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Problem Sheet 4

String Theory II

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1 Spin Connections and Killing Spinors

B/C

[*Conventions for this questions are as in BLT Section 14.8.*]

The spin connection ω is a connection for the local Lorentz symmetry in a given representation and can be expanded in terms of 1-forms

$$\omega = \omega_\mu(x)\mathrm{d}x^\mu.$$

As for gauge connections in Yang–Mills theory we can define the curvature 2-form as

$$\mathcal{R} = \mathrm{d}\omega + \omega \wedge \omega.$$

Infinitesimal local Lorentz transformations map

$$\delta_\Lambda \omega = \mathrm{d}\Lambda + [\omega, \Lambda]$$

and so $\delta_\Lambda \mathcal{R} = [\mathcal{R}, \Lambda]$. Let e^a_μ be the viel-bein where $a, b, \cdots$ are the flat indices and $\mu, \nu \cdots$ the curved indices. Let ∇_μ be the covariant derivative with Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).$$

The spin connection is then the 1-form valued in the local Lorentz algebra (ω has a μ curved index, and is a matrix valued object with a, b flat indices) that satisfies

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\mu b} e^b_\nu = 0$$

In components it is given by

$$\omega^{ab}_\mu = \frac{1}{2} \left(\Omega_{\mu\nu\rho} - \Omega_{\nu\rho\mu} + \Omega_{\rho\mu\nu} \right) e^{\nu a} e^{\rho b} \quad \text{where} \quad \Omega_{\mu\nu\rho} = \left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu \right) e_{a\rho}.$$

Question 1.1

Determine the components of the curvature 2-tensor ${R_{\mu\nu}}^{ab}$ in terms of ω.

Proof. The curvature 2-form is defined as

$$R = \mathrm{d}\omega + \omega \wedge \omega.$$

In terms of the frame bundle indices, this becomes a equation of real-valued 2-forms on the manifold:

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$$

Expanding in terms of the coordinate indices:

$$\begin{aligned} {R_{\mu\nu}}^a{}_b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu &= \mathrm{d}({\omega_\mu}^a{}_b \mathrm{d}x^\mu) + {\omega_\mu}^a{}_c \mathrm{d}x^\mu \wedge {\omega_\nu}^c{}_b \mathrm{d}x^\nu \\ &= \left(\partial_\mu {\omega_\nu}^a{}_b - \partial_\nu {\omega_\mu}^a{}_b + {\omega_\mu}^a{}_c {\omega_\nu}^c{}_b - {\omega_\nu}^a{}_c {\omega_\mu}^c{}_b \right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \end{aligned}$$

Therefore we have

$${R_{\mu\nu}}^{ab} = \partial_\mu {\omega_\nu}^{ab} - \partial_\nu {\omega_\mu}^{ab} + {\omega_\mu}^a{}_c {\omega_\nu}^{cb} - {\omega_\nu}^a{}_c {\omega_\mu}^{cb}. \quad ✓$$

□

Question 1.2

Consider 10 d spacetime $\mathbb{R}^{1,3} \times M_6$, with local Lorentz group $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$ in the spinor representation as in the lecture (i.e. the generators are Γ^{ab}). Let ϵ be a 16 component spinor of $\mathrm{SO}(1,9)$. Compute $[\nabla_\mu, \nabla_\nu]\,\epsilon$.

Proof. The $\mathfrak{so}(1,9)$ generators T_{ab} satisfies the usual Lorentz algebra:

$$[T_{ab}, T_{cd}] = \mathrm{i}\,(\eta_{ac}T_{bd} - \eta_{ad}T_{bc} - \eta_{bc}T_{ad} + \eta_{bd}T_{ac})\,.$$

In the spinor representation, the generators are given by $T_{ab} = -\frac{\mathrm{i}}{4}[\Gamma_a, \Gamma_b]$, where Γ^a satisfy the (flat) Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\,\mathrm{id}$. Let ϵ be a 16 component Weyl spinor of $\mathfrak{so}(1,9)$. The covariant derivative of ϵ is given by

$$\nabla_\mu \epsilon = \left(\partial_\mu \epsilon + \frac{\mathrm{i}}{2}\omega_\mu{}^{ab}T_{ab}\epsilon\right)\mathrm{d}x^\mu$$

which gives the curvature $R_{\mu\nu}\epsilon = [\nabla_\mu, \nabla_\nu]\epsilon$ (*as expected*).

yes, but you had to do this computation!

