence on $[0,1]\times[1,\infty)$.

By symmetry, g is integrable over $\mathbb{R}^2$ if and only if it is integrable over the first quadrant $[0,\infty)\times[0,\infty)$. In the first quadrant we have $g < f$ everywhere. Since f is integrable over $[0,1]\times[1,\infty)\cup[1,\infty)\times[0,\infty)$, so is g. In addition g is bounded on $[0,1]\times[0,1]$ and hence is integrable over the square. We conclude that g is integrable over the first quadrant.

(b) In the polar coordinates:

$$\iint_{\mathbb{R}^2} g(x,y)\,\mathrm{d}(x\times y) = \int_0^{2\pi}\mathrm{d}\theta \int_0^\infty \frac{r}{(1+r^2)^\alpha}\,\mathrm{d}r = \pi\cdot\frac{1}{1-\alpha}(1+r^2)^{1-\alpha}\Big|_0^\infty = \frac{\pi}{\alpha-1}$$

Hence g is integrable over $\mathbb{R}^2$ by Theorem 7.9. □

Question 6

For $p>0$, calculate $||f||_p$ when f is (i) $\chi_{(0,1)}$, (ii) $\chi_{(1,2)}$, (iii) $\chi_{(0,2)}$.

Now assume that $0<p<1$. Is $||\cdot||_p$ a norm on L^p?

For $f,g\in L^p(\mathbb{R})$, let $d_p(f,g)=\int_{\mathbb{R}}|f-g|^p$. Show that d_p is a metric on $L^p(\mathbb{R})$.

Proof. (i) $||\chi_{(0,1)}||_p = \left(\int_0^1 \mathrm{d}x\right)^{1/p} = 1;$

(ii) $||\chi_{(1,2)}||_p = \left(\int_1^2 \mathrm{d}x\right)^{1/p} = 1;$

(iii) $||\chi_{(0,2)}||_p = \left(\int_0^2 \mathrm{d}x\right)^{1/p} = 2^{1/p}.$

$||\cdot||_p$ is not a norm for $0<p<1$. If $p<1$, then

$$||\chi_{(0,1)}||_p + ||\chi_{(0,1)}||_p = 2 < 2^{1/p} = ||\chi_{(0,2)}||_p = ||\chi_{(0,1)}+\chi_{(1,2)}||_p$$

The triangular inequality is violated.

Suppose that $d_p(f,g)=\int_{\mathbb{R}}|f-g|^p$. By definition d_p is symmetric and positive definite on $L^p(\mathbb{R})$. To proof the triangular inequality, we shall first prove the following inequality:

$$|a+b|^p \leqslant |a|^p + |b|^p$$

For $0<p<1$, the function $t\mapsto t^p$ has negative second derivative for $t>0$ and hence is concave. By Jensen's inequality:

$$(\lambda s + (1-\lambda)t)^p \geqslant \lambda s^p + (1-\lambda)t^p$$

for all $s,t\geqslant 0$ and $\lambda\in[0,1]$.

Take $s = |a| + |b|$, $t = 0$, and $\lambda = |a|/(|a| + |b|)$. We obtain:

$$|a|^p \geqslant \frac{|a|}{|a| + |b|}(|a| + |b|)^p$$

Take $s = |a| + |b|$, $t = 0$, and $\lambda = |b|/(|a| + |b|)$. We obtain:

$$|b|^p \geqslant \frac{|b|}{|a| + |b|}(|a| + |b|)^p$$

Combine the two expressions and we have:

$$|a + b|^p \leqslant (|a| + |b|)^p \leqslant |a|^p + |b|^p$$

In particular for $f, g \in \mathcal{L}^p(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$|f(x) + g(x)|^p \leqslant |f(x)|^p + |g(x)|^p$$

Integrate on both sides, we obtain:

$$\int_{\mathbb{R}} |f + g|^p \leqslant \int_{\mathbb{R}} |f|^p + \int_{\mathbb{R}} |g|^p$$

For $f, g, h \in L^p(\mathbb{R})$:

$$d_p(f, h) = \int_{\mathbb{R}} |f - h|^p = \int_{\mathbb{R}} |(f - g) + (g - h)|^p \leqslant \int_{\mathbb{R}} |f - g|^p + \int_{\mathbb{R}} |g - h|^p = d_p(f, g) + d_p(g, h)$$

hence proving the trigular inequality. d_p is a metric on $L^p(\mathbb{R})$ for $0 < p < 1$. □

Question 7

Consider the relation $\sim$ on the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ given by $f \sim g \iff f = g$ a.e.

