Topology & Groups

Michaelmas 2016

Question Sheet 5

Questions with an asterisk $*$ beside them are optional.

1. If $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$, supply (with proof) a presentation for $G_1 \times G_2$. Deduce that if G_1 and G_2 are finitely presented, then so is $G_1 \times G_2$.

2. Show that $\langle a, b \mid aba^{-1}b^{-1}, a^5b^2, a^2b \rangle$ is the trivial group. [Hint: don't try to use Tietze transformations.]

3. Show that the group of symmetries of a regular n-sided polygon is $\langle a, b \mid a^n, b^2, abab \rangle$. [Hint: you will find it useful to show that $\langle a, b \mid a^n, b^2, abab \rangle$ has at most $2n$ elements.]

4. Show that $\langle a, b \mid abab^{-1} \rangle \cong \langle c, d \mid c^2d^2 \rangle$, by setting up an explicit isomorphism between them. [Hint: Note that in the first group, $(ab)(ab) = (aba)b = b^2$.]

5. Prove that the push-out of
$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$
is isomorphic to $\mathbb{Z}$.

6. Show that the group $\langle x, y \mid xyx = yxy \rangle$ is isomorphic to the push-out of
$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 3} & & \\ \mathbb{Z} & & \end{array}$$
[*Hint: consider the elements xy and yxy.*] Is this an amalgamated free product?

$*$ 7. A group-theoretic property P is known as *semi-decidable* if there is an algorithm that starts with a finite presentation of a group G and *either* terminates with a 'yes' answer if G has property P, *or* does not terminate if G does not have property P. Prove that the following properties of a group are semi-decidable:

(i) being abelian;

(ii) being free;

(iii) being a specific finite group (which is given by its multiplication table);

(iv) being finite.

Topology & Groups 5

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1. Let $[X_1, X_2] := \{aba^{-1}b^{-1} \in F(X_1 \sqcup X_2) : a \in X_1,\ b \in X_2\}$.

We claim that $G_1 \times G_2 = \langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup [X_1, X_2] \rangle$

Let H be the group with presentation on the RHS.

We shall show that $H \cong G_1 \times G_2$:

$$\begin{tikzcd}
& G_1 \times G_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 & & G_2 \\
& H \arrow[ul, "\varphi_1"] \arrow[uu, "\varphi"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Let $f_1 : X_1 \sqcup X_2 \to G_1$ be a function such that :

$f_1(a) = a$, $f_1(b) = \cancel{b}\, e_1$ $\forall a \in X_1$, $\forall b \in X_2$.

f_1 induces a group homomorphism $\sigma_1 : F(X_1 \sqcup X_2) \to G_1$

For $r \in R_1$: $\sigma_1(r) = r = e_1$ given that $G_1 = \langle X_1 \mid R_1 \rangle$

For $r \in R_2$: $\sigma_1(r) = e_1$ by definition of f_1 .

For $aba^{-1}b^{-1} \in [X_1, X_2]$: $\sigma_1(aba^{-1}b^{-1}) = ae_1a^{-1}e_1 = e_1$.

Then by Lemma 5.11, σ_1 induces a group homomorphism :

$\varphi_1 : H \to G_1$. Moreover, φ_1 is surjective .

Similarly we define $f_2 : X_1 \sqcup X_2 \to G_2$ by :

$f_2(a) = e_2$, $f_2(b) = b$ $\forall a \in X_1$, $\forall b \in X_2$.

This gives an epimorphism $\varphi_2 : H \to G_2$.

Let $\varphi : H \to G_1 \times G_2$ given by $\varphi := (\varphi_1, \varphi_2)$.

It suffices to show that φ is bijective :

Surjectivity : For $(g_1, g_2) \in G_1 \times G_2$:

$g_1 g_2 \in H$. And $\varphi(g_1 g_2) = (\varphi_1(g_1 g_2), \varphi_2(g_1 g_2))$

$= (\varphi_1(g_1), \varphi_2(g_2)) = (g_1, g_2)$.

Hence φ is surjective .

Injectivity : Suppose $h \in \operatorname{Ker} \varphi$. Then $\varphi_1(h) = e_1$, $\varphi_2(h) = e_2$.

Since $[X_1, X_2]$ is a relation in H, we have
$x_1 x_2 = x_2 x_1 \in H$ for any $x_1 \in X_1$, $x_2 \in X_2$.
$h \in H$ is a finite string in $X_1 \sqcup X_2$. We can insert the relation $x_2^{-1} x_1 x_2 x_1^{-1}$ 所 between each $x_2 x_1$ with $x_1 \in X_1$ and $x_2 \in X_2$. Therefore we have $h = g_1 g_2$ with g_1 being a string in X_1 and g_2 being a string in X_2.

$\varphi_1(h) = e_1 \Rightarrow \varphi_1(g_1 g_2) = \varphi_1(g_1) = e_1 \Rightarrow g_1 = e_1$

$\varphi_2(h) = e_2 \Rightarrow \varphi_2(g_1 g_2) = \varphi_2(g_2) = e_2 \Rightarrow g_2 = e_2$.

