Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 2

B2.1: Introduction to Representation Theory

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Throughout this sheet, k denotes a field, A denotes a ring and G denotes a finite group.

Question 1

For each $a \in A$, let $r_a : A \to A$ be the left A-linear map given by $r_a(b) = ba$ for each $b \in A$.

Prove that the map $r : A^{\mathrm{op}} \to \mathrm{End}_A(A)$ given by $r(a) = r_a$ is an isomorphism of rings.

Proof.

- r is a ring homomorphism: For $a, b, c \in A$:

$$r(a \star b)(c) = r_{ba}(c) = cba = r(b) \circ r(a)(c)$$

Hence $r(a \star b) = r(b) \circ r(a)$. $= r(a) \circ r(b)$

$$r(a+b)(c) = r_{a+b}c = c(a+b) = ca + cb = (r_a + r_b)(c)$$

Hence $r(a+b) = r(a) + r(b)$.

- r is injective: For $a \in \ker r$, $\forall c \in A: \; ca = 0$. In particular $1_A a = 0$. Hence $a = 0$. $\ker r = \{0\}$.
- r is surjective: Let $\omega : A \to A$ be an A-module homomorphism. Let $\omega(1_A) = \omega_0 \in A$. Then for any $a \in A$, $\omega(a) = \omega(a1_A) = a\omega(1_A) = a\omega_0$. Hence $\omega = r(\omega_0) \in \mathrm{im}\, r$.

Hence $r : A^{\mathrm{op}} \to \mathrm{End}_A(A)$ is an isomorphism of rings. A- □

Question 2

(a) Suppose that $|G| \neq 0$ in k and let $e := \frac{1}{|G|}\sum_{g\in G} g \in kG$. Prove that e is a central idempotent.

(b) Let $G = C_3 = \langle x \rangle$ be a cyclic group of order 3. Suppose that $\mathrm{char}(k) \neq 3$ and that k contains a primitive cube root of unity ω. Find an explicit isomorphism of k-algebras $k \times k \times k \xrightarrow{\cong} kC_3$.

Proof. (a) Fix $h \in G$. Note that $\rho_h : G \to G$ given by $\rho_h(g) = h^{-1}gh$ is a group isomorphism. Then

$$eh = \frac{1}{|G|}\sum_{g\in G} gh = \frac{1}{|G|}\sum_{g\in G} h\rho_h(g) = \frac{1}{|G|}\sum_{\rho_h(g)\in G} h\rho_h(g) = \frac{1}{|G|}\sum_{g\in G} hg = he$$

Extending this linearly, we have $ea = ae$ for any $a \in k[G]$. Hence $e \in \mathrm{Z}(k[G])$.

Next we notice that, for $g \in G$, the left multiplication by g^{-1} is a group isomorphism. Hence

not true $e \not\mapsto e$ its just a bijection!

$$e^2 = \frac{1}{|G|^2}\left(\sum_{g\in G} g\right)\left(\sum_{h\in G} h\right) = \frac{1}{|G|^2}\left(\sum_{g\in G} g\right)\left(\sum_{h\in G} g^{-1}h\right) = \frac{1}{|G|^2}\sum_{g\in G}\sum_{h\in G} h = \frac{1}{|G|}\sum_{h\in G} h = e$$

We conclude that e is a central idempotent.

(b) Note that by Chinese Remainder Theorem we have

$$k[C_3] \cong \frac{k[x]}{\langle x^3 - 1\rangle} \cong \frac{k[x]}{\langle x - 1\rangle} \times \frac{k[x]}{\langle x - \omega\rangle} \times \frac{k[x]}{\langle x - \omega^2\rangle} \cong k^3$$

To find the explicit isomorphism, we need to find $e_1 \in \langle x-1\rangle$, $e_2 \in \langle x - \omega\rangle$ and $e_3 \in \langle x - \omega^2\rangle$ such that $e_1 + e_2 + e_3 = 1 \in k[x]$. One choice is:

$$e_1 = -\frac{1}{3}(x-1), \quad e_2 = -\frac{1}{3}\omega^2(x-\omega), \quad e_3 = -\frac{1}{3}\omega(x-\omega^2)$$

need to choose different e_1, e_2, e_3!