State which properties of null sets are used to prove each of the following true statements:

(i) $f \sim f$;

(ii) $f \sim g \implies g \sim f$;

(iii) $f \sim g \ \wedge \ g \sim h \implies f \sim h$;

(iv) $\forall\, n \in \mathbb{N} :\ f_n \sim g_n \implies \sup f_n \sim \sup g_n$;

(v) $f \sim g \implies h \circ f \sim h \circ g$.

Give an example where h is injective, $f \sim g$, but $f \circ h \nsim g \circ h$.

Proof. (i) It follows from the reflexivity of equality.

(ii) It follows from the symmetry of equality.

(iii) If $f \sim g \ \wedge \ g \sim h$, then $A := \{x \in \mathbb{R} :\ f(x) \neq g(x)\}$ and $B := \{x \in \mathbb{R} :\ g(x) \neq h(x)\}$ are both null sets. Then $A \cup B$ is also a null set and $\{x \in \mathbb{R} :\ f(x) \neq h(x)\} \subseteq A \cup B$. Since the subset of a null set is also null, $f \sim h$.

(iv) Let $A_n := \{x \in \mathbb{R} :\ f_n(x) \neq g_n(x)\}$. A_n are null sets. Since the countable union of null sets is null, we know that $\bigcup_{n \in \mathbb{N}} A_n$ is a null set. For $x \in \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} A_n$, $f_n(x) = g_n(x)$ for all $n \in \mathbb{N}$. Hence $\sup f_n = \sup g_n$ for $x \in \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} A_n$. In other words, $\sup f_n \sim \sup g_n$.

(v) The result follows from a simple observation that $\{x \in \mathbb{R} :\ f(x) = g(x)\} \subseteq \{x \in \mathbb{R} :\ h \circ f(x) = h \circ g(x)\}$ and that a subset of a null set is null.

Consider an uncountable null set (e.g. a Cantor set) $C \subseteq \mathbb{R}$. There exists an injection $h : \mathbb{R} \to C$. Let f and g be measurable functions such that $f(x) = g(x) \Leftrightarrow x \in \mathbb{R}\backslash C$. Then we have $f \circ h(x) \neq g \circ h(x)$ for all $x \in \mathbb{R}$ which implies that $f \circ h \not\sim g \circ h$.

First we observe that $h_1(x) := \dfrac{1}{2} + \dfrac{1}{\pi} \arctan x$ is a bijection from $\mathbb{R}$ to $(0, 1)$. Let $\{a_n\}$ be the binary expansion of a number $x \in (0, 1)$. That is, $x = \sum_{n=1}^{\infty} a_n 2^{-n}$, $a_n \in \{0, 1\}$. The map $h_2 : \sum_{n=1}^{\infty} a_n 2^{-n} \mapsto \sum_{n=1}^{\infty} a_n 3^{-n}$ is an injection from $(0, 1)$ to C (defined in Sheet 1). Hence $h := h_2 \circ h_1$ is the map with desired properties. □

Question 8

Let $p > 1$. Give examples of sequences $\{f_n\}$ and $\{g_n\}$ in $L^p(0, 1)$ such that

(i) $\lim_{n\to\infty} f_n(x) = 0$ a.e. but $\lim_{n\to\infty} ||f_n||_p \neq 0$.

(ii) $\lim_{n\to\infty} ||g_n||_p = 0$ but $\lim_{n\to\infty} g_n(x)$ does not exist for any $x \in (0, 1)$.

For each $\varepsilon > 0$ find a measurable subset E_ε of $[0, 1]$ such that $m(E_\varepsilon) < \varepsilon$ and $f_n(x) \to 0$ uniformly on $[0, 1]\backslash E_\varepsilon$.

Find a subsequence $\{g_{n_r}\}$ such that $\lim_{r\to\infty} g_{n_r}(x) = 0$ a.e.