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{\mathrm{i}}{2}R_{\mu\nu}{}^{ab}T_{ab}\epsilon = \frac{1}{8}R_{\mu\nu\rho\sigma}e^{a\rho}e^{b\sigma}[\Gamma_a, \Gamma_b]\epsilon = \frac{1}{8}R_{\mu\nu\rho\sigma}[\Gamma^\mu, \Gamma^\nu]\epsilon$$

it was the point of the exercise

where Γ^μ satisfy the (curved) Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}\,\mathrm{id}$.

The decomposition under $\mathfrak{so}(1,3) \oplus \mathfrak{so}(6)$ is given by $\mathbf{16} = \mathbf{2}_L \otimes \overline{\mathbf{4}} \oplus \mathbf{2}_R \otimes \mathbf{4}$, where $\mathbf{2}_L, \mathbf{2}_R$ are Weyl spinors of $\mathfrak{so}(1,3)$ and $\mathbf{4}, \overline{\mathbf{4}}$ are Weyl spinors of $\mathfrak{so}(6)$. Since $\mathbb{R}^{1,3}$ is flat, we only need to concern the compactified manifold M_6. *It has been shown in the lectures that the covariantly constant condition $\nabla_\mu \epsilon = 0$ on the spinor impose restrictions on the metric such that the manifold M_6 is Ricci-flat. More specifically,*

$$\begin{aligned}
&[\nabla_m, \nabla_n]\epsilon = \frac{1}{8}R_{mnpq}[\Gamma^p, \Gamma^q]\epsilon = 0 \\
&\implies 0 = 2R_{mnpq}\Gamma^n[\Gamma^p, \Gamma^q]\epsilon = R_{mnpq}(\Gamma^{[n}\Gamma^p\Gamma^{q]} + g^{np}\Gamma^q - g^{nq}\Gamma^p)\epsilon = 2R_{mq}\Gamma^q\epsilon,
\end{aligned}$$

where $m, n, p, q = 4, ..., 9$. The Riemannian signature on M_6 implies that the coefficients of $\Gamma^q\epsilon$ should vanish. That is, the Ricci components $R_{mq} = 0$.

I am not sure if this is what the question asks. A further decomposition of the Weyl spinor in $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ depends on the holonomy of the manifold M_6. If M_6 is Calabi–Yau, then it has $\mathrm{SU}(3)$ holonomy, and the Weyl spinor decomposes into a singlet and a triplet: $\mathbf{4}_{\mathfrak{su}(4)} = (\mathbf{1} \oplus \mathbf{3})_{\mathfrak{su}(3)}$ under the decomposition $\mathfrak{su}(4) \to \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. This provides $\mathcal{N} = 1$ supersymmetry on $\mathbb{R}^{1,3}$. If M_6 is a general manifold, I am not sure what can be said here... □

2 Kähler Manifolds, Projective Space

A/B

Question 2.1

Let X_n be a complex n-dimensional Kähler manifold. Determine the non-trivial Christoffel symbols and the Ricci tensor for X_n.

Proof. By definition, a Kähler manifold (X_n, g, J) is a complex manifold with an Hermitian fundamental form, locally given by $\omega = \mathrm{i}g_{\mu\overline{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu$, which is closed: $\mathrm{d}\omega = 0$.