$\Rightarrow h = e_H$. $Ker\, \varphi = \{e_H\}$.

Hence φ is injective. $\varphi : H \to G_1 \times G_2$ is an isomorphism.

$G_1 \times G_2 \cong H = \langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup [X_1, X_2] \rangle$

If G_1, G_2 are finitely presented, then X_1, X_2, R_1, R_2 are all finite sets.
Then $\text{card}[X_1, X_2] = \text{card}(X_1 \times X_2)$ 所 is finite.
$\Rightarrow X_1 \sqcup X_2$ is finite, $R_1 \cup R_2 \cup [X_1, X_2]$ is finite
$\Rightarrow G_1 \times G_2$ is finitely presented.

2. The relations indicate that:

$aba^{-1}b^{-1} = e \quad \cdots ①$ $\qquad a^5 b^2 = e \quad \cdots ②$ $\qquad a^2 b = e \quad \cdots ③$

① implies that $ab = ba$ or a and b commutes.
③ implies that $b = a^{-2}$ or $b^{-1}a^{-2} = e$.
Then $a^5 b^2 = e \Rightarrow a^5 b^2 b^{-1} a^{-2} b^{-1} a^{-2} = e$.

$\Rightarrow a^5 b a^{-2} b^{-1} a^{-2} = e \Rightarrow a^5 a^{-1} b a a^{-2} b^{-1} a^{-2} = e$

$\Rightarrow a^4ba^{-1}b^{-1}a^{-2}=e \Rightarrow a^4a^{-1}baa^{-1}ba^{-2}=e$

$\Rightarrow a^3bb^{-1}a^{-2}=e \Rightarrow a^3a^{-2}=e \Rightarrow a=e$

Then $b=a^{-2}=e^{-2}=e$.

Then $\langle a, b \mid aba^{-1}b^{-1}, a^5b^2, a^2b\rangle$

$=\langle a, b \mid a, b, aba^{-1}b^{-1}, a^5b^2, a^2b\rangle = \{e\}$

is the trivial group.

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3. Let D_{2n} be the group of symmetries of a regular n-sided polygon. ~~Let a be~~ We use $\{0, 1, \cdots, n-1\}$ to denote the vertices of the polygon. Let a be the anti-clockwise rotation by $2\pi/n$ and Let b be the reflection that fixes vertex 0. We shall show that

$D_{2n}=\langle a, b \mid a^n, b^2, abab\rangle$

① D_{2n} has $2n$ elements:

The position of vertex 0 (n choices) and the position of vertex 1 (either at LHS or RHS of 0) uniquely determines the symmetry. So there are $2n$ symmetries in total.

② D_{2n} is generated by $\{a, b\}$:

In a symmetry, if $0 \mapsto k \in \{0, \cdots, 1-n\}$ and the orientation does not change, then it is given by ~~r^k~~ a^k;

if $0 \mapsto k \in \{0, \cdots, 1-n\}$ and the orientation is reversed, then it is given by ~~r^ks~~ a^kb.

$\Rightarrow D_{2n}=\{e, a, \cdots, a^{n-1}, b, ab, \cdots, a^{n-1}b\}$.

③ From geometry we see that $a^n=e$ (rotation by 2π),

$b^2 = e$ (reflection twice), $abab = e$.

Then we have $D_{2n} \leq \langle a, b \mid a^n, b^2, abab \rangle$

④ $|\langle a, b \mid a^n, b^2, abab\rangle| \leq 2n$:

Consider a word of the form :

$$a^{i_1} b^{j_1} a^{i_2} b^{j_2} \in \langle a, b \mid a^n, b^2, abab \rangle$$

Since $a^n = b^2 = e$, we may assume that $j_1, j_2 \in \{0, 1\}$

and ~~fi~~ $i_1, i_2 \in \{0, \cdots, 1-n\}$.

Since $abab = e$, $b^2 = e$, we have $ab = ba^{-1}$

Inductively $a^m b = ba^{-m}$ for $m \in \mathbb{Z}$.

$\cancel{\Rightarrow a^m b^n = b^n a^{-m} \text{ for } m, n \in \mathbb{Z}}$.