Solution. (i) Consider $f_n := n\chi_{(0,\ 1/n)}$. Then $f_n(x) \to 0$ as $n \to \infty$ for all $x \in (0, 1)$. But

$$||f_n||_p = \left(\int_0^{1/n} n \,\mathrm{d}x \right)^{1/p} = 1$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} ||f_n||_p = 1 \neq 0$.

(ii) We define a double sequence of functions on $\mathbb{R}$ by $y_k^{(i)} = \chi_{\left(\frac{i-1}{2^k},\ \frac{i}{2^k}\right)}$. We define $\{g_n\}$ by $g_1 = y_1^{(1)}$, $g_2 = y_2^{(1)}$, $g_3 = y_2^{(2)}$, $g_4 = y_3^{(1)}, \cdots$ In general,

$$g_{2^k+n} = y_{k+1}^{(n+1)}, \qquad k \in \mathbb{N},\ 1 \leqslant n \leqslant 2^k$$

Then $\{g_n\}$ is a sequence of functions on $(0, 1)$. We have:

$$\left|\left| y_k^{(n)} \right|\right|_p = \left(\int_{(i-1)2^{-k}}^{i2^{-k}} \mathrm{d}x \right)^{1/p} = 2^{-k/p} \to 0$$

as $k \to \infty$. Hence $||g_n||_p \to 0$ as $n \to \infty$.

On the other hand, $\{g_n\}$ has no pointwise limit. In fact for each $x \in (0, 1)$ we can find a subsequence of $\{g_n\}$ converging to 0 and a subsequence converging to 1.

(iii) Let $E_\varepsilon = (0, \varepsilon)$ for any $\varepsilon > 0$ so that $[0, 1]\backslash E_\varepsilon = [\varepsilon, 1]$. For $n > 1/\varepsilon$, $f_n(x) = 0$ for all $x \in [\varepsilon, 1]$. Hence $f_n \to 0$ uniformly on $[0, 1]\backslash E_\varepsilon$.

(iv) We have a subsequence of $\{g_n\}$ given by $g_{2^k}(x) = y_{k+1}^{(1)} = \chi_{(0,\ 2^{-k})}$. It is clear that $g_{2^k}(x) \to 0$ as $k \to \infty$ for all $x \in (0, 1)$.

□

Question 9

A function $g\colon [0,+\infty) \to \mathbb{R}$ is *convex* if

$$g(x) = \sup\{\alpha x + \beta :\ \forall\, y \in [0,+\infty)\ (\alpha y + \beta \leqslant g(y))\}$$

If g is continuous on $[0,+\infty)$ with non-negative second derivative on $(0,+\infty)$, then g is convex.

Let $f : [0,1] \to [0,+\infty)$ be bounded, measurable, and $M_n = \displaystyle\int_0^1 f^n = ||f||_n^n$. Show that

(i) $g\left(\displaystyle\int_0^1 f(x)\,\mathrm{d}x\right) \leqslant \displaystyle\int_0^1 g(f(x))\,\mathrm{d}x$ for every convex function g;

(ii) $M_n^2 \leqslant M_{n+1}M_{n-1}$;

(iii) $||f||_n \leqslant M_{n+1}/M_n \leqslant ||f||_\infty$;

(iv) $\lim\limits_{n\to\infty} M_{n+1}/M_n = ||f||_\infty$.

Proof. First, we shall prove that any convex function is continuous.

To begin with, the definition leads to a simple fact that

$$\frac{f(b)-f(a)}{b-a} \leqslant \frac{f(c)-f(b)}{c-b} \leqslant \frac{f(c)-f(a)}{c-a}$$

for any $0 \leqslant a \leqslant b \leqslant c$.

First we fix $b > 0$. We choose $0 \leqslant a < b$ and a sequence $\{c_n\}$ such that $c_n \leqslant b$ and $\lim\limits_{n\to\infty} c_n = b$. Notice that

$$g_n := \frac{f(c_n)-f(b)}{c_n - b}$$

is a decreasing sequence with a lower bound $(f(b)-f(a))/(b-a)$. By Monotone Convergence Theorem, $\lim g_n$ exists. In fact we have

$$\lim_{n\to\infty} f(c_n) - f(b) = 0$$

and hence f is right continuous at b. Similarly we can prove that f is left continuous at b so f is continuous at b.