The fundamental form ω is related to the Riemannian metric g on X_n by

$$\omega(X, Y) = g(JX, Y), \qquad X, Y \in \Gamma(\mathrm{T}X_n),$$

where $J \in \mathrm{End}(\mathrm{T}X_n)$ satisfying $J^2 = -\mathrm{id}$ is the almost complex structure. This justifies the above local expansion $\omega_{\mu\overline{\nu}} = \mathrm{i}g_{\mu\overline{\nu}}$ (*cf.* §1.2 of Huybrechts). The closedness of ω imposes the constraints on the Hermitian metric g:

$$\mathrm{d}\omega = \partial\omega + \overline{\partial}\omega = 0 \implies \partial\omega = \overline{\partial}\omega = 0 \implies \partial_\mu g_{\nu\overline{\rho}} = \partial_\nu g_{\mu\overline{\rho}} \text{ and } \partial_{\overline{\mu}} g_{\nu\overline{\rho}} = \partial_{\overline{\nu}} g_{\mu\overline{\rho}}.$$

Next we compute the Christoffel symbols. On a Kähler manifold the Chern connection coincides with the Levi-Civita connection, so there is no ambiguity. By positive definite Hermitian metric g we have $g_{\mu\nu} = 0$ and $g_{\overline{\mu}\overline{\nu}} = 0$. Combining with the Kähler condition, we can show by the Koszul formulae that all Christoffel symbols with mixed indices vanish:

$$\begin{aligned}
\Gamma^{\rho}{}_{\overline{\mu}\overline{\nu}} &= \frac{1}{2} g^{\overline{\sigma}\rho}(\partial_\mu g_{\overline{\sigma}\overline{\nu}} + \partial_{\overline{\nu}} g_{\overline{\sigma}\overline{\mu}} - \partial_{\overline{\sigma}} g_{\overline{\mu}\overline{\nu}}) = 0 \\
\Gamma^{\overline{\rho}}{}_{\mu\nu} &= \frac{1}{2} g^{\sigma\overline{\rho}}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) = 0 \\
\Gamma^{\rho}{}_{\mu\overline{\nu}} &= \frac{1}{2} g^{\overline{\sigma}\rho}(\partial_\mu g_{\overline{\sigma}\overline{\nu}} + \partial_{\overline{\nu}} g_{\overline{\sigma}\mu} - \partial_{\overline{\sigma}} g_{\mu\overline{\nu}}) = \frac{1}{2} g^{\overline{\sigma}\rho}\partial_\mu g_{\overline{\sigma}\overline{\nu}} = 0 \\
\Gamma^{\overline{\rho}}{}_{\mu\overline{\nu}} &= \frac{1}{2} g^{\sigma\overline{\rho}}(\partial_\mu g_{\sigma\overline{\nu}} + \partial_{\overline{\nu}} g_{\sigma\mu} - \partial_\sigma g_{\mu\overline{\nu}}) = \frac{1}{2} g^{\sigma\overline{\rho}}\partial_{\overline{\nu}} g_{\sigma\mu} = 0
\end{aligned}$$

The non-zero Christoffel symbols are given by

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2} g^{\rho\overline{\sigma}}(\partial_\mu g_{\overline{\sigma}\nu} + \partial_\nu g_{\overline{\sigma}\mu} - \partial_{\overline{\sigma}} g_{\mu\nu}) = g^{\rho\overline{\sigma}}\partial_\mu g_{\nu\overline{\sigma}}, \\
\Gamma^{\overline{\rho}}{}_{\overline{\mu}\overline{\nu}} &= \frac{1}{2} g^{\overline{\rho}\sigma}(\partial_{\overline{\mu}} g_{\sigma\overline{\nu}} + \partial_{\overline{\nu}} g_{\sigma\overline{\mu}} - \partial_\sigma g_{\overline{\mu}\overline{\nu}}) = g^{\overline{\rho}\sigma}\partial_{\overline{\mu}} g_{\overline{\nu}\sigma}.
\end{aligned}$$

