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**Problem Sheet 2**

# ASO: Projective Geometry

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In these questions, $F$ denotes the base field.

**Question 1**

Write down the dual to the Pappus' Theorem.

*Theorem.* Suppose that $L_A, L_B, L_C$ and $L_{A'}, L_{B'}, L_{C'}$ are two triples of concurrent lines. Let $X = L_A \cap L_{B'}$, $X' = L_{A'} \cap L_B$, $Y = L_B \cap L_{C'}$, $Y' = L_{B'} \cap L_C$, $Z = L_C \cap L_{A'}$, and $Z' = L_{C'} \cap L_A$. Then the lines $XX'$, $YY'$ and $ZZ'$ are concurrent. □

**Question 2**

Let $P_0, P_1, P_2, P_3$ be four distinct points in a projetive plane $\mathbb{P}(V)$. Show that $P_0, P_1, P_2, P_3$ are in general position if and only if the lines $P_0P_1, P_1P_2, P_2P_3, P_3P_0$ are in general postition in $\mathbb{P}(V^*)$.

*Proof.* Suppose that $P_0 = \langle v_0 \rangle$, $P_1 = \langle v_1 \rangle$, $P_2 = \langle v_2 \rangle$, and $P_3 = \langle v_3 \rangle$, where $v_0, v_1, v_2, v_3 \in V$. In the dual space $V^*$, we have

$$(P_0P_1)^* = \langle v_0, v_1 \rangle^\circ \qquad (P_1P_2)^* = \langle v_1, v_2 \rangle^\circ \qquad (P_2P_3)^* = \langle v_2, v_3 \rangle^\circ \qquad (P_3P_0)^* = \langle v_3, v_0 \rangle^\circ$$

Suppose that the four points are not in general position. Without loss of generality we assume that $(P_0P_1)^*$, $(P_1P_2)^*$ and $(P_2P_3)^*$ are collinear. Then we have

$$\dim((P_0P_1)^* + (P_1P_2)^* + (P_2P_3)^*) = 2$$

But by Proposition 8.1 we know that

$$\dim((P_0P_1)^* + (P_1P_2)^* + (P_2P_3)^*) = \dim(\langle v_0, v_1 \rangle^\circ + \langle v_1, v_2 \rangle^\circ + \langle v_2, v_3 \rangle^\circ) = \dim(\langle v_0, v_1 \rangle \cap \langle v_1, v_2 \rangle \cap \langle v_2, v_3 \rangle)^\circ = 2$$

so that

$$\dim(\langle v_1 \rangle \cap \langle v_2, v_3 \rangle) = \dim(\langle v_0, v_1 \rangle \cap \langle v_1, v_2 \rangle \cap \langle v_2, v_3 \rangle) = \dim V - 2 = 1$$

Hence $v_1 \in \langle v_2, v_3 \rangle$. In $\mathbb{P}(V)$, $P_1 \in P_2P_3$. So the four points are not in general position. □

**Question 3**

Use the general position argument to show that given five points in the projective plane, such that no three are collinear, there is a unique conic through these five points.

*Proof.* Let $A, B, C, D, E$ be the given five points. By assumption $A, B, C, D$ are in general position. By applying a projective transformation we may assume that $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $C = [0 : 0 : 1]$ and $D = [1 : 1 : 1]$. Suppose that $E = [\alpha_0 : \alpha_1 : \alpha_2]$. Let $\mathcal{C} : \sum_{i,j=0}^{2} \lambda_{i,j} x_i x_j = 0$ be a conic that contains the five points.

$A, B, C \in \mathcal{C}$ implies that $\lambda_{0,0} = \lambda_{1,1} = \lambda_{2,2} = 0$. So $\mathcal{C}$ has the form

$$\lambda_{0,1} x_0 x_1 + \lambda_{1,2} x_1 x_2 + \lambda_{2,1} x_2 x_0 = 0$$

$D, E \in \mathcal{C}$ implies that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (1, 1, 1) = 0$, $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (\alpha_0, \alpha_1, \alpha_2) = 0$. Since $D \neq E$, $\langle (1, 1, 1) \rangle \neq \langle (\alpha_0, \alpha_1, \alpha_2) \rangle$. We deduce that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \in \langle (1, 1, 1),\ (\alpha_0, \alpha_1, \alpha_2) \rangle^\perp$, which is a 1-dimensional subspace. Hence the coefficients of the quadric is uniquely determined up to rescaling by a constant. The conic determined by the quadric is unique. □

**Question 4**

Let $C, D$ be conics in a projective plane $\mathbb{P}(V)$, where $V$ is a 3-dimensional real vector space, and suppose that $C \cap D = \{p_1, p_2, p_3, p_4\}$, where $p_1, ..., p_4$ are dinstinct points in $\mathbb{P}(V)$.