Furthermore, we observe that $\lim\limits_{n\to\infty} g_n$ is the right derivative of f at b. We see that f is both left and right differentiable at any $x > 0$, with the right derivative larger than or equal to the left derivative. It implies that f' is non-decreasing with at most countably many discontinuities. By Lebesgue Theorem for differention we have that f'' exists a.e. and is non-negative.

(i) Since g is continuous and f is bounded measurable, $g \circ f$ is also bounded measurable and hence is integrable over $(0,1)$.

Suppose that $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha y + \beta \leqslant g(y)$ for all $y \geqslant 0$. Then $\alpha f(x) + \beta \leqslant g \circ f(x)$ for all $x \in [0,1]$. Then:

$$\alpha \int_0^1 f(x)\,\mathrm{d}x + \beta \leqslant \int_0^1 g \circ f(x)\,\mathrm{d}x$$

Hence

$$g\left(\int_0^1 f(x)\,\mathrm{d}x\right) = \sup\left\{\alpha \int_0^1 f(x)\,\mathrm{d}x + \beta\right\} \leqslant \int_0^1 g \circ f(x)\,\mathrm{d}x$$

(ii) Since f is bounded measurable over a bounded domain, $f^k \in \mathcal{L}^p[0,1]$ for any $k > 0$ and $p \geqslant 1$. By Hölder's Inequality with $p = q = 2$:

$$\left|\left|f^{\frac{n+1}{2}}\right|\right|_1 \left|\left|f^{\frac{n-1}{2}}\right|\right|_1 \leqslant \left|\left|f^{\frac{n+1}{2}}\right|\right|_2 \left|\left|f^{\frac{n-1}{2}}\right|\right|_2$$

Hence

$$\int_0^1 f^n \leqslant \sqrt{\int_0^1 f^{n+1}} \cdot \sqrt{\int_0^1 f^{n-1}}$$

Taking square on both sides, we obtain:

$$M_n^2 \leqslant M_{n+1} M_{n-1}$$

(iii) Since $t \mapsto t^{\frac{n+1}{n}}$ has non-negative second derivative, it is convex. By part (i):

$$\left(\int_0^1 f^n\right)^{\frac{n+1}{n}} \leqslant \int_0^1 (f^n)^{\frac{n+1}{n}} = \int_0^1 f^{n+1}$$

where LHS $= ||f||_n^{n+1} = M_n \, ||f||_n$ and RHS $= M_{n+1}$. We have $||f||_n \leqslant M_{n+1}/M_n$.

On the other hand, since $f \leqslant ||f_\infty||$ a.e., we have

$$M_{n+1} = \int_0^1 f \cdot f^n \leqslant \int_0^1 ||f||_\infty \, f^n = ||f||_\infty \, M_n$$

and hence $M_{n+1}/M_n \leqslant ||f||_\infty$.

(iv) Fix $\varepsilon > 0$. Let $S_\varepsilon := \{x \in [0,1] : \ f(x) \geqslant ||f||_\infty - \varepsilon\}$. S_ε is measurable as f is measurable. We have:

$$||f||_n = \left(\int_0^1 f^n\right)^{1/n} \geqslant \left(\int_{S_\varepsilon} f^n\right)^{1/n} \geqslant \left(\int_{S_\varepsilon} (||f||_\infty - \varepsilon)^n\right)^{1/n} = m(S_\varepsilon)^{1/n} \, (||f||_\infty - \varepsilon)$$

As $\liminf$ preserves weak limit, we have:

$$\liminf_{n\to\infty} ||f||_n \geqslant \liminf_{n\to\infty} m(S_\varepsilon)^{1/n} \, (||f||_\infty - \varepsilon) = ||f||_\infty - \varepsilon$$

As $\varepsilon > 0$ is arbitrary, we have $\liminf_{n\to\infty} ||f||_n \geqslant ||f||_\infty$. By part (iii) we know that $||f||_n \leqslant ||f||_\infty$ for all $n > 1$. We conclude that $\lim_{n\to\infty} ||f||_n$ exists and equals to $||f||_\infty$.

By part (iii) and sandwich lemma, we conclude that $\lim_{n\to\infty} M_{n+1}/M_n = ||f||_\infty$.

$\square$