Next we compute the Ricci curvature. The components with pure indices $R_{\mu\nu}$, $R_{\overline{\mu},\overline{\nu}}$ vanish in the same reason as those of g. More specifically, the Ricci curvature is J-invariant: $\mathrm{Ric}(X,Y) = \mathrm{Ric}(JX,JY)$, which follows from the compatibility equation $\nabla J = 0$. The components with mixed indices[1] are given by:

$$R_{\mu\overline{\nu}} = -\partial_{\overline{\nu}}\Gamma^{\rho}{}_{\mu\rho} = -\partial_{\overline{\nu}}(g^{\rho\overline{\sigma}}\partial_\mu g_{\rho\overline{\sigma}}) = -\frac{1}{2}\partial_{\overline{\nu}}\,\mathrm{tr}\big(g^{-1}\partial_\mu g\big) = -\frac{1}{\underbrace{2}}\partial_\mu\partial_{\overline{\nu}}(\log\det g). \qquad \square$$

should be 1

Question 2.2

Consider $\mathbb{P}^n = \mathbb{C}^{n+1}/\sim$, n-dimensional projective space, where the equivalence relation is $\left(z^0, \cdots, z^n\right) \sim \left(w^0, \cdots, w^n\right)$ if $\exists\lambda \neq 0$ such that $\left(z^0, \cdots, z^n\right) = \lambda\left(w^0, \cdots, w^n\right)$. An open cover is given by $U_r = \{z^r \neq 0\}$, with local coordinates in U_r given by $z^i_{(r)}, i = 1, \cdots, n$.

i. Determine charts ϕ_r and transition functions $\phi_r \circ \phi_s^{-1}$ which make this into a complex manifold.

ii. Determine the metric that follows from the Kähler potential in the open patch U_r

$$K_{(r)} = \log\left(1 + \sum_{i=1}^{n}\left|z^i_{(r)}\right|^2\right)$$

This is the Fubini–Study metric for $\mathbb{P}^n$.

iii. Compute the Ricci form for this metric.

[1] In general, the components of the Ricci curvature are given by

$$R_{ij} = \partial_a\Gamma^{a}{}_{ij} - \partial_j\Gamma^{a}{}_{ai} + \Gamma^{a}{}_{ab}\Gamma^{b}{}_{ij} - \Gamma^{a}{}_{ib}\Gamma^{b}{}_{aj}.$$

We see that all terms except for the second one vanish in this case because they contain Christoffel symbols with mixed indices.

iv. Show that for $n = 1$ the space is (as a real manifold) a 2-sphere S^2.

Proof. i. The chart $\varphi_r \colon U_r \to \mathbb{C}^n$ is given by

$$\varphi_r([z^0 : \cdots : z^n]) = \left(\frac{z^0}{z^r}, \ldots, \widehat{\frac{z^r}{z^r}}, \ldots, \frac{z^n}{z^r}\right),$$

with inverse given by

$$\varphi_r^{-1}(z^1, \ldots, z^n) = [z^1 : \cdots : z^r : 1 : z^{r+1} : \cdots : z^n],$$

where the hat on a term denotes the negligence of that term in the expression. For $r < s$, the transition map $\varphi_r \circ \varphi_s^{-1} \colon \mathbb{C}^n \setminus \{z^r = 0\} \to \mathbb{C}^n \setminus \{z^s = 0\}$ is given by

$$\varphi_r \circ \varphi_s^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^r}, \ldots, \widehat{\frac{z^r}{z^r}}, \ldots, \frac{z^s}{z^r}, \frac{1}{z^r}, \frac{z^{s+1}}{z^r}, \ldots, \frac{z^n}{z^r}\right).$$

This is a bi-holomorphism with holomorphic inverse $\varphi_s \circ \varphi_r^{-1}$. This makes $\mathbb{CP}^n$ a complex manifold.

ii. It can be shown that $K_{(r)}$ are compatible on the overlap $U_r \cap U_s$, which glues to a global Kähler potential K. (*cf.* Example 3.1.9 of Huybrechts)

To compute the Fubini–Study metric, we look at the fundamental form $\omega_{(r)} = \mathrm{i}\partial\overline{\partial}K_{(r)}$.