Then it is clear from the proof of Chinese Remainder Theorem that $\sigma : k^3 \to k[C_3]$ given by

$$\sigma(\alpha, \beta, \gamma) = (\alpha e_1 + \beta e_2 + \gamma e_3)$$

is an isomorphism of rings. It is clear that it is also an isomorphism of k-modules. Hence it is an isomorphism of algebras. □

$e_1^2 = \frac{1}{9}(x-1)(x-1)$
$= \frac{1}{9}(x^2 - 2x + 1) \neq 1$

But $(1,0,0)^2 = (1,0,0)$

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Question 3

(a) Prove that every representation $\rho : G \to GL(V)$ extends to a k-algebra homomorphism $\tilde{\rho} : kG \to \mathrm{End}_k(V)$.

(b) Let $G = S_3$ and let $\rho : G \to GL(W)$ be the degree 2 representation from Example 1.20. Prove that $\tilde{\rho} : kG \to \mathrm{End}_k(W)$ is surjective, provided $\mathrm{char}(k) \neq 3$.

(c) Assume that the characteristic of k is not 2 or 3. Using part (b), prove that there is an isomorphism of k-algebras

$$kS_3 \xrightarrow{\cong} k \times k \times M_2(k)$$

Does such an isomorphism exist when $\mathrm{char}(k) = 3$? ← You didn't answer.

Hint: Consider $Z(kG)$ and $Z(k \times k \times M_2(k))$

Proof. (a) Define $\tilde{\rho} : k[G] \to \mathrm{End}_k(V)$ by

$$\tilde{\rho}\left(\sum_{g\in G} a_g g\right) := \sum_{g \in G} a_g \rho(g)$$

For $\sum_{g\in G} a_g g, \sum_{h \in G} b_h h \in k[G]$, $\alpha, \beta \in k$,

$$\tilde{\rho}\left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h\right) = \tilde{\rho}\left(\sum_{g\in G}\sum_{h\in G} a_g b_h gh\right) = \sum_{g\in G}\sum_{h\in G} a_g b_h \rho(gh) = \left(\sum_{g\in G} a_g \rho(g)\right)\left(\sum_{h\in G} b_h \rho(h)\right)$$
$$= \tilde{\rho}\left(\sum_{g\in G} a_g g\right)\tilde{\rho}\left(\sum_{h\in G} b_h h\right)$$
$$\tilde{\rho}\left(\alpha\sum_{g\in G} a_g g + \beta \sum_{h\in G} b_h h\right) = \tilde{\rho}\left(\sum_{g\in G}(\alpha a_g + \beta b_g) g\right) = \sum_{g\in G}(\alpha a_g + \beta b_g)\rho(g) = \alpha \sum_{g\in G} a_g \rho(g) + \beta \sum_{h\in H} b_h \rho(h)$$
$$= \alpha\tilde{\rho}\left(\sum_{g\in G} a_g g\right) + \beta\tilde{\rho}\left(\sum_{h\in G} b_h h\right)$$

(also $\tilde{\rho}(e_G) = \mathrm{id}_V$.)

Extra: Show this is unique

Look up universal property of kG!

Hence $\tilde{\rho}$ is a k-algebra homomorphism with $\tilde{\rho}|_G = \rho$.