(a) Show that $p_1, ..., p_4$ are in general position. Prove that there exist homogeneous coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}(V)$ for which

$$p_1 = [1 : 1 : 1], \qquad p_2 = [1 : -1 : 1] \qquad p_3 = [1 : 1 : -1] \qquad p_4 = [1 : -1 : -1]$$

(b) Show that any conic through $p_1, ..., p_4$ has equation $\lambda x_0^2 + \mu x_1^2 + \nu x_2^2 = 0$, where $\lambda + \mu + \nu = 0$.

(c) Find four projective transformations $\tau$ of $\mathbb{P}(V)$ that form a group, and for which $\tau(C) = C$ and $\tau(D) = D$.

*Proof.* (a) Let $p_1 = \langle v_1 \rangle$, $p_2 = \langle v_2 \rangle$, $p_3 = \langle v_3 \rangle$, and $p_4 = \langle v_4 \rangle$, where $v_1, v_2, v_3, v_4 \in V$. Suppose for contradiction that $p_1$, $p_2$ and $p_3$ are collinear. By rescaling we may assume that $v_3 = v_1 + v_2$. Let $\langle Bx, x \rangle = 0$ be the equation of $C$. Then we have

$$\langle Bv_1, v_1 \rangle = 0 \qquad \langle Bv_2, v_2 \rangle = 0 \qquad \langle B(v_1 + v_2), (v_1 + v_2) \rangle = 0$$

We expand the third equation:

$$\langle B(v_1 + v_2), v_1 + v_2 \rangle = \langle Bv_1, v_1 \rangle + 2 \langle Bv_1, v_2 \rangle + \langle Bv_2, v_2 \rangle = 0.$$

Hence we have $\langle Bv_1, v_2 \rangle = 0$. In particular, for any $\mu v_1 + \nu v_2$,

$$\langle B(\mu v_1 + \nu v_2), \mu v_1 + \nu v_2 \rangle = \mu^2 \langle Bv_1, v_1 \rangle + 2\mu\nu \langle Bv_1, v_2 \rangle + \nu^2 \langle Bv_2, v_2 \rangle = 0$$

We deduce that the conic $C$ contains the whole projective line $p_1 p_2 p_3$. Similarly $D$ also contains $p_1 p_2 p_3$. Then $C \cap D$ is an infinite set. Contradiction. Then $p_1, ..., p_4$ are in general position.

Since $[1 : 1 : 1], [1 : -1 : 1], [1 : 1 : -1], [1 : -1 : -1]$ are in general position, by general position theorem there exists a projective transformation which maps the standard basis to a basis in which $p_1 = [1 : 1 : 1]$, $p_2 = [1 : -1 : 1]$, $p_3 = [1 : 1 : -1]$ and $p_4 = [1 : -1 : -1]$.

(b) Suppose that the quadric has equation $\sum_{i,j=0}^{2} \alpha_{i,j} x_i x_j = 0$. Since the quadric passes through $p_1, ..., p_4$, we have

$$\begin{cases} \alpha_{0,0} + \alpha_{1,1} + \alpha_{2,2} + 2\alpha_{0,1} + 2\alpha_{1,2} + 2\alpha_{2,0} = 0 \\ \alpha_{0,0} + \alpha_{1,1} + \alpha_{2,2} - 2\alpha_{0,1} - 2\alpha_{1,2} + 2\alpha_{2,0} = 0 \\ \alpha_{0,0} + \alpha_{1,1} + \alpha_{2,2} + 2\alpha_{0,1} - 2\alpha_{1,2} - 2\alpha_{2,0} = 0 \\ \alpha_{0,0} + \alpha_{1,1} + \alpha_{2,2} - 2\alpha_{0,1} + 2\alpha_{1,2} - 2\alpha_{2,0} = 0 \end{cases}$$

In matrix form the equations are

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{0,1} \\ \alpha_{1,2} \\ \alpha_{2,0} \end{pmatrix} = \eta \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where $\eta$ satisfies that $\alpha_{0,0} + \alpha_{1,1} + \alpha_{2,2} = -2\eta$.