If $j_1 = 0$, then $a^{i_1} b^{j_1} a^{i_2} b^{j_2} = a^{i_1+i_2} b^{j_2} = b^{j_2} a^{-(i_1+i_2)}$

$\cancel{\text{If } j_1 = 1 \text{, then } a^{i_1} b^{j_1} a^{i_2} b^{j_2} = b}$ $\quad = a^{i_1+i_2} b^{j_2}$

If $j_1 = 1$, then $a^{i_1} b^{j_1} a^{i_2} b^{j_2} = ba^{-i_1+i_2} b^{j_2} = b^{j_1+j_2} a^{-i_2+i_1}$

$= a^{-i_1+i_2} b^{j_1}$

We can in fact show that $a^{i_1} b^{j_1} a^{i_2} b^{j_2}$ equals to :

	$j_2 = 0$	$j_2 = 1$
$j_1 = 0$	$a^{i_1+i_2}$	$a^{i_1+i_2} b$
$j_1 = \cancel{0}\,1$	$a^{i_1-i_2} b$	$a^{i_1-i_2}$

Inductively, for any finite string $a^{i_1} b^{j_1} \cdots a^{i_r} b^{j_r}$ in the alphabet $\{a, b\}$, we can reduce it to

$a^i b^j$ $\quad (i \in \{0, \cdots, 1-n\}, j \in \{0, 1\})$

Hence $|\langle a, b \mid a^n, b^2, abab \rangle| \leq 2n$.

But $|D_{2n}| = 2n$. We conclude that

$D_{2n} = \langle a, b \mid a^n, b^2, abab \rangle$.

4. Let $G_1 = \langle c, d \mid c^2d^2 \rangle$, $G_2 = \langle a, b \mid abab^{-1} \rangle$.

Define $f : \{c, d\} \to G_2$ by :

$$f(c) = ab, \quad f(d) = b^{-1}.$$

This induces a group homomorphism $\sigma : F(\{c, d\}) \to G_2$.

$$\sigma(c^2d^2) = (ab)^2(b^{-1})^2 = ababb^{-2} = abab^{-1} = e.$$

By Lemma 5.11, it induces a group homomorphism

$$\varphi : G_1 \to G_2.$$

Since G_2 is generated by $a = \varphi(cd)$ and $b = \varphi(d^{-1})$,

φ is surjective.

~~For $g \in \ker\varphi$,~~

Conversely, we can construct epimorphism $\psi : G_2 \to G_1$

such that $\psi(a) = cd$, $\psi(b) = d^{-1}$. You don't need to show these are surjective

$\Rightarrow \varphi \circ \psi = id_{G_2}$, $\psi \circ \varphi = id_{G_1}$.

$\Rightarrow \varphi : G_1 \to G_2$ is an isomorphism

$\langle c, d \mid c^2d^2 \rangle \cong \langle a, b \mid abab^{-1} \rangle$ NT

5. For the following push-out :

$$\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}.$$

The group on the left is $G_1 := \mathbb{Z} = \langle x \mid \ \rangle$ where $x = 1$

The group on the right is $G_2 := \mathbb{Z} = \langle y \mid \ \rangle$ where $y = 2$

The push-out $G_1 *_{\mathbb{Z}} G_2$, by Lemma V.20. is isomorphic to

$\langle x, y \mid \{ id(g) = 2g : g \in \mathbb{Z} \} \rangle$

$\cong \langle x, y \mid \underline{2x} = y \rangle$ or $\langle x, y \mid \overset{2xy^{-1}}{\cancel{2yx^{-1}}} \rangle$ This additive notation works best for commutative groups

$\cong \langle \overset{y}{\cancel{x}} \mid \ \rangle$ by Tietze transformation 5

$\cong \mathbb{Z}$ A

6. Consider the push-out :

$$G_1 \xleftarrow{\times 3} \mathbb{Z} \xrightarrow{\times 2} G_2$$

where $G_1 = \langle a \mid \ \rangle$, $G_2 = \langle b \mid \ \rangle$. $(a = 3,\ b = 2)$

Then $G_1 *_{\mathbb{Z}} G_2 = \langle a, b \mid 2a = 3b \rangle = \langle a, b \mid a^3 b^{-2} \rangle$

We show that $G_1 *_{\mathbb{Z}} G_2$ is isomorphic to $\langle x, y \mid xyx = yxy \rangle$

by a sequence of Tietze transformations :

~~<a, b | 2a =~~

$$\begin{aligned}
&\langle x, y \mid xyx = yxy \rangle = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle \\
&\cong \langle x, y, a, b \mid xyxy^{-1}x^{-1}y^{-1},\ xya^{-1},\ yxyb^{-1} \rangle && (T5) \\
&\cong \langle x, y, a, b \mid xyx(yxy)(yxy)^{-2},\ xya^{-1},\ yxyb^{-1} \rangle && (T3) \\
&\cong \langle x, y, a, b \mid a^3 b^{-2},\ xya^{-1},\ yxyb^{-1} \rangle && (T4) \\
&\cong \langle x, y, a, b \mid a^3 b^{-2},\ xya^{-1},\ yab^{-1} \rangle && (T4) \\
&\cong \langle x, y, a, b \mid a^3 b^{-2},\ xba^{-2},\ yab^{-1} \rangle && (T4) \\
&\cong \langle x, y, a, b \mid a^3 b^{-2},\ a^2 b^{-1} x^{-1},\ ba^{-1} y^{-1} \rangle && (T4) \\
&\cong \langle a, b \mid a^3 b^{-2} \rangle && (T5)
\end{aligned}$$

which completes the proof.

This is an amalgamated free product because both $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ are injective. Great

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