$$\begin{aligned}
\omega_{(r)} &= \mathrm{i}\partial\overline{\partial}\log\left(1 + \sum_{i=1}^n |z^i|^2\right) = \mathrm{i}\partial\left(\frac{\sum_{j=1}^n z^j \mathrm{d}\overline{z}^j}{1 + \sum_{i=1}^n |z^i|^2}\right) = \mathrm{i}\sum_{k=1}^n \partial_k\left(\frac{\sum_{j=1}^n z^j \mathrm{d}\overline{z}^j}{1 + \sum_{i=1}^n |z^i|^2}\right)\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^j \\
&= \mathrm{i}\sum_{k=1}^n \frac{(1 + \sum_{i=1}^n |z^i|^2)\delta_{jk} - z^j\overline{z}^k}{(1 + \sum_{i=1}^n |z^i|^2)^2}\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^j.
\end{aligned}$$

Therefore the metric components in U_r is given by

$$g_{i\overline{j}} = \frac{(1 + \sum_{k=1}^n |z^k|^2)\delta_{ij} - \overline{z}^i z^j}{(1 + \sum_{k=1}^n |z^k|^2)^2}.$$

iii. The Ricci form is defined by $\rho(X, Y) := \mathrm{Ric}(JX, Y)$, or $\rho = \mathrm{i}R_{\mu\overline{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu$ in local coordinates. To compute the Ricci components, we need to compute $\det(g_{i\overline{j}})$. Following some change-of-basis trick, we have:

$$\begin{aligned}
\det(g_{i\overline{j}}) &= \frac{1}{(1 + \sum_{i=1}^n |z^i|^2)^{2n}} \det\left(\left(1 + \sum_{i=1}^n |z^i|^2\right)\mathrm{id} - \overline{\boldsymbol{z}}\boldsymbol{z}^\top\right) \\
&= \frac{1}{(1 + \sum_{i=1}^n |z^i|^2)^{2n}} \det\left(\left(1 + \sum_{i=1}^n |z^i|^2\right)\mathrm{id} - \begin{pmatrix} \sum_{i=1}^n |z^i|^2 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}\right) \\
&= \frac{1}{(1 + \sum_{i=1}^n |z^i|^2)^{n+1}}.
\end{aligned}$$

Therefore

$$R_{\mu\overline{\nu}} = -\frac{1}{2}\partial_\mu\partial_{\overline{\nu}}\log\left(\frac{1}{(1 + \sum_{i=1}^n |z^i|^2)^{n+1}}\right)$$

$$
\begin{aligned}
&= \frac{n+1}{2}\partial_\mu \partial_{\overline{\nu}} \log\left(1+\sum_{i=1}^{n}|z^i|^2\right) = \frac{n+1}{2}\partial_\mu \partial_{\overline{\nu}} K_{(r)} \\
&= \frac{n+1}{2} g_{\mu\overline{\nu}}
\end{aligned}
$$

1 again

Fubini–Study metric makes $\mathbb{CP}^n$ an Einstein manifold! The Ricci form is given locally by

$$
\rho = \frac{(n+1)\mathrm{i}}{2}\frac{(1+\sum_{k=1}^{n}|z^k|^2)\delta_{ij} - \overline{z}^i z^j}{(1+\sum_{k=1}^{n}|z^k|^2)^2}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j.
$$

iv. It is obvious that $\mathbb{CP}^1$ is diffeomorphic to the 2-sphere S^2 by the stereographic projection (*cf.* A2. Complex Analysis). We may show further that $\mathbb{CP}^1$ with the Fubini–Study metric is isometric to S^2 with (a rescaling of) the round metric.

By the theorem of space forms in Riemannian geometry (*cf.* Theorem 8.9 in C3.11. Riemannian Geometry), to show the isometry it suffices to show that $\mathbb{CP}^1$ with the Fubini–Study metric has constant sectional curvature $K > 0$. Since $\dim_{\mathbb{R}} \mathbb{CP}^1 = 2$, the Riemann curvature has only one independent component. Therefore this is equivalent to that $\mathbb{CP}^1$ has constant Ricci curvature, that is, $\mathrm{Ric}(X,X) = \text{const}$ for any unit tangent vector field X. But we have shown that $\mathbb{CP}^1$ is Einstein, which implies that $\mathrm{Ric}(X,X) = g(X,X) = 1$ is constant. This concludes the proof. □