We perform elementary row operations to the coefficients matrix. It is not hard to verify that the system has only trival solution: $\alpha_{0,1} = \alpha_{1,2} = \alpha_{2,0} = \eta = 0$. Then we deduce that the quadric has the equation $\lambda x_0^2 + \mu x_1^2 + \nu x_2^2 = 0$ where $\lambda + \mu + \nu = 0$.

(c) We consider the subgroup of the permutation group of $\{p_1, p_2, p_3, p_4\}$ which induces a group of projective transformations. We consider the subgroup generated by the double transpositions $(1\ 2)(3\ 4)$ and $(1\ 3)(2\ 4)$, which corresponds to the subgroup generated by reflections $x_1 \mapsto -x_1$ and $x_2 \mapsto -x_2$. These transfomrations fix $C$ and $D$, because the equations of the quadrics have no off-diagonal terms. The action of the group on $p_1, ..., p_4$ are:

$$\{\text{id},\ (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$$

The subgroup is isomorphic to $V_4$. □

**Question 5**

Let $F = (x_0, x_1, x_2)$ be a homogneous polynomial of degree $n$. Let $\mathcal{C}$ be the set of points $[a_0 : a_1 : a_2]$ in $\mathbb{RP}^2$ such that $F(a_0 : a_1 : a_2) = 0$. Let $\boldsymbol{a}$ be a point on $\mathcal{C}$. Provided that $\boldsymbol{\nabla} F(\boldsymbol{a}) \neq \boldsymbol{0}$, the *tangent line* to $\mathcal{C}$ at $\boldsymbol{a} = [a_0 : a_1 : a_2]$ is the line

$$x_0 \frac{\partial F}{\partial x_0}(\boldsymbol{a}) + x_1 \frac{\partial F}{\partial x_1}(\boldsymbol{a}) + x_2 \frac{\partial F}{\partial x_2}(\boldsymbol{a}) = 0$$

in $\mathbb{RP}^2$ and $\boldsymbol{a}$ is said to be *singular* if $\boldsymbol{\nabla} F(\boldsymbol{a}) = 0$.

(i) Show that $\boldsymbol{a}$ lies on the tangent line to $\boldsymbol{a}$.

(ii) Given a $3 \times 3$ symmetric real matrix $B$ its associated *conic* is the set of solutions to the equation $\boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} = 0$ where $\boldsymbol{x} = [x_0 : x_1 : x_2]$ and the conic is said to be *singular* if $B$ is singular. Show that a conic is singular if and only if it has a singular point.

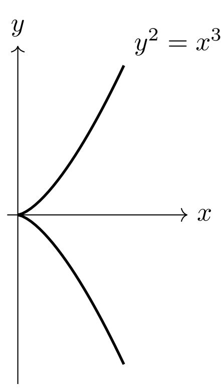
(iii) Sketch the curves $y^2 = x^3$ and $y^2 = x^2(x+1)$ in $\mathbb{R}^2$. What singular points fo these curves have? Show that $y = x^3$ has a singular point at infinity.

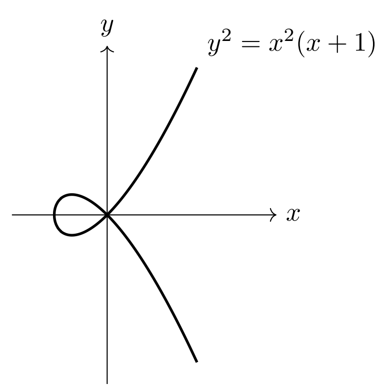
*Proof.* (i) By Intro Manifolds Sheet 1 Question 2, $F$ is homogeneous of degree $n$ implies that $\langle \boldsymbol{\nabla} F(\boldsymbol{x}), \boldsymbol{x} \rangle = nF(\boldsymbol{x})$. Hence $F(\boldsymbol{a}) = 0$ implies that $\langle \boldsymbol{\nabla} F(\boldsymbol{a}), \boldsymbol{a} \rangle = 0$. That is, $\boldsymbol{a}$ lies on the tangent line to $\boldsymbol{a}$.