(b) We claim that $\dim\ker\tilde{\rho} = 2$. We shall prove this by brute force. Let $\sum_{\sigma\in S_3} a_\sigma \sigma \in \ker\tilde{\rho}$. We know that $\{e_1 - e_2, e_2 - e_3\}$ is a basis of W. Then we have

$$\tilde{\rho}\left(\sum_{\sigma\in S_3} a_\sigma\sigma\right)(e_1 - e_2) = \tilde{\rho}\left(\sum_{\sigma\in S_3} a_\sigma\sigma\right)(e_2 - e_3) = 0$$

which implies that

$$\sum_{\sigma\in S_3} a_\sigma e_{\sigma(1)} = \sum_{\sigma\in S_3} a_\sigma e_{\sigma(2)} = \sum_{\sigma\in S_3} a_\sigma e_{\sigma(3)}$$

Since $S_3 = \{e, (12), (13), (23), (123), (132)\}$, we expand the expression above and equate the coefficients of e_1, e_2 and e_3:

If interested Try and prove this using (Jacobson Density theorem).

$$a_e + a_{(23)} = a_{(12)} + a_{(132)} = a_{(13)} + a_{(123)};$$
$$a_{(12)} + a_{(123)} = a_e + a_{(13)} = a_{(23)} + a_{(132)};$$
$$a_{(13)} + a_{(132)} = a_{(23)} + a_{(123)} = a_e + a_{(12)}.$$

Then a_σ must satisfy $a_e = a_{(123)} = a_{(132)}$ and $a_{(12)} = a_{(23)} = a_{(13)}$. Hence the $\dim\ker\tilde{\rho} \leqslant 2$. By first isomorphism theorem, $\dim\mathrm{im}\,\tilde{\rho} = \dim k[S_3] - \dim\ker\tilde{\rho} \geqslant 4$. Since $\mathrm{im}\,\tilde{\rho} \subseteq \mathrm{End}_k(W)$ and $\dim\mathrm{End}_k(W) = 4$. We deduce that $\mathrm{im}\,\tilde{\rho} = \mathrm{End}_k(W)$. Therefore $\tilde{\rho}$ is surjective.

(c) We don't know if k is algebraically closed, so we cannot use the Artin-Wedderburn Theorem.

The group S_3 has irreducible representations on k: $\rho_1 : S_3 \to GL(k)$ given by $\rho_1(\sigma) = 1_k$ and $\rho_2 : S_3 \to GL(k)$ given by $\rho_2(\sigma) = (-1)^\sigma 1_k$, where

$$(-1)^\sigma = \begin{cases} -1, & \sigma \in S_3 \text{ is an odd permtuation;} \\ 1, & \sigma \in S_3 \text{ is an even permtuation.} \end{cases}$$

The representations induce surjective k-algebra homomorphisms $\widetilde{\rho_1} : k[S_3] \to k$ and $\widetilde{\rho_2} : k[S_3] \to k$. Since ρ_1, ρ_2, ρ are distinct sub-representations of the left regular representation $k[S_3]$, we have the surjective $k[S_3]$-module homomor-

How do you know surjective?

This is not immediate.

phism:

$$\tau : k[S_3] \to k \times k \times M_2(k)$$

which is also k-linear. Notice that $\dim_k(k[S_3]) = \dim_k(k \times k \times M_2(k)) = 6$. Then τ is in fact a bijection. Therefore it is a $k[G]$-module isomorphism, which is also a k-algebra isomorphism. □

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Question 4

Find an example of a ring A that contains a field F such that A is not an F-algebra.

Proof. The ring of quaternions

$$\mathbb{H} = \{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : \ a, b, c, d \in \mathbb{R},\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1\}$$

contains the complex field $\mathbb{C}$, but is not a $\mathbb{C}$-algebra. The inclusion $\iota : \mathbb{C} \hookrightarrow \mathbb{H}$ defines a $\mathbb{C}$-module structure on $\mathbb{H}$. Suppose that it is a $\mathbb{C}$-algebra. Then

$$-1 = \mathrm{k}^2 = (\mathrm{ij})^2 = \mathrm{i}^2\mathrm{j}^2 = -1 \cdot -1 = 1$$

which is a contradiction.