3 Calabi–Yau Manifolds F

Question 3.1

Consider the Type IIB supergravity in 10d compactified on a Calabi–Yau three-fold W_6 — with Hodge numbers $h^{p,q}(W_6)$, in particular $h^{1,1}(W_6), h^{1,2}(W_6)$. Determine the bosonic field content by expanding the 10d fields into harmonic forms along W_6. Confirm that the massless spectrum agrees with this of IIA on the mirror Calabi–Yau M_6, where $h^{1,1}(W_6) = h^{1,2}(M_6)$ and $h^{1,2}(W_6) = h^{1,1}(M_6)$.

Proof. (*cf.* §14.3, 14.6 of BLT.) In the 10-dimensional type-IIB supergravity, the bosonic fields from the (R,R)-sector are the p-forms C_p, with $p = 0, 2, 4$; those from the (NS,NS)-sector are the metric g, the anti-symmetric metric 2-form B, and the dilaton Φ. Upon compactification on the Calabi–Yau three-fold, we split the coordinate indices as $(\mu, i, \bar{\imath})$, where μ is the 4d flat spacetime indices and $(i, \bar{\imath})$ are the CY indices.

The decomposition of a p-form ω in $\mathbb{R}^{1,3} \times W_6$ as a direct sum is given by

$$
\omega(x) = \sum_{r=0}^{p} \omega^{\mathrm{M}}_{(p-r)}(x^\mu) \cdot \omega^{\mathrm{CY}}_{(r)}(z^i, \overline{z}^i),
$$

where $\omega^{\mathrm{M}}_{(p-r)}$ is a $(p-r)$-form on $\mathbb{R}^{1,3}$ and $\omega^{\mathrm{CY}}_{(r)}$ is an (r)-form on W_6. The zero-mode equation on W_6 is the Laplace equation

$$
\Delta_6 \omega^{\mathrm{CY}}_{(r)} = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega^{\mathrm{CY}}_{(r)} = 0,
$$

which implies that $\omega^{\mathrm{CY}}_{(r)}$ is a harmonic r-form of W_6. The number of r-forms is the Betti number $b^r(W_6)$, and each such of them gives rise to a massless field representated by a $(p-r)$-form in $\mathbb{R}^{1,3}$. The decomposition of the metric g is similar. In summary, the bosonic fields in 4d are:

$$
g_{\mu\nu},\ g_{ij},\ g_{i\bar{\jmath}},\ g_{\bar{\imath}\bar{\jmath}},\ B_{\mu\nu},\ B_{i\bar{\jmath}},\ \Phi,\ C_0,\ (C_2)_{\mu\nu},\ (C_2)_{i\bar{\jmath}},\ (C_4)_{\mu\nu i\bar{\jmath}},\ (C_4)_{\mu ijk},\ (C_4)_{\mu ij\bar{k}}.
$$

In particular we don't have $(C_2)_{\mu i}$ because $b^1(W_6) = 0$ from the Hodge diamond. These fields are grouped into multiplets (with their fermionic counterparts neglected), as shown below: (*cf.* Table 14.1 of BLT)

Expand a bit more
why $B_{\mu\nu}$ counts as a scalar?

Multiplet	Fields	Multiplicity
Gravity	$g_{\mu\nu}$, $(C_4)_{\mu ijk}$	1
Hyper	Φ, $B_{\mu\nu}$, C_0, $(C_2)_{\mu\nu}$	1
Hyper	$g_{i\bar{j}}$, $B_{i\bar{j}}$, $(C_2)_{i\bar{j}}$, $(C_4)_{\mu\nu i\bar{j}}$	$h^{1,1}(W_6)$
Vector	g_{ij}, $g_{\bar{i}\bar{j}}$, $(C_4)_{\mu ij\bar{k}}$	$h^{1,2}(W_6)$