(ii) Let $F(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x}$. Then $\boldsymbol{\nabla} F(\boldsymbol{x}) = 2B\boldsymbol{x}$. We have:

$$\begin{aligned} \mathcal{C} \text{ has a singular point } \boldsymbol{a} &\iff \boldsymbol{\nabla} F(\boldsymbol{a}) = 2B\boldsymbol{a} = 0 \\ &\iff \ker B \neq \{0\} \\ &\iff B \text{ is singular} \\ &\iff \mathcal{C} \text{ is singular} \end{aligned}$$

(iii) The sketch of the two curves:




We observe that both curves have a singularity at $(0,0)$. For the curve $y = x^3$ in $\mathbb{R}^2$, we embed $\mathbb{R}^2$ into $\mathbb{RP}^2$ by identifying $(x, y)$ with $[1 : x : y]$. We make the equation of the curve homogeneous (of degree 3) by setting $F(x, y, z) = x^3 - yz^2$. Then clearly the vanishing loci of $F$ on $\mathbb{R}^2$ is the curve $y = x^3$.

On the line of infinity, $z = 0$. $F(x, y, 0) = x^3 = 0 \implies x = 0$. The locus of $F$ at infinity is $[0 : 1 : 0]$. We compute that gradient of $F$:

$$\boldsymbol{\nabla} F(x, y, z) = (3x^2, -z^2, -2yz).$$

Then $\nabla F(0,1,0) = 0$. We conclude that $[0:1:0]$ is a singularity of the curve. □

**Question 6**

Find all rational numbers $x, y$ such that $x^2 + y^2 - xy = 1$.

*Solution.* We embed $\mathbb{Q}^2$ into $\mathbb{QP}^2$ via $(x,y) \mapsto [x:y:1]$. Then the curve $x^2+y^2-xy=1$ is the loci of the quadratic form $F(x,y,z) = x^2+y^2-xy-z^2$. Let $\mathcal{C}$ be the conic represented by $F$ in $\mathbb{QP}^2$. The Gram matrix associated with the quadratic form is given by

$$B = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We choose $x = (1,1,1)$, a zero of $F$ in $\mathbb{Q}^3$, and $X = \langle x \rangle \in \mathbb{QP}^2$. Consider the projectiv line $L: \ z = 0$ in $\mathbb{QP}^2$. Clearly $X \notin L$. Then by Theorem 9.10 in the lecture notes, there is a bijection $\alpha : L \to \mathcal{C}$ such that for each $Y \in L$, $X, Y, \alpha(Y)$ are collinear. For $Y = \langle y \rangle = \langle (\lambda_1, \lambda_2, 0) \rangle$, $\alpha$ is given explicitly by:

$$\alpha(y) = \langle By, y \rangle\, x - 2\, \langle Bx, y \rangle\, y$$

(*The point $X$ and the line $L$ are chosen deliberately such that the solutions are symmetric in the parameters $\lambda_1$ and $\lambda_2$.*)

We compute $\langle By, y\rangle$ and $\langle Bx, y\rangle$:

$$\langle By, y \rangle = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2$$

$$\langle Bx, y \rangle = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{2}(\lambda_1 + \lambda_2)$$

Therefore

$$\alpha(y) = (\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2)(1,1,1) - (\lambda_1+\lambda_2)(\lambda_1,\lambda_2,0) = (\lambda_2^2 - \lambda_1\lambda_2,\ \lambda_1^2-\lambda_1\lambda_2,\ \lambda_1^2+\lambda_2^2-\lambda_1\lambda_2)$$

For $\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2 \neq 0$, we deduce that $\left( \frac{\lambda_2^2-\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2}, \frac{\lambda_1^2-\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2} \right) \in \mathbb{Q}^2$ are on $\mathcal{C}$ for $\lambda_1, \lambda_2 \in \mathbb{Z}$.

Moreover, we claim that any rational solution of $x^2+y^2-xy=1$ can be expressed in the given form. This is because $\alpha$ is surjective. □