From a higher perspective, $\mathbb{H}$ is a simple ring in the sense that it has no non-trivial two sided ideals. If it is a $\mathbb{C}$-algebra, then by Artin-Wedderburn Theorem we have $\mathbb{H} \cong M_n(\mathbb{C})$ as $\mathbb{C}$-algebras for some $n \in \mathbb{N}$. But

$$\dim_{\mathbb{C}} \mathbb{H} = \dim_{\mathbb{R}} \mathbb{H} / \dim_{\mathbb{R}} \mathbb{C} = 2 \neq n^2 = \dim_{\mathbb{C}} M_n(\mathbb{C})$$

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for any $n \in \mathbb{N}$, which is also a contradiction. Good □

Question 5

Let V be an A-module, let $D := \mathrm{End}_A(V)$ and let $n \geq 1$.

(a) Use the inclusion maps $\sigma_j : V \hookrightarrow V^n$ and the projection maps $\pi_j : V^n \to V (j = 1, \cdots, n)$ to construct an explicit ring isomorphism $M_n(D) \xrightarrow{\cong} \mathrm{End}_A(V^n)$.

(b) Prove that $M_n(S)^{\mathrm{op}}$ is isomorphic to $M_n\left(S^{\mathrm{op}}\right)$ for any ring S.

Proof. (a) Let $\rho : \mathrm{End}_A(V^n) \to M_n(D)$ given by $\rho(f) = \{f_{ij}\}$, where $f_{ij} = \pi_i \circ f \circ \sigma_j \in \mathrm{End}_A(V) = D$.

- ρ is a ring homomorphism: For $f, g \in \mathrm{End}_A(V^n)$, let $h = f \circ g$. We have

$$h_{ij} = \pi_i \circ f \circ g \circ \sigma_j = \pi_i \circ f \circ \mathrm{id}_{V^n} \circ g \circ \sigma_j = \sum_{k=1}^{n} (\pi_i \circ f \circ \sigma_k) \circ (\pi_k \circ g \circ \sigma_j) = \sum_{k=1}^{n} f_{ik} \circ g_{kj}$$

  in which we used the identity $\mathrm{id}_{V^n} = \sum_{k=1}^{n} \sigma_k \circ \pi_k$. Hence ρ is a ring homomorphism.

better to define inverse

- ρ is injective: Suppose that $f \in \ker \rho$. Write $f(v_1, ..., v_n) = (w_1, ..., w_n)$. Then $w_i = \sum_{j=1}^{n} f_{ij}(v_j) = 0$. Hence $f = 0$. ρ is injective.

- ρ is surjective: For $\{f_{ij}\} \in M_n(D)$, define $f : V^n \to V^n$ by $f(v_1, ..., v_n) = \sum_{j=1}^{n} (f_{1j}(v_j), ..., f_{nj}(v_j))$. Then $f \in \mathrm{End}_A(V)$. Hence f is surjective.

We conclude that $\rho : \mathrm{End}_A(V^n) \to M_n(D)$ is an isomorphism.

(b) It suffice to check that the identity map $\tau : M_n(S)^{\mathrm{op}} \to M_n(S^{\mathrm{op}})$ is a ring homomorphism. For $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ in $M_n(S)$,

$$\tau(A \star B) = A \star B = BA = \{b_{ik} a_{kj}\}_{i,j=1}^{n} = \{a_{kj} \star b_{ik}\}_{i,j=1}^{n} = \tau(A) \star \tau(B)$$

Hence τ is an isomorphism. □

$(BA)_{ij} = \Sigma_k b_{ik} a_{kj} = \Sigma_k a_{kj} \cdot_{op} b_{ik}$

$\neq$

Instead: $A \mapsto A^T$

$(AB)_{ij} = \Sigma_k a_{ik} b_{kj}$

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Question 6

Let V be a finite dimensional kG-module.

(a) Let W be a one-dimensional kG-module. Prove that $V \otimes W$ is simple if and only if V is simple.

(b) Prove that V is simple if and only if V^* is simple.

Proof. (a) *I think "W is a one-dimensional kG-module" in fact means that "W is a kG-module such that* $\dim_k(W) = 1$". Yes.

Since W is one-dimensional over k, we have k-module isomorphisms $V \otimes_k W \cong V \otimes_k k \cong V$.