On the other hand, for the 10-dimensional type-IIA supergravity, the bosonic fields from the (NS,NS)-sector are the same, and those from the (R,R)-sector are the p-forms C_p, with $p = 1, 3$. Upon compactification, the bosonic fields in $\mathbb{R}^{1,3}$ are given by

$$g_{\mu\nu},\ g_{ij},\ g_{i\bar{j}},\ g_{\bar{i}\bar{j}},\ B_{\mu\nu},\ B_{i\bar{j}},\ \Phi,\ (C_1)_\mu\ (C_3)_{\mu i\bar{j}}\ (C_3)_{ijk}\ (C_3)_{ij\bar{k}}\ (C_3)_{i\bar{j}\bar{k}}\ (C_3)_{\bar{i}\bar{j}\bar{k}}.$$

These fields are grouped into multiplets (with their fermionic counterparts neglected), as shown below:

Multiplet	Fields	Multiplicity
Gravity	$g_{\mu\nu}$, $(C_1)_\mu$	1
Hyper	Φ, $B_{\mu\nu}$, $(C_3)_{ijk}$, $(C_3)_{\bar{i}\bar{j}\bar{k}}$	1
Hyper	g_{ij}, $g_{\bar{i}\bar{j}}$, $(C_2)_{i\bar{j}}$, $(C_3)_{ij\bar{k}}$, $(C_3)_{i\bar{j}\bar{k}}$	$h^{1,2}(M_6)$
Vector	$g_{i\bar{j}}$, $B_{i\bar{j}}$, $(C_3)_{\mu i\bar{j}}$	$h^{1,1}(M_6)$

Note that if $h^{1,1}(W_6) = h^{1,2}(M_6)$ and $h^{1,2}(W_6) = h^{1,1}(M_6)$, the two models produce the same massless spectrum. This is exactly the mirror symmetry between M_6 and W_6. ✓ □

Question 3.2

Consider now a Calabi–Yau 2-fold (real 4d space), also called a K3-surface. The Hodge diamond is completely fixed in this case and has non-trivial entries

$$h^{2,2} = h^{0,0} = h^{2,0} = h^{0,2} = 1, \quad h^{1,1} = 20.$$

By Hodge duality, the $(1, 1)$ forms are self-dual.

i. Determine the degrees of freedom that the following bosonic fields have in 6d: scalar, anti-symmetric tensor, graviton, vector.

ii. IIB has two spinors ϵ, ϵ' in the **16**. Decompose the spinors appropriate for a compactification to $\mathbb{R}^{1,5} \times M_4$, where M_4 is (1) a generic 4d manifold and (2) a K3 surface. What is the supersymmetry of the 6d theory obtained from IIB on K3?

iii. By expanding the IIB bosonic supergravity fields determine the massless spectrum of IIB on K3.

[*Note: be careful about the self-duality of* F_5.]

Proof. i. (*cf.* §19.8 of Polchinski.)

ii. The local Lorentz group is decomposed by $\mathrm{SO}(1, 9) \to \mathrm{SO}(1, 5) \times \mathrm{SO}(4)$. Note that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The 16 component Weyl spinors are decomposed as

$$\mathbf{16} = \mathbf{4}_L \otimes \overline{\mathbf{2}} \oplus \mathbf{4}_R \otimes \mathbf{2}, \qquad \overline{\mathbf{16}} = \mathbf{4}_L \otimes \mathbf{2} \oplus \mathbf{4}_R \otimes \overline{\mathbf{2}},$$

where $\mathbf{4}_L, \mathbf{4}_R$ are Weyl spinors of $\mathfrak{so}(1, 5)$ and $\mathbf{2}, \overline{\mathbf{2}}$ are Weyl spinors of $\mathfrak{so}(4)$.

- If M_4 is a generic manifold, then the Killing spinor condition $\nabla_\mu \epsilon = 0$ is violated and the supersymmetry is broken...?

- If M_4 is the K3 surface, then it has SU(2) holonomy. The compactification of IIB on K3 produces the chiral $(2,0)$-supersymmetry in $\mathbb{R}^{1,5}$. ✓

iii. $\square$