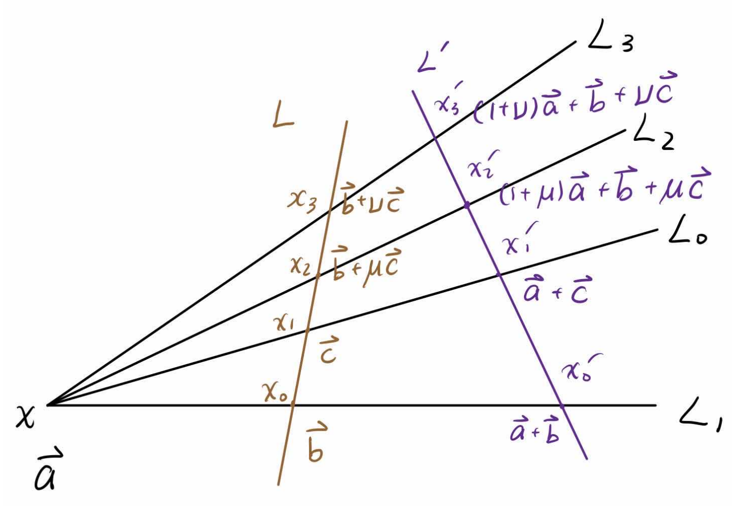
**Question 7**

Let $V$ be a 3-dimensional real vector space and suppose that $L_0, L_1, L_2, L_3$ are four lines in the projective plane $\mathbb{P}(V)$ all intersecting in a common point $x$. Explain why

(i) if $L$ is a line in $\mathbb{P}(V)$ that does not pass through $x$, but intersects $L_i$ in a point $x_i$ (so $x_0, x_1, x_2, x_3$ are four distinct collinear points), then the cross-ratio $(x_0x_1 : x_2x_3)$ is independent of the choice of $L_i$.

(ii) the cross-ratio defined in (i) equals the cross-ratio $(L_0L_1 : L_2L_3)$ formed by regarding $L_0, L_1, L_2, L_3$ as collinear points of the dual projetive plane $\mathbb{P}(V^*)$.

*Proof.* (i) Consider two different lines $L$ and $L'$, which corresponds to two sets of points: $\{x_0, x_1, x_2, x_3\}$ and $\{x_0', x_1', x_2', x_3'\}$. Let $x = \langle a \rangle$, $x_1 = \langle b \rangle$ and $x_2 = \langle c \rangle$, where $a, b, c \in V$ are linearly independent. Since $x_0'$ lies on the projective




line $xx_0$, we can set $x_0' = \langle a+b \rangle$. Since $x_1'$ lies on the projective line $xx_1$, we can set $x_1' = \langle a+c \rangle$. Since $x_2, x_3$ lie on the line $x_0x_1$, we have $x_2 = \langle b+\mu c \rangle$, $x_3 = \langle b+\nu c \rangle$ for some $\mu, \nu \in \mathbb{R}\backslash\{0\}$. From the figure we observe that $x_2' = xx_2 \cap x_0'x_1' = \langle (1+\mu)a + b + \mu c \rangle$ and $x_3' = xx_3 \cap x_0'x_1' = \langle (1+\nu)a + b + \nu c \rangle$.

Let $L = \mathbb{P}(U) = \mathbb{P}\langle b, c \rangle$ and $L' = \mathbb{P}(W) = \mathbb{P}\langle a+b, a+c \rangle$. Then the linear transformation $\varphi : U \to W$ given by

$$\varphi(b) = a+b \qquad \varphi(c) = a+c$$

induces a projective transformation $\tilde{\varphi} : L \to L'$, which sends $x_0, x_1, x_2, x_3$ to $x_0', x_1', x_2', x_3'$ respectively. Since the cross-ratio is preserved by a projective transformation, $(x_0x_1 : x_2x_3) = (x_0'x_1' : x_2'x_3')$. We conclude that the cross-ratio is independent of the choice of $L$.

(ii) We consider $\{a, b, c\}$ as a basis of $V$. Then the points have coordinates:

$$x_0 = [0:1:0] \qquad x_1 = [0:0:1] \qquad x_2 = [0:1:\mu], \qquad x_3 = [0:1:\nu]$$

Let $f_a$, $f_b$, $f_c$ be the dual basis of $a, b, c$. In the dual space $\mathbb{P}(V*)$, the dual of the lines are given by:

$$L_0^* = \langle a, b \rangle^\circ \qquad L_1^* = \langle a, c \rangle^\circ \qquad L_2^* = \langle a, b+\mu c \rangle^\circ, \qquad L_3^* = \langle a, b+\nu c \rangle^\circ$$

In the dual basis, these dual lines have coordinates

$$L_0^* = [0:0:1] \qquad L_1^* = [0:1:0] \qquad L_2^* = [0:\mu:-1], \qquad L_3^* = [0:\nu:-1]$$

Finally, by explicit calculations, we verify that

$$(x_0x_1 : x_2x_3) = (L_0^*L_1^* : L_2^*L_3^*) = \mu/\nu$$

□