"$\Longrightarrow$": Suppose that $V \otimes_k W$ is a simple $k[G]$-module. Let $U \subseteq V$ be a sub-$k[G]$-module. Then $U \otimes_k W$ is a sub-$k[G]$-module of $V \otimes_k W$. Hence $U = V$ or $U = \{0\}$. We deduce that V is a simple $k[G]$-module.

"$\Longleftarrow$": Suppose that V is a simple $k[G]$-module. Let $U \subseteq V \otimes_k W$ be a sub-$k[G]$-module. If $U \neq \{0\}$, then there exists $\sum_i \lambda_i v_i \otimes_k w_i \neq 0$ in U ($w_i \neq 0$). Note that $W = \langle w_0 \rangle$ as k-modules for some $w_0 \in W$. There exists $\eta_i \in k$ such that $w_i = \eta_i w_0$. Then

$$\sum_i \lambda_i v_i \otimes_k w_i = \left(\sum_i \lambda_i \eta_i v_i\right) \otimes_k w_0 \in U \Longrightarrow \left\langle \sum_i \lambda_i \eta_i v_i \right\rangle_{k[G]} \otimes_k \langle w_0 \rangle_{k[G]} \subseteq U$$

more details please !!.

Both V and W are simple $k[G]$-modules. Hence $U = V \otimes_k W$. We deduce that $V \otimes_k W$ is a simple $k[G]$-module.

(b) We have the canonical isomorphism $V \cong V''$ (of kG-Modules) so it suffices to prove one direction. Suppose that V^* is a simple $k[G]$-module.

Let $U \subseteq V$ be a sub-$k[G]$-module. Then U is a sub-k-module of V, and the annihilator

$$U^0 := \{\varphi \in V' : \forall\, u \in U(\varphi(u) = 0)\} \subseteq V'$$

is a sub-k-module of V'. It is also a sub-$k[G]$-module of V', because for $\varphi \in U^0$, $u \in U$ and $\sum_{a \in G} a_g g \in k[G]$,

$$\left(\sum_{a \in G} a_g g\right) \cdot \varphi(u) = \sum_{a \in G} a_g \varphi\left(g^{-1} \cdot u\right) = 0 \qquad (u \in U \Longrightarrow g^{-1} \cdot u \in U)$$

so $\left(\sum_{a \in G} a_g g\right) \cdot \varphi \in U^0$. If V' is simple, then $U^0 = V'$ or $\{0\}$. So $U = V$ or $\{0\}$. Hence V is a simple $k[G]$-module. A- □

Question 7

Recall the maps $\alpha : V^* \otimes W \to \mathrm{Hom}(V, W)$ and $\beta : \mathrm{Hom}(V, W) \to V^* \otimes W$ from Lemma 4.11.

(a) Prove that $\beta \circ \alpha = 1_{V^* \otimes W}$.

(b) Prove that α is a homomorphism of kG-modules.

Proof. (a) A Remainder:

$$\alpha(\varphi \otimes_k w) : v \mapsto \varphi(v) w, \qquad \beta(T) = \sum_{i=1}^{n} v_i' \otimes_k T(v_i)$$

Then

$$\beta \circ \alpha(\varphi \otimes_k w) = \sum_{i=1}^{n} v_i' \otimes_k \alpha(\varphi \otimes_k w)(v_i) = \sum_{i=1}^{n} v_i' \otimes_k \varphi(v_i) w$$

Since $w = \sum_{j=1}^{n} v_j'(w) v_j$ and $\varphi = \sum_{j=1}^{n} \varphi(v_j) v_j'$,

$$\sum_{i=1}^{n} v_i' \otimes_k \varphi(w) v_i = \sum_{i=1}^{n} v_i' \otimes_k \varphi(v_i) \left(\sum_{j=1}^{n} v_j'(w) v_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i' \otimes_k v_j'(w) \varphi(v_i) v_j = \left(\sum_{i=1}^{n} \varphi(v_i) v_i'\right) \otimes_k \left(\sum_{j=1}^{n} v_j'(w) v_j\right) = \varphi \otimes_k w$$

By linearity, we deduce that $\beta \circ \alpha = \mathrm{id}_{V' \otimes_k W}$.

(b) α is already k-linear. For $g \in G$, $\varphi \in V'$, $w \in W$ and $v \in V$,

$$\alpha(g \cdot (\varphi \otimes_k w))(v) = \alpha((g \cdot \varphi) \otimes_k (g \cdot w))(v) = (g \cdot \varphi)(v)(g \cdot w) = \varphi(g^{-1} \cdot v)(g \cdot w) = (g \cdot \alpha(\varphi \otimes_k w))(v)$$

Hence $\alpha(g \cdot (\varphi \otimes_k w)) = g \cdot \alpha(\varphi \otimes_k w)$. Extending this linearly on $k[G]$, we deduce that α is a $k[G]$-module homomorphism. □

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Question 8

Let U, V, W be finite dimensional kG-modules. Prove $\mathrm{Hom}(U \otimes V, W)$ is isomorphic to $\mathrm{Hom}(U, \mathrm{Hom}(V, W))$ as kG-modules.

Proof. For $\sigma \in \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$, $\sigma(u) \in \mathrm{Hom}_k(V, W)$. Thus σ defines a k-bilinear map $\varphi : U \times V \to W$ given by $\varphi(u, v) = \sigma(u)(v)$. By universal property of $U \otimes_k V$, there exists a unique k-linear map $\overline{\varphi} : U \otimes_k V \to W$ such that $\overline{\varphi}(u \otimes_k v) = \sigma(u)(v)$. We claim that $\mathscr{F} : \sigma \mapsto \overline{\varphi}$ defines a $k[G]$-module isomorphism from $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$ to $\mathrm{Hom}_k(U \otimes V, W)$.

We shall prove that $\mathscr{F}$ is a $k[G]$-module homomorphism. For $\sigma \in \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$, $u \in U$, $v \in V$ and $g \in G$,

$$\begin{aligned}(g \cdot \sigma)(u)(v) &= \left(g \cdot \sigma(g^{-1} \cdot u)\right)(v) = g \cdot \sigma(g^{-1} \cdot u)(g^{-1} \cdot v) \\ &= g \cdot \overline{\varphi}((g^{-1} \cdot u) \otimes_k (g^{-1} \cdot v)) = (g \cdot \overline{\varphi})(g^{-1} \cdot (u \otimes_k v)) = (g \cdot \overline{\varphi})(u \otimes_k v)\end{aligned}$$

By extending g linearly in $k[G]$, we deduce that $\mathscr{F}$ is a $k[G]$-module homomorphism.

$\mathscr{F}$ is injective:

$$\sigma \in \ker \mathscr{F} \implies \overline{\varphi} = 0 \implies \forall u \in U\ \forall v \in V\ \overline{\varphi}(u \otimes_k v) = 0 \implies \forall u \in U\ \forall v \in V\ \sigma(u)(v) = 0 \implies \forall u \in U\ \sigma(u) = 0 \implies \sigma = 0$$

Hence $\ker \mathscr{F} = \{0\}$.

So surj! by dimension!

$\mathscr{F}$ is surjective: Let $\overline{\varphi} \in \mathrm{Hom}_k(U \otimes_k V, W)$. For $u \in U$, $\sigma_u : v \mapsto \overline{\varphi}(u \otimes_k v)$ is k-linear. In addition $\sigma : u \mapsto \sigma_u$ is also k-linear. Hence there is $\sigma \in \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$ such that $\mathscr{F}(\sigma) = \overline{\varphi}$.

We conclude that $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \cong \mathrm{Hom}_k(U \otimes V, W)$ as $k[G]$-modules. □

Remark. The functor $- \otimes_k V$ is the left adjoint to the functor $\mathrm{Hom}_k(V, -)$. Yes, you need to show naturality in U, W too for this!

It seems to me that the definition of $k[G]$-module structure on k-vector spaces given in the notes can be formulated in a more categorical way. For example, it may be defined as a faithful functor from k-Vect to $k[G]$-Mod such that it commutes with the tensor functors.

not sure what you mean by